Quantization of Singular Systems in Canonical Formalism

Bachelor thesis by
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Conventions and Nomenclature

Throughout this thesis the Einstein summation convention is used. The range of each summation is either explicitly mentioned or emerges from the context. For first class constraints only latin indices are used and for second class constraints greek indices.

An equality on the submanifold, induced by the constraints, is indicated by the symbol $\approx$. It shall not be mistaken for an approximation, for which we use $\simeq$.

In Section 8 we use the usual four-vector notation, where the indices μ,ν,λ run from 0 to 3 and i,j,k from 1 to 3. The components of a contravariant vector will be denoted with superscripts

$$x^\mu = (x^0, x^1, x^2, x^3)^T \,, \tag{0.1}$$

and the covariant components with subscripts

$$x_\mu = (x_0, x_1, x_2, x_3)^T \,. \tag{0.2}$$

An index is raised or lowered by the Minkowski metric

$$x^\mu = \eta^{\mu\nu} x_\nu \,, \qquad x_\mu = \eta_{\mu\nu} x^\nu \,, \tag{0.3}$$

where the latter is given by

$$\eta_{\mu\nu} = \eta^{\mu\nu} \;:=\; \mathrm{diag}(+1, -1, -1, -1) \,. \tag{0.4}$$

The partial derivative with respect to the contravariant coordinates is denoted as

$$\frac{\partial \varphi}{\partial x^\mu} = \partial_\mu \varphi = \varphi_{,\mu} \,, \tag{0.5}$$

especially the time derivative by a dot

$$\partial_0 \varphi = \dot{\varphi} \,. \tag{0.6}$$

The latter does not contain a factor c^{-1} since we are working in Heaviside units. Within this four-formalism the spatial components A^i of a four-vector A^μ just differ from their covariant components by a minus sign.

The three dimensional Levi-Civita tensor is defined as

$$\epsilon^{ijk} \;:=\; \begin{cases} +1 & \text{if } (ijk) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (ijk) \text{ is an odd permutation of } (1,2,3) \\ \;\;0 & \text{if two or more indices are identical} \end{cases} \tag{0.7}$$

The Kronecker symbol in three dimensions

$$\delta^i{}_j = \delta_j{}^i = \delta^i_j \,, \qquad \delta^{ij} = \delta_{ij} \tag{0.8}$$

is defined to be equal to 1 if $i = j$ and 0 else. A generalization to the four-formalism is obvious.

1 Introduction

In the last few decades there has been rapid developments in the realm of quantum field theory and especially gauge field theory. To quote one of the greatest physicists of our time:

> *Three principles - the conformability of nature to herself, the applicability of the criterion of simplicity, and the 'unreasonable effectiveness' of certain parts of mathematics in describing physical reality - are thus consequences of the underlying law of the elementary particles and their interactions. Those three principles need not be assumed as separate metaphysical postulates. Instead, they are emergent properties of the fundamental laws of physics.* - Murray Gell-Mann: "Beauty and truth in physics", talk in march 2007 in Monterey, California on TED TV

Gauge theory is a tremendous accomplishment of the human mind and combines a plenitude of mathematical notions and techniques. But as physicists we do not study theories just because of their beauty, we have to keep an eye on physical application. For instance, theoretical elementary particle physics utilizes the methods of gauge theory to construct a description of the fundamental particles and their interactions. The discovery of the $SU(3)$ symmetry and the development of the theory, today known as Quantum-Chromodynamics, mainly by Gell-Mann, was a milestone in that physical field. We might be hopeful that a unification of *all* fundamental forces is accessible in a description with gauge fields.

Now, one is tempted to ask, why we are talking about gauge field theory. What is the relation between gauge field theory and singular systems? As we will expose within this thesis, gauge field theories are just a special class of singular Lagrangian, respectively Hamiltonian, systems. The physical systems we are dealing with contain gauge theories as a subset. Hence this thesis is supposed to be a fundamental work, which may illuminate the underlying structure of gauge theory from another perspective.

Section 2 clarifies the notions of a singular physical system and constraints on the classical phase space. It turns out that the presence of singularities causes such constraints and vice versa, so that we may use these notions synonymously. The following sections attend to the study of the constraints. Within the Dirac-Bergmann algorithm we identify all the constraints. There are in general two possibilities to differentiate the emerging constraints. The classification into primary and secondary refers to the origin of the specific constraint, and the classification into first and second class refers to their properties. The precise meaning of these concepts will become apparent in the thesis. Especially the study of first class constraints will be interesting, since here we encounter the relation to gauge theory. After studying the properties of the constraints, our ultimate goal is to perform the transition to quantum theory. Section 6 briefly recalls the canonical quantization program and describes the problems arising in the presence of constraints. Furthermore, this section presents a consistent way to incorporate the classical restrictions on phase space into a quantum theory. The limiting process to an infinite number of degrees of freedom to arrive at a field theoretical description is done in Section 7. Finally, we want to complete the treatment by discussing Maxwell's electrodynamics as an example.

This description of singular systems and their quantization is mainly conform to those of K. Sundermeyer [24] and M.Henneaux/C. Teitelboim [16]. Furthermore, we extensively used the textbooks of T. Kugo [20] and W. Greiner [13]. Of course, these textbooks are just a small excerpt of the list of references used to compose this thesis.

2 Singular Systems

This section serves as an initiation to the concept of singularities in the Lagrange formalism. We will introduce some basic notions such as constraints arising due to the singularities and the definition of the canonical momenta.

We will start our discussion of constrained systems with the principle of least action. According to the latter a physical system can be described by a function L depending on the positions and velocities

$$L = L(q(t), \dot{q}(t)) \,. \tag{2.1}$$

We assume for the sake of simplicity that this *Lagrange function* exhibits no explicit time dependence. The abbreviations $q(t)$ and $\dot{q}(t)$ stand for the set of *all* positions $q(t) = \{q^i(t)\}$ and velocities $\dot{q}(t) = \{\dot{q}^i(t)\}$, respectively, with $i = 1, ..., N$. The system's motion proceeds in a way that the action integral

$$\mathcal{A} = \int_{t_1}^{t_2} dt\, L(q(t), \dot{q}(t)) \tag{2.2}$$

becomes stationary under infinitesimal variations $\delta q^i(t)$. Assuming that the end points are fixed during the variation, i.e. $\delta q^i(t_1) = \delta q^i(t_2) = 0$, yields the equations of motion for the classical path

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0 \,. \tag{2.3}$$

Executing the total time derivative gives

$$\ddot{q}^j \frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i} = \frac{\partial L}{\partial q^i} - \dot{q}^j \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i} \,. \tag{2.4}$$

In this form we recognize that the accelerations $\ddot{q}^i$ can be uniquely expressed by the positions q^i and the velocities $\dot{q}^i$ if and only if the Hessian matrix

$$H_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \tag{2.5}$$

is invertible. In other words its determinant must not vanish

$$\det H_{ij} \neq 0 \,. \tag{2.6}$$

Since we are interested in the Hamiltonian formulation, we have to perform a Legendre transformation from the velocities to the momenta. The latter are defined as

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \,. \tag{2.7}$$

In the case that the determinant vanishes, the Hessian (2.5) is singular and some of the accelerations are not determined by the velocities and positions. This means that some of the variables are not independent from each other. The singularity of the Hessian is equivalent to the noninvertibility of (2.7). As a consequence, in a singular system we are not able to display the velocities as functions of the momenta and the positions. This gives rise to the existence of R relations between the positions and momenta

$$\phi_r(p, q) = 0, \qquad r = 1, ..., R \,, \tag{2.8}$$

if the rank of (2.5) is $(N - R)$. These conditions, which obviously cannot be equations of motion, are called primary constraints. They follow directly from the structure of the Lagrangian and the definition of the momenta (2.7). The interesting point is that these functions are real restrictions on the phase space. All constraints together define a submanifold Γ_P in phase space of dimension $(2N - R)$, which contains all physically achievable variables. In the following it is not the structure of this manifold which we are interested in, but simply the physical implications due to the existence of such a manifold.

There is an ambiguity in the functional form of (2.8). With $\phi_r = 0$ we also have that $(\phi_r)^2 = 0$. This means the submanifold can be overdetermined, there exist many equivalent ways in representing Γ_P. In order to have a minimal set of constrained functions we have to impose regularity conditions. That is simply, the Jacobian matrix

$$J = \frac{\partial(\phi_1, ..., \phi_R)}{\partial(\{q^i\}, \{p_i\})}\,, \tag{2.9}$$

has to be of rank R on the constraint surface. Alternatively, one can characterize the regularity conditions by the statement that the gradients of ϕ_r have to be *linearly* independent. In the following we will assume that this requirement is fulfilled and the ϕ_r are independent of each other.

3 Dirac-Bergmann Algorithm

Knowing about the concept of singularities in the Lagrange formalism, we work out the consequences for the transition to the Hamiltonian formulation.

3.1 Equations of Motion with Primary Constraints

The transition to the Hamilton formulation of mechanics is given by the Legendre transformation

$$H(p(t), q(t)) = p_i(t)\dot{q}^i(t) - L(q(t), \dot{q}(t))\,. \tag{3.1}$$

Although some of the velocities $\dot{q}^i(t)$ cannot be expressed as functions of the positions and momenta, the Hamilton function is *not* a function of the velocities. This can be seen as follows:

$$\begin{aligned} dH &= d(p_i\dot{q}^i) - dL \\ &= \dot{q}^i dp_i + p_i d\dot{q}^i - \frac{\partial L}{\partial q^i} dq^i - \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i \\ &= \dot{q}^i dp_i - \frac{\partial L}{\partial q^i} dq^i\,, \end{aligned} \tag{3.2}$$

where we used the definition of the momenta (2.7) from the second to the third line. We recognize that the velocity differentials completely disappear, only differentials of the positions and momenta are left. As a consequence, the Hamilton function *cannot* be a function of the velocities. Using (3.1), one can write the action as

$$\mathcal{A} = \int dt\, \Big[p_i(t)\dot{q}^i(t) - H(p(t), q(t))\Big]\,. \tag{3.3}$$

Recall that in the Lagrange formalism constraints are taken into account by coupling them to the Lagrangian via multipliers $\lambda^r(t)$

$$L(q^i(t), \dot{q}^i(t)) + \lambda^r(t)\,\phi_r(q(t), \dot{q}(t))\,. \tag{3.4}$$

Herein the multipliers act as new dynamical variables. This can also be done within the Hamilton formalism. If we perform the Legendre transformation and couple the primary constraints (2.8), we arrive at the action

$$\mathcal{A} = \int dt\, \Big[p_i(t)\dot{q}^i(t) - H(p(t), q(t)) - \lambda^r(t)\,\phi_r(p(t), q(t))\Big]\,. \tag{3.5}$$

The variation of this action gives

$$\begin{aligned} \delta\mathcal{A} &= \delta \int dt\, \Big[p_i\dot{q}^i - H(p,q) - \lambda^r \phi_r(p,q)\Big] \\ &= \int dt\, \Big[\delta p_i \dot{q}^i + p_i \delta\dot{q}^i - \delta H - \delta\lambda^r \phi_r - \lambda^r \delta\phi_r\Big] \\ &= \int dt\, \left[\delta p_i \dot{q}^i - \dot{p}_i \delta q^i - \frac{\partial H}{\partial q^i}\delta q^i - \frac{\partial H}{\partial p_i}\delta p_i - \delta\lambda^r \phi_r - \lambda^r \left(\frac{\partial \phi_r}{\partial p^i}\delta q^i + \frac{\partial \phi_r}{\partial p_i}\delta p_i\right)\right] \\ &= \int dt\, \left[\left(-\dot{p}_i - \frac{\partial H}{\partial q^i} - \lambda^r \frac{\partial \phi_r}{\partial q^i}\right)\delta p_i + \left(\dot{q}^i - \frac{\partial H}{\partial p_i} - \lambda^r \frac{\partial \phi_r}{\partial p_i}\right)\delta q^i - \phi_r \delta\lambda^r\right]\,. \end{aligned} \tag{3.6}$$

Since the classical path is obtained by the principle of least action, the variation has to vanish. Because the variations of the positions δq^i, momenta δp_i and multipliers $\delta\lambda^r$ are regarded as independent of each other, the three integrands have to vanish separately

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} - \lambda^r \frac{\partial \phi_r}{\partial q^i}\,, \tag{3.7a}$$

$$\dot{q}^i = \frac{\partial H}{\partial p_i} + \lambda^r \frac{\partial \phi_r}{\partial p_i}\,, \tag{3.7b}$$

$$\phi_r = 0\,. \tag{3.7c}$$

The first two sets of these equations are the Hamiltonian equations of motion and the third set are just the primary constraints. We recognize that the theory is invariant under the transformation $H \to H + \mu^r \phi_r$, since it would result in a renaming of the multipliers $\lambda^r \to \lambda^r + \mu^r$.

Defining the Poisson bracket of two phase space functions $F(p, q)$ and $G(p, q)$ as

$$\{F, G\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q^i}\frac{\partial F}{\partial p_i}\,, \tag{3.8}$$

we can write the fundamental Hamilton equations of motion (3.7a), (3.7b) as

$$\dot{p}_i = \{p_i, H\} + \lambda^r \{p_i, \phi_r\} \tag{3.9a}$$

$$\dot{q}^i = \{q^i, H\} + \lambda^r \{q^i, \phi_r\}\,. \tag{3.9b}$$

For a general phase space function $F(p, q)$ the equation of motion reads

$$\begin{aligned}\dot{F} &= \frac{\partial F}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial F}{\partial q^i}\frac{dq^i}{dt} \\ &= \frac{\partial F}{\partial p_i}\left(-\frac{\partial H}{\partial q^i} - \lambda^r \frac{\partial \phi_r}{\partial q^i}\right) + \frac{\partial F}{\partial q^i}\left(\frac{\partial H}{\partial p_i} + \lambda^r \frac{\partial \phi_r}{\partial p_i}\right) \\ &= \{F, H\} + \lambda^r \{F, \phi_r\}\,.\end{aligned} \tag{3.10}$$

At this stage we introduce the so called primary Hamiltonian

$$H_P = H + \lambda^r \phi_r\,. \tag{3.11}$$

This is nothing but the canonical Hamiltonian with the constraints coupled via Lagrange multipliers. Using this Hamilton function the equation of motion gains its ordinary form as if there were no constraints

$$\begin{aligned}\dot{F} &= \{F, H_P\} - \{F, \lambda^r\}\phi_r \\ &\approx \{F, H_P\}\,.\end{aligned} \tag{3.12}$$

Here we used the fact that the constraints vanish on the physical submanifold. This is indicated by the symbol $\approx$ meaning weak equality, that is an equality on the manifold induced by the constraints. Consequently, we should write the constraints as $\phi_r \approx 0$. Thus, in our context the symbol $\approx$ has *not* the meaning of an approximation. The above result is not very astonishing, since we already recognize this primary Hamiltonian in the action (3.5).

3.2 Consistency Checks and Secondary Constraints

To get reasonable results one has to demand for consistency, that the system must not leave the physical submanifold at any time. In other words, the time derivative of the primary constraints has to vanish at least weakly:

$$\dot{\phi}_r = \{\phi_r, H\} + \lambda^s\{\phi_r, \phi_s\} \overset{!}{\approx} 0\,. \tag{3.13}$$

In order to fulfill this equation we can distinguish four different cases. Defining the matrix $P_{rs} \equiv \{\phi_r, \phi_s\}$ we have:

(*i*) $\{\phi_r, H\} \not\approx 0$, $\det P_{rs} \not\approx 0$
All Lagrange multipliers are (weakly) fixed $\lambda^s = -P^{rs}\{\phi_r, H\}$ and the equations of motion become $\dot{F} = \{F, H\} - \{F, \phi_r\}P^{rs}\{\phi_s, H\}$.

(*ii*) $\{\phi_r, H\} \not\approx 0$, $\det P_{rs} \approx 0$
Will be discussed further.

(*iii*) $\{\phi_r, H\} \approx 0$, $\det P_{rs} \not\approx 0$
Only the trivial solution $\lambda^r \approx 0$ exists. If $\{\phi_r, H\} \approx 0$ originates from $H = 0$, one has to impose $\det P_{rs} \approx 0$, otherwise it is hard to find a physical interpretation, since a vanishing Hamiltonian does not allow for any time development at all. Indeed, the vanishing of the canonical Hamiltonian is a quite interesting case, because this is what happens in the description of a relativistic free particle. A discussion of this system as well as its quantization may be found in Ref. [24].

(*iv*) $\{\phi_r, H\} \approx 0$, $\det P_{rs} \approx 0$
Some of the multipliers are fixed, depending on the rank of P_{rs}.

Note that the inverse of the matrix P_{rs} is denoted by P^{rs}, so that they satisfy $P^{rr'}P_{r's} = \delta^r{}_s$. We will adopt this notation for any other matrix and its inverse respectively. In the following we will discuss the second case further.

The vanishing of the determinant causes the existence of $(R - E)$ nulleigenvectors $e^r_{(\varrho)}$, if E is the rank of the matrix P_{rs}:

$$\{\phi_r, \phi_s\}e^r_{(\varrho)} \approx 0, \qquad \varrho = 1, ..., (R - E)\,. \tag{3.14}$$

Multiplying (3.13) with such a vector we obtain

$$\begin{aligned} \dot{\phi}_r e^r_{(\varrho)} &\approx \{\phi_r, H\}e^r_{(\varrho)} + \lambda^s\{\phi_r, \phi_s\}e^r_{(\varrho)} \\ &\approx \{\phi_r, H\}e^r_{(\varrho)} \overset{!}{\approx} 0\,. \end{aligned} \tag{3.15}$$

For each nulleigenvector these equations *could* yield conditions on the phase space variables which are independent of the primary constraints. Let there be K' of such new independent conditions which we will call secondary constraints $\chi_{k'}$. In contrast to the primary ones, we had to make explicitly use of the equations of motion. Now all constraints, no matter if primary or secondary, form a new submanifold Γ' in phase space.

The consistency requirement demands that the time derivatives of those secondary constraints also have to vanish

$$\dot{\chi}_{k'} \approx \{\chi_{k'}, H\} + \lambda^s\{\chi_{k'}, \phi_s\} \overset{!}{\approx} 0\,, \qquad k' = 1, ..., K'\,, \tag{3.16}$$

where the weakly vanishing now refers to the submanifold Γ'. Observe that the matrix $\{\chi_{k'}, \phi_s\}$ is not quadratic anymore and consequently the above procedure using nulleigenvectors breaks down. Although we cannot specify an algorithm as above, (3.16) may lead to further constraints.

Up to now there are $K'' \geq K'$ secondary and R primary constraints, all restricting the motion of the system to the submanifold Γ''. This procedure has to be repeated until no more independent relations arise. After a finite number of steps, we arrive at the following linear system of equations for the Lagrange multipliers

$$\{\phi_r, H\} + \lambda^s \{\phi_r, \phi_s\} \approx 0, \qquad r = 1, ..., R\,, \tag{3.17a}$$
$$\{\chi_k, H\} + \lambda^s \{\chi_k, \phi_s\} \approx 0, \qquad k = 1, ..., K \geq ... \geq K'' \geq K'\,. \tag{3.17b}$$

All primary and secondary constraints together form our physical submanifold, we will simply call Γ, in phase space.

To complete this subsection we state an important property of a weakly vanishing function. Namely a phase space function F, vanishing on the submanifold Γ, can be expressed as a "linear" combination of the constraints

$$F \approx 0 \quad \Longleftrightarrow \quad F = f^r \phi_r + g^k \chi_k\,. \tag{3.18}$$

In general the coefficients f^i and g^k can be functions of the phase space variables. A proof of that property may be found in Ref. [16].

3.3 Classification to First and Second Class

According to Dirac [6] it is possible to distinguish two different kinds of phase space functions. A function is called first class if its Poisson brackets with all constraints vanish at least weakly

$$\{F, \phi_r\} \approx 0\,, \tag{3.19a}$$
$$\{F, \chi_k\} \approx 0\,, \tag{3.19b}$$

otherwise it is called to be of second class. In the following we will see that the distinction between primary and secondary constraints is not substantial. However, mathematically as well as physically significant differences between the constraints arise when we perform the decomposition into first and second class.

$$\phi_r = (\phi_a, \phi_\alpha) \quad \Longrightarrow \quad \lambda^r \phi_r = \eta^a \phi_a + \kappa^\alpha \phi_\alpha\,, \tag{3.20a}$$
$$\chi_k = (\chi_b, \chi_\beta)\,. \tag{3.20b}$$

The latin indices $a, a', ...$ $(b, b', ...)$ cover the primary (secondary) first class constraints and the greek indices $\alpha, \alpha', ...$ $(\beta, \beta', ...)$ the primary (secondary) second class constraints. This decomposition yields for the consistency relations

$$\{\phi_a, H\} + \eta^{a'} \{\phi_a, \phi_{a'}\} + \kappa^{\alpha'} \{\phi_a, \phi_{\alpha'}\} \approx \{\phi_a, H\} \approx 0\,, \tag{3.21a}$$
$$\{\chi_b, H\} + \eta^{a'} \{\chi_b, \phi_{a'}\} + \kappa^{\alpha'} \{\chi_b, \phi_{\alpha'}\} \approx \{\chi_b, H\} \approx 0\,, \tag{3.21b}$$
$$\{\phi_\alpha, H\} + \eta^{a'} \{\phi_\alpha, \phi_{a'}\} + \kappa^{\alpha'} \{\phi_\alpha, \phi_{\alpha'}\} \approx \{\phi_\alpha, H\} + \kappa^{\alpha'} \{\phi_\alpha, \phi_{\alpha'}\} \approx 0\,, \tag{3.21c}$$
$$\{\chi_\beta, H\} + \eta^{a'} \{\chi_\beta, \phi_{a'}\} + \kappa^{\alpha'} \{\chi_\beta, \phi_{\alpha'}\} \approx \{\chi_\beta, H\} + \kappa^{\alpha'} \{\chi_\beta, \phi_{\alpha'}\} \approx 0\,. \tag{3.21d}$$

Here we see that it is possible to solve the linear system of equations for such multipliers, which couple the (primary) second class constraints to the canonical Hamiltonian. The first class multipliers completely disappear. Later we will see that this disappearance has a precise physical meaning. In order to distinguish easier between the two classes of constraints, we rename all first class constraints as $(\phi_a, \chi_b) = \gamma_m$ and the second class constraints as $(\phi_\alpha, \chi_\beta) = \xi_\mu$. With this we can write the consistency relations (3.21) in a more compact form

$$\{\gamma_m, H\} = 0\,, \tag{3.22a}$$
$$\{\xi_\mu, H\} + \kappa^\alpha \{\xi_\mu, \xi_\alpha\} = 0\,. \tag{3.22b}$$

Introducing the matrix consisting of the Poisson brackets of all second class constraints

$$\Delta_{\mu\nu} = \begin{pmatrix} \{\phi_\alpha, \phi_{\alpha'}\} & \{\phi_\alpha, \chi_{\beta'}\} \\ \{\chi_\beta, \phi_{\alpha'}\} & \{\chi_\beta, \chi_{\beta'}\} \end{pmatrix} = \Big(\{\xi_\mu, \xi_\nu\}\Big)\,, \tag{3.23}$$

yields for the second class part of the consistency relations (3.22b)

$$\{\xi_\mu, H\} + \kappa^\alpha \Delta_{\mu\alpha} \approx 0\,. \tag{3.24}$$

The Δ-matrix is, indeed, non-singular and hence possesses an inverse $\Delta^{\mu\nu}$. The multiplication of (3.24) with the inverse Δ-matrix leads to

$$\kappa^\alpha \approx -\Delta^{\alpha\nu}\{\xi_\nu, H\}\,, \tag{3.25a}$$
$$\kappa^\beta \approx -\Delta^{\beta\nu}\{\xi_\nu, H\} \approx 0\,. \tag{3.25b}$$

Note the important fact that, although the *secondary second class* constraints are not explicitly included in (3.11), they are encoded in the Lagrange multipliers κ^μ. The multipliers coupling the secondary second class constraints vanish weakly, according to (3.25b), and we can symmetrize (3.24) with respect to the Δ-matrix

$$\{\xi_\mu, H\} + \kappa^\nu \Delta_{\mu\nu} \approx 0\,. \tag{3.26}$$

Inserting the multipliers fixed that way into the equations of motion gives

$$\begin{aligned} \dot{F} &\approx \{F, H\} + \lambda^r \{F, \phi_r\} \\ &\approx \{F, H\} + \eta^a \{F, \phi_a\} + \kappa^\alpha \{F, \phi_\alpha\} \\ &\approx \{F, H\} + \eta^a \{F, \gamma_a\} - \{F, \phi_\alpha\}\Delta^{\alpha\nu}\{\xi_\nu, H\}\,. \end{aligned} \tag{3.27}$$

And with the help of (3.25b) in the symmetrized form

$$\dot{F} \approx \{F, H\} + \eta^a \{F, \gamma_a\} - \{F, \xi_\mu\}\Delta^{\mu\nu}\{\xi_\nu, H\}\,. \tag{3.28}$$

Now these are the equations of motion at which we arrive with the Dirac-Bergmann algorithm. Herein all constraints appear, except for the *secondary first class* ones. In the following sections we study the structure of these equations and the constraints.

There is one thing left to be mentioned in the discussion of this algorithm. In a similar treatment as presented in Ref. [20], one could have regarded the secondary constraints formal as primary ones and included them in the Hamiltonian (3.11) by coupling via new multipliers. After proceeding this way, one would have looked for more constraints arising from this new "primary" Hamiltonian. This may be confusing, since we would have changed the Hamiltonian, and hence the equations of motion, during the algorithm weakly. Nevertheless the determination of the multipliers would have led immediately to the equations of motion using the extended Hamiltonian to be discussed later.

4 Second Class Constraints

In this section we study the case as if there were no first class constraints, so that the equations of motion become

$$\dot{F} \approx \{F, H\} - \{F, \xi_\mu\}\Delta^{\mu\nu}\{\xi_\nu, H\}\,. \tag{4.1}$$

4.1 The Dirac Bracket

First of all we define the Dirac bracket of two phase space functions F and G as

$$\{F, G\}_{D(\Delta)} = \{F, H\} - \{F, \xi_\mu\}\Delta^{\mu\nu}\{\xi_\nu, H\}\,. \tag{4.2}$$

The Δ in the index is used to emphasize that the Dirac bracket is evaluated with the Δ-matrix. If there will be no confusions we will omit the Δ in the index. With the help of this definition the equations of motion (4.1) read

$$\dot{F} \approx \{F, H\}_{D(\Delta)}\,. \tag{4.3}$$

The Dirac bracket exhibits several algebraic identities:

(i)	$\{F, G\}_D = -\{G, F\}_D,$	Antisymmetry
(ii)	$\{c_1F_1 + c_2F_2, G\}_D = c_1\{F_1, G\}_D + c_2\{F_2, G\}_D,$	Linearity
(iii)	$\{c, F\}_D = 0,$	$c =$ const.
(iv)	$\{F, \{G, H\}_D\}_D + \{G, \{H, F\}_D\}_D + \{H, \{F, G\}_D\}_D = 0$	Jacobi identity
(v)	$\{FG, H\}_D = F\{G, H\}_D + \{F, H\}_D G$	Product rule

They are essentially the same as for the Poisson bracket. The only but crucial difference is that the Dirac bracket does not obey the fundamental commutation relations. Soon it will become clear that this circumstance is not a defect but a necessity for the consistency of the quantum theory for constrained systems.

It is possible to define for every phase space function F, which may be second class, a first class conjugate F^*

$$F^* = F - \xi_\mu\Delta^{\mu\nu}\{\xi_\nu, F\}\,. \tag{4.4}$$

To proof the first class property of F^* its Poisson bracket with all constraints has to vanish

$$\begin{aligned}\{\,\cdot\,, F^*\} &= \{\,\cdot\,, F\} - \{\,\cdot\,, \xi_\mu\Delta^{\mu\nu}\{\xi_\nu, F\}\} \\ &= \{\,\cdot\,, F\} - \{\,\cdot\,, \xi_\mu\}\Delta^{\mu\nu}\{\xi_\nu, F\} - \xi_\mu\{\,\cdot\,, \Delta^{\mu\nu}\{\xi_\nu, F\}\}\,.\end{aligned} \tag{4.5}$$

The “dot” in the Poisson brackets serves as a placeholder for first or second class constraints. For second class constraints we have

$$\begin{aligned}\{\xi_\kappa, F^*\} &= \{\xi_\kappa, F\} - \{\xi_\kappa, \xi_\mu\}\Delta^{\mu\nu}\{\xi_\nu, F\} - \xi_\mu\{\xi_\kappa, \Delta^{\mu\nu}\{\xi_\nu, F\}\} \\ &= \{\xi_\kappa, F\} - \delta_\kappa{}^\nu\{\xi_\nu, F\} - \xi_\mu\{\xi_\kappa, \Delta^{\mu\nu}\{\xi_\nu, F\}\} \\ &\approx 0\,.\end{aligned} \tag{4.6}$$

The first two terms cancel identically and the last term vanishes weakly, since $\xi\mu$ is constrained to vanish. The Poisson brackets with first class constraints yield

$$\begin{aligned}\{\gamma_m, F^*\} &= \{\gamma_m, F\} - \{\gamma_m, \xi_\mu\}\Delta^{\mu\nu}\{\xi_\nu, F\} - \xi_\mu\{\gamma_m, \Delta^{\mu\nu}\{\xi_\nu, F\}\} \\ &\approx \{\gamma_m, F\}\,.\end{aligned} \tag{4.7}$$

Here the second term vanishes weakly due to the first class property of the γ's and the last one vanishes weakly as before. Now we are left with one term, which in general does not vanish. However, we will see in the next section that this term can be omitted, because it is connected to a gauge transformation. If the phase space function is the Hamilton function, this term has to vanish due to the consistency relations (3.22a).

With the definition of the Hamilton function H^* we can write the equations of motion (4.1) as

$$\dot{F} \approx \{F, H^*\}\,, \tag{4.8}$$

and identify

$$\{F, G\}_D \approx \{F, G^*\}\,. \tag{4.9}$$

The proof of that does not cause any effort:

$$\begin{aligned} \{F, G^*\} &= \{F, G\} - \{F, \xi_\mu \Delta^{\mu\nu} \{\xi_\nu, G\}\} \\ &= \{F, G\} - \{F, \xi_\mu\} \Delta^{\mu\nu} \{\xi_\nu, G\} - \xi_\mu \{F, \Delta^{\mu\nu} \{\xi_\nu, G\}\} \\ &\approx \{F, G\}_D\,. \end{aligned} \tag{4.10}$$

We just used the definition of the Dirac bracket (4.2) and the constraints ξ_μ.

4.2 Connection to Reduced Phase Space

Up to now we are dealing with the entire phase space, where we restricted the dynamics in an embedded submanifold. This is due to the constraints, because not all coordinates and their conjugate momenta are independent. But actually we just want to work in the reduced phase space containing the true degrees of freedom. Unfortunately, in a given situation it is hard if not impossible to identify the canonical variables spanning this reduced vector space. In the following we will see the Dirac bracket in another light as just writing the equations of motion in a compact way.

Consider a constrained system with canonical variables q^i and p_i. It was shown by Maskawa and Nakajima [21], that one can construct two sets of variables Q^k, $\mathcal{Q}^m$ with their conjugates P_k, $\mathcal{P}_m$ by a *canonical* transformation, such that the constraints read $\mathcal{Q}^m = 0$ and $\mathcal{P}_m = 0$. The left variables Q^k and P_k are free and, thus, form the reduced phase space. If one defines $\xi_{1m} = \mathcal{Q}^m$ and $\xi_{2m} = \mathcal{P}_m$, the Δ matrix (3.23) can be written as

$$\Delta = \begin{pmatrix} \{\mathcal{Q}^m, \mathcal{Q}^n\} & \{\mathcal{Q}^m, \mathcal{P}_n\} \\ \{\mathcal{P}_m, \mathcal{Q}^n\} & \{\mathcal{P}_m, \mathcal{P}_n\} \end{pmatrix} = \begin{pmatrix} 0 & \delta^m{}_n \\ -\delta_m{}^n & 0 \end{pmatrix}. \tag{4.11}$$

The inverse is

$$\Delta^{-1} = -\Delta\,, \tag{4.12}$$

so that we have for the Dirac brackets

$$\begin{aligned} \{F, G\}_{D(\Delta)} &= \{F, G\} + \{F, \mathcal{Q}^m\}\{\mathcal{P}_m, G\} - \{F, \mathcal{P}_m\}\{\mathcal{Q}^m, G\} \\ &= \{F, G\} - \frac{\partial F}{\partial \mathcal{Q}^m}\frac{\partial G}{\partial \mathcal{P}_m} + \frac{\partial G}{\partial \mathcal{Q}^m}\frac{\partial F}{\partial \mathcal{P}_m} \\ &= \frac{\partial F}{\partial Q^k}\frac{\partial G}{\partial P_k} - \frac{\partial G}{\partial Q^k}\frac{\partial F}{\partial P_k}\,, \end{aligned} \tag{4.13}$$

which is nothing else but the Poisson bracket for the unconstrained variables. Now it becomes clear that the fundamental Dirac brackets must not be fulfilled, otherwise we immediately recognize that the calculations in (4.13) would not hold. The construction of the canonical transformation would not be possible, since a canonical transformation does not change the fundamental Poisson brackets. We draw the important conclusion that the Dirac bracket acts in the constrained phase space as the Poisson bracket in the reduced phase space. Hence, the Dirac bracket is closely connected to the reduced phase space.

5 First Class Constraints

We will study the properties of the equation of motion only containing first class constraints. Without second class constraints the equations of motion (3.28) reduce to

$$\dot{F} \approx \{F, H\} + \eta^a \{F, \gamma_a\}\,, \tag{5.1}$$

only containing primary first class constraints.

5.1 Generators for Gauge Transformations

The multipliers η^a in the above equation of motion are completely free. Therefore, we may expect arbitrary functions of time in the general solution. Of course, this arbitrariness cannot have any physical meaning and there has to be a transformation mediating between the different time developments of the phase space function F.

Consider two phase space functions F_η and $F_{\eta'}$ with two different multipliers η and η' evolving from the same initial value F_0. Expanding these functions for small times

$$F(t) \simeq F(0) + \frac{dF}{dt}\bigg|_{t=0} t + \frac{1}{2}\frac{d^2F}{dt^2}\bigg|_{t=0} t^2 + \dots\,, \tag{5.2}$$

and inserting the equations of motion (5.1), one obtains up to first order in time

$$F(t) \simeq F_0 + (\{F_0, H\} + \mu^a \{F_0, \gamma_a\})t + \dots\,. \tag{5.3}$$

The difference between the two functions is

$$F_\eta - F_{\eta'} \simeq (\eta^a - \eta'^a) t \{F_0, \gamma_a\}\,. \tag{5.4}$$

For an infinitesimal time evolution δt the difference is given by

$$\delta F = \epsilon^a \{F, \gamma_a\}\,, \tag{5.5}$$

which represents the demanded gauge transformation. Of course we can expect, that not only the sum of all first class constraints generates a gauge transformation, but that each first class constraint does this by itself. Therefore we can write

$$\delta_\epsilon F = \epsilon \{F, \gamma_a\}\,. \tag{5.6}$$

In a special case, where the phase space functions are just the canonical variables, we have

$$p'_i = p_i - \epsilon \frac{\partial \gamma_a}{\partial q^i}\,, \tag{5.7}$$

$$q'^i = q^i + \epsilon \frac{\partial \gamma_a}{\partial p_i}\,. \tag{5.8}$$

Here we immediately recognize that the infinitesimal *gauge* transformations are associated with infinitesimal *canonical* transformations with the first class constraints being the generating functions.

So far we proved that *primary first class* constraints generate gauge transformations. One may ask, what about the *secondary first class* constraints? Originally, Dirac conjectured in his lectures [6] to take all first class constraints into account. Indeed, there are several hints to proceed that way. For instance, as we will see later in the application of the Maxwell field, we have to consider all first class constraints to relate them with the local $U(1)$ gauge transformations of the four-potential A^μ. Therefore, one may include *all* constraints to the canonical Hamiltonian and obtain the extended Hamiltonian

$$H_E = H + \eta^m \gamma_m + \kappa^\mu \xi_\mu = H^* + \eta^m \gamma_m \,. \tag{5.9}$$

Using this Hamilton function the equations of motion become

$$\dot{F} \approx \{F, H_E\} \approx \{F, H\} + \mu^m \{F, \gamma_m\} - \{F, \xi_\mu\} \Delta^{\mu\nu} \{\xi_\nu, H\} \,. \tag{5.10}$$

This is the final equation of motion containing *all* constraints. Especially we have that all first class constraints generate gauge transformations. In the next section, when we come to the quantization of the theory, we will make use of these equations of motion and particularly the first class constraints will play an important role.

5.2 Group Properties of First Class Constraints

In this subsection we deepen our understanding in the gauge generating characteristic of the first class constraints. For this purpose we have to prove a potentially obvious property of the Poisson bracket. Namely the Poisson bracket of two first class functions is again first class.

We already know, that in general a weakly vanishing function is a linear combination of all constraints

$$\{F, \phi_i\} = f_i{}^j \phi_j \,, \qquad \{G, \phi_i\} = g_i{}^j \phi_j \,. \tag{5.11}$$

Here the summation over j is meant to cover all the constraints. The proof is straightforward, just using the Jacobi identity and the product rule yields the desired result:

$$\begin{aligned} \{\{F, G\}, \phi_i\} &= -\{\{G, \phi_i\}, F\} - \{\{\phi_i, F\}, G\} \\ &= \{F, f_i{}^j \phi_j\} - \{G, g_i{}^j \phi_j\} \\ &= f_i{}^j \{F, \phi_j\} + \{F, f_i{}^j\} \phi_j - g_i{}^j \{G, \phi_j\} - \{G, g_i{}^j\} \phi_j \\ &= f_i{}^j f_j{}^k \phi_k + \{F, f_i{}^j\} \phi_j - g_i{}^j g_j{}^k \phi_k - \{G, g_i{}^j\} \phi_j \\ &\approx 0 \,. \end{aligned} \tag{5.12}$$

Especially the Poisson bracket of two first class constraints is first class

$$\{\gamma_m, \gamma_n\} = f_{mn}{}^\ell \gamma_\ell \,. \tag{5.13}$$

So we can draw the important conclusion, that first class constraints form a closed Lie algebra and therewith represent a subgroup in the group of constraints with the Poisson bracket as the underlying product operation. This property will be very important in the following discussion. Consider the infinitesimal canonical transformation (5.6)

$$\delta_\epsilon F = \epsilon \{F, \gamma_m\} \,.$$

This can be written in the form

$$\delta_\epsilon F = \epsilon \widehat{T}_m F \,, \tag{5.14}$$

where we define the operator

$$\widehat{T}_m \equiv \{\,\cdot\,, \gamma_m\} \,. \tag{5.15}$$

The calculation of the commutator is straightforward:

$$\begin{aligned}\left[\widehat{T}_m, \widehat{T}_n\right] &= \{\{\,\cdot\,, \gamma_n\}, \gamma_m\} - \{\{\,\cdot\,, \gamma_m\}, \gamma_n\} \\ &= -\{\{\gamma_n, \gamma_m\}, \,\cdot\,\} - \{\{\gamma_m, \,\cdot\,\}, \gamma_n\} - \{\{\,\cdot\,, \gamma_m\}, \gamma_n\} \\ &= \{\,\cdot\,, \{\gamma_n, \gamma_m\}\} \\ &= \{\,\cdot\,, f_{nm}{}^{\ell}\gamma_\ell\} \\ &= \{\,\cdot\,, f_{nm}{}^{\ell}\}\gamma_\ell + f_{nm}{}^{\ell}\{\,\cdot\,, \gamma_\ell\}\,. \end{aligned} \tag{5.16}$$

If the structure coefficients $f_{mn}{}^{\ell}$ are functions of the phase space variables, the group property holds at least weakly

$$\left[\widehat{T}_m, \widehat{T}_n\right] \approx f_{nm}{}^{\ell}\widehat{T}_\ell\,. \tag{5.17}$$

The operators $\widehat{T}_m$ together with the commutation relation define a Lie algebra. In this description it is obvious that each first class constraint is a generator of a gauge transformation $\delta_\epsilon F$, because it is part of the operator generating the symmetry transformation. We have to emphasize, that the closed algebraic structure of the first class constraints was essential in the derivation of the commutator. In the Appendix A we give some remarks on Lie algebras, Lie groups, and their relation to gauge transformations.

6 Quantization of Constrained Dynamics

After the classification of the constraints and studying their properties, it is our aim to perform the transition from classical to quantum theory.

6.1 The Canonical Quantization Program

Canonical quantization was one of the first steps in the development of the new quantum theory of finite dimensional regular systems. The main purpose of this thesis is to present a consistent way, how to extend the formalism to singular systems. To this end we briefly recall the postulates for the canonical quantization program[1]:

(i) Describe the quantum state of a physical system by a (normed) vector $|\psi\rangle$. It is an element of a finite/infinite dimensional Hilbert space $\mathbb{H}$, with a scalar product $\langle\varphi|\psi\rangle$.

(ii) Any measurable quantity has to be represented by a linear, hermitean operator $\widehat{A}$. In general the operators could exhibit an explicit time dependence, whereas for our purposes they shall not.

The outcome of a measurement is always an eigenvalue of such an operator, because of the hermiticity these values are real. For the sake of simplicity, we assume that the eigenvalues are non-degenerate. If $|a_i\rangle$ denotes the eigenvectors of the operator $\widehat{A}$

$$\widehat{A}\,|a_i\rangle = a_i\,|a_i\rangle \ , \tag{6.1}$$

one can expand each vector $|\psi\rangle$ in a series of these eigenvectors

$$|\psi\rangle = \sum_i c_i\,|a_i\rangle \ . \tag{6.2}$$

The probability, that the eigenvalue a_i occurs in a single measurement, is given by

$$|\,\langle a_i|\psi\rangle\,|^2 = |\,\langle a_i|\sum_j c_j\,|a_j\rangle\,|^2 = |\sum_j c_j\,\langle a_i|a_j\rangle\,|^2 = |c_i|^2\,. \tag{6.3}$$

(iii) The dynamical evolution of a state is given by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\,|\psi\rangle = \widehat{H}\,|\psi\rangle \ . \tag{6.4}$$

The Schrödinger equation implies the equation of motion for the expectation value of an operator $\langle\widehat{F}\rangle = \langle\psi|\,\widehat{F}\,|\psi\rangle$, where we neglect an explicit time dependence:

$$\begin{aligned}
\frac{d}{dt}\,\langle\psi|\,\widehat{F}\,|\psi\rangle &= \left(\frac{d}{dt}\,\langle\psi|\right)\widehat{F}\,|\psi\rangle + \langle\psi|\,\widehat{F}\left(\frac{d}{dt}\,|\psi\rangle\right)\\
&= -\frac{1}{i\hbar}\,\langle\psi|\,\widehat{H}\widehat{F}\,|\psi\rangle + \langle\psi|\,\widehat{F}\frac{1}{i\hbar}\widehat{H}\,|\psi\rangle\\
&= \frac{1}{i\hbar}\,\langle\psi|\,\widehat{F}\widehat{H} - \widehat{H}\widehat{F}\,|\psi\rangle\\
&= \frac{1}{i\hbar}\,\langle[F,H]\rangle \ .
\end{aligned} \tag{6.5}$$

[1] In this subsection we shall not use the summation convention and therefore explicitly mention when there is a summation.

This equation is formally identical to the classical Hamiltonian equation of motion for an unconstrained system, if we perform the substitution

$$\{F, H\} \longrightarrow \frac{1}{i\hbar}\left[\widehat{F}, \widehat{H}\right] . \tag{6.6}$$

In general we can retain, that the transfer to quantum theory is done by the following correspondence rule

$$\left[\widehat{F}, \widehat{G}\right] = i\hbar\widehat{\{F, G\}} . \tag{6.7}$$

However, it must be remarked that this canonical quantization method even in the regular case, where we do not consider constraints, is too naive and works only for some special Hamiltonians. In general we encounter operator ordering problems, because classical observables commute, the corresponding quantum operators do not. If there are constraints reducing the classical phase space, the situation becomes even more severe.

6.2 Problems in Quantization with Constraints

The above description gives us a simple tool to quantize regular systems. There exist several ways how to consider constraints and just as many problems with these descriptions. In the following we discuss some of them.

To take into account that the classical phase space is reduced by the constraints, one may quantize the theory as mentioned above and restrict that part of the Hilbert space as physically relevant, by imposing certain relations on the states. One possibility of this approach is to demand the vanishing of matrix elements of each constraint

$$\langle\psi'|\, \widehat{\sigma}_\kappa \,|\psi\rangle = 0 , \tag{6.8}$$

whereas $\widehat{\sigma}_\kappa$ have to be understood as the quantum analogues of the classical constraints. And already here we face a serious problem of this approach, since the operator analogue of a classical constraint may be not well-defined. This is mainly due to the ordering problem of the operator formalism. Furthermore, the above condition is not very handy for practical calculations as one needs to know the full spectrum of the Hamiltonian first before deciding which states belong to the physical Hilbert space. Despite these difficulties this quantization method may work. For instance, it was used by S. Gupta [14] and K. Bleuler [3] for the quantization of the Maxwell field in the relativistic covariant Lorentz gauge.

Another possibility is that we impose direct conditions on the states according to

$$\widehat{\sigma}_\kappa \,|\psi\rangle = 0 . \tag{6.9}$$

Of course these equations imply (6.8) but they are significantly stronger conditions on the states as just the vanishing of matrix elements. In the classical theory we obtained the result that first class constraints form a closed algebra. Naively one could expect that the quantum analogue of relation (5.13) would be

$$[\widehat{\gamma}_m, \widehat{\gamma}_n] = \widehat{f}_{mn}{}^{\ell}\widehat{\gamma}_\ell . \tag{6.10}$$

But there is no reason that the "structure-operators" $\widehat{f}_{mn}{}^{\ell}$ are always to the left of the constraints $\widehat{\gamma}_\ell$. And therefore we may have

$$[\widehat{\gamma}_m, \widehat{\gamma}_n]\,|\psi\rangle \neq 0 , \tag{6.11}$$

for some combinations of m and n. But this is in sharp contradiction to the consistency, since we have to have from (6.9) that the commutator of two first class constraints applied to any physical state vector vanishes

$$[\widehat{\gamma}_m, \widehat{\gamma}_n]\,|\psi\rangle = 0 . \tag{6.12}$$

Nevertheless it is possible that this approach works for some special, "docile" constraints. An important example is the treatment of the free relativistic particle, as done in Ref. [24], leading to the Klein-Gordon equation.

If we are dealing with second class constraints the situation becomes even more severe. On the classical level the Poisson bracket of two second class constraints in general does not vanish. After the transition to quantum mechanics we would expect some relations

$$\left[\widehat{\xi}_\mu, \widehat{\xi}_\nu\right] |\psi\rangle \neq 0 \tag{6.13}$$

But this is already in contradiction to (6.9), since we have to demand the same consistency relation for the second class commutator as in the above first class case. In this approach there is no possibility that we get a consistent quantum theory if second class constraints arise on the classical level.

These examples illustrate that the quantization of a theory containing constraints is non-trivial and we have to face many problems which partially go beyond the operator ordering problem. In the next section we will present *one* way out of this misery.

6.3 Gauge Reduction

Our ultimate goal is to quantize a singular system. But this cannot be possible if we work in the constrained phase space. If we would apply standard operator methods and quantize the system by imposing the commutation relation (6.7), inconsistencies arise immediately. Since there are dependent variables, this would create uncertainties where none should be. In a theory, where gauge freedom is present, it turns out that one *must* remove this gauge freedom. Indeed, first class constraints generate gauge transformations, as a consequence we are forced to remove the first class constraints from our theory. Before we explain how to proceed, we want to motivate this gauge reduction.

The best way to understand the *physical significance* of this reduction process gives the path integral formulation of quantum mechanics. All the physical information of a quantum mechanical system is given by the transition amplitude. This amplitude can be expressed as an integral over all possible paths a particle can move [18]. The original expression was derived in configuration space by Feynman

$$\langle q_f, t_f | q_i, t_i \rangle = \int \mathcal{D}q \, \exp\left\{ \frac{i}{\hbar} \int dt \, L(q(t), \dot{q}(t)) \right\} . \tag{6.14}$$

Since we are working in phase space, the Hamiltonian representation would be more apropriate

$$\langle q_f, t_f | q_i, t_i \rangle = \int \mathcal{D}q \, \mathcal{D}p \, \exp\left\{ \frac{i}{\hbar} \int dt \, \left[p_j(t) \dot{q}^j(t) - H(p(t), q(t)) \right] \right\} . \tag{6.15}$$

We recognize the action integral in the exponent of the phase. The path integral describes nothing but the transition amplitude of a particle from an initial state $|q_i, t_i\rangle$ to a final state $|q_f, t_f\rangle$ as a sum over all possible paths. In a regular system each path corresponds to one single physical configuration. However, in a singular system there are superfluous functions. One single physical configuration is described by more than one canonical pair of variables p and q. If there would be no gauge reduction, the path integral is allowed to sum over all possible configurations, especially over the physically equivalent ones. As a consequence the path integral cannot converge. To enforce the possibility of convergence we have to fix the gauge somehow. This fixation picks out one physical "gauge orbit".

There are several methods for this gauge fixation. One possibility is given by adding a so called gauge fixing term to the Lagrangian density, so that there are no first class constraints in the first place. In the case of the Maxwell theory this could be the well known, Lorentz covariant Fermi term $-\frac{\alpha}{2}(\partial_\mu A^\mu)^2$ [9]. The parameter α should not be interpreted as a Lagrange multiplier, since it is no new dynamical variable of the theory. It is just a c-number which can be freely chosen. Nevertheless the particular choice represents different gauges, for example the choice $\alpha = 1$ is known as the *Feynman gauge*. Further discussions about this gauge fixing term and its implications for propagators can be found in Refs. [13, 17]. Within the path integral formalism, this method has been extended by Fadeev and Popov [7, 8]. It is quite elegant and its justification is based on the BRS-transformation [1] of non-abelian Yang-Mills fields, whereat the Maxwell theory is just a special abelian case with the only infinitesimal generator equal to unity $\widehat{T}^a = 1$. We will not go into this transformation any further because this would go way beyond this thesis.

The method we prefer is simply to "choose" a particular gauge. This implicitly means to fix the undetermined multipliers in the equations of motion by imposing "external" conditions Ω_m. There are as many gauge conditions as there are first class constraints. In order to fix the gauge completely, the conditions have to fulfill two restrictions:

(i) Invertibility:

$$\det\Big(\{\Omega_m, \gamma_n\}\Big) = \det \widetilde{G}_{mn} \neq 0\,. \tag{6.16}$$

(ii) Attainability: There must exist a transformation from the arbitrary values of the gauge variables to those fulfilling Ω_m.

Admissible gauge constraints also have to fulfill the consistency relations, which are now formulated with the extended Hamiltonian:

$$\dot{\Omega}_m \approx \{\Omega_m, H_E\} \approx \{\Omega_m, H^*\} + \eta^n\{\Omega_m, \gamma_n\} \overset{!}{\approx} 0\,. \tag{6.17}$$

Due to the restrictions on Ω_m one can solve these equations for η^n:

$$\eta^n \approx -\widetilde{G}^{mn}\{\Omega_m, H^*\}\,, \tag{6.18}$$

with the inverse matrix

$$\widetilde{G}^{mn} = \Big(\{\Omega_m, \gamma_n\}\Big)^{-1}\,. \tag{6.19}$$

The equations of motion become

$$\dot{F} \approx \{F, H_E\} \approx \{F, H^*\} - \{F, \gamma_m\}\widetilde{G}^{mn}\{\Omega_n, H^*\}\,. \tag{6.20}$$

Unfortunately, the constraints appear in an asymmetric manner. To symmetrize the equations we collect the first class and gauge constraints in a new vector $\varphi_v = (\gamma_m, \Omega_n)$ and then introduce a new matrix

$$\bar{G}_{vw} = \Big(\{\varphi_v, \varphi_w\}\Big) = \begin{pmatrix} \{\gamma_m, \gamma_n\} & \{\gamma_m, \Omega_n\} \\ \{\Omega_m, \gamma_n\} & \{\Omega_m, \Omega_n\} \end{pmatrix}\,. \tag{6.21}$$

We recognize that the matrix separates into independent subblocks and, because of the first class property of the γ's, one of them vanishes

$$\bar{G}_{vw} = \begin{pmatrix} 0 & (-\widetilde{G}_{nm}) \\ (\widetilde{G}_{mn}) & (G'_{mn}) \end{pmatrix} = \begin{pmatrix} 0 & -\widetilde{G}^T \\ \widetilde{G} & G' \end{pmatrix}\,. \tag{6.22}$$

The block structure makes it easy for us to find the inverse

$$\bar{G}^{vw} = \begin{pmatrix} G'' & \widetilde{G}^{-1} \\ -(\widetilde{G}^T)^{-1} & 0 \end{pmatrix} . \tag{6.23}$$

Herein G' and G'' are just some matrices whose concrete form will not be of any interest. They just have to fulfill the following identity

$$\widetilde{G}G'' - G'(\widetilde{G}^T)^{-1} = 0 . \tag{6.24}$$

Using (6.23) we can symmetrize the second term in the equations of motion (6.20), which can be seen as follows

$$\begin{aligned} \{F, \phi_v\}\bar{G}^{vw}\{\phi_w, H^*\} = &\{F, \gamma_m\}G''^{mn}\{\gamma_n, H^*\} \\ &- \{F, \Omega_m\}\widetilde{G}^{nm}\{\gamma_n, H^*\} \\ &+ \{F, \gamma_m\}\widetilde{G}^{mn}\{\Omega_n, H^*\} \\ \approx &\{F, \gamma_m\}\widetilde{G}^{mn}\{\Omega_n, H^*\} . \end{aligned} \tag{6.25}$$

So the equations of motion read

$$\dot{F} \approx \{F, H^*\}_{D(\bar{G})} \approx \{F, H^*\} - \{F, \varphi_v\}\bar{G}^{vw}\{\varphi_w, H^*\} . \tag{6.26}$$

Due to the restriction on the gauge constraints $\det(\{\Omega, \gamma\}) \approx 0$ the first class constraints have to become second class. Now there are no more generators for gauge transformations and the equation of motion is unique. By arranging all constraints, now being second class, in the vector $\sigma_\kappa = (\gamma_m, \Omega_n, \xi_\mu)$ and introducing the final matrix

$$G_{\kappa\omega} = \Big(\{\sigma_\kappa, \sigma_\omega\}\Big) , \tag{6.27}$$

we finally arrive at the equations of motion, where all multipliers are fixed

$$\dot{F} \approx \{F, H\}_{D(G)} \approx \{F, H\} - \{F, \sigma_\kappa\}G^{\kappa\omega}\{\sigma_\omega, H\} . \tag{6.28}$$

The proof, that this equation follows from (6.26), is quite long and does not give any new insight. Nevertheless, we can convince ourselves that this form has to hold. Due to the “external” gauge conditions the former first class constraints γ_m become second class. Now *all* the constraints σ_κ are second class as mentioned before. But (6.28) is exactly the form for second class constraints only.

After removing all first class constraints, by construction we are only left with second class constraints. One may ask how to deal with this kind of constraints, since in the last section it seemed they cause more troubles than first class constraints. The answer is given by the following discussion.

In Section 4.2 we studied the connection of the reduced phase space with the Dirac bracket. It turned out that the Poisson brackets, calculated with the reduced set of variables, equal the Dirac brackets. But the reduced phase space is nothing but the physical space of independent variables. Indeed, after the gauge reduction on the classical level all gauge freedom has been removed and the physical phase space is uniquely determined. Therefore, it seems reasonable that we can consistently quantize a singular system by changing the third postulate in Section 6.1 into

$$\Big[\widehat{F}, \widehat{G}\Big] = i\hbar\widehat{\{F, G\}}_{D(G)} . \tag{6.29}$$

This recipe works quite well, but unfortunately it is not *the* golden rule for quantization with constraints. This canonical substitution rule is suited to implement constraints, nevertheless it contains all the difficulties of the operator formalism, for instance operator ordering. Therefore modern methods use path integral quantization as an adequate technique [18]. But this formalism is highly sophisticated and an extensive treatment would go beyond the scope of this thesis.

7 Translation to Field Theory

Up to now we are able to quantize a singular system with a finite number of degrees of freedom. Regrettably, nearly all physically interesting examples are in the range of field theory. Actually, there are examples in point mechanics but most of these are technically constructed and do not have a great physical meaning. In field theory we are dealing with an infinite number of degrees of freedom, hence the developed theory is not applicable without any modification. The naive generalization is straightforward, the formulae we get are very similar to the point mechanic case and the translation rules to an infinite number of degrees of freedom are quite obvious. Nevertheless due to this limiting process some subtle peculiarities arise. But before we study these special features of constrained field theories, we develop the Hamilton formalism for continuous regular systems. Then the generalization taking constraints into account will not cause any difficulties.

First of all, consider the Lagrange function, now being a functional of the fields and their time derivatives

$$L(t) = L[\varphi^\mu(\boldsymbol{x},t); \dot{\varphi}^\mu(\boldsymbol{x},t)], \qquad \mu = 1, ..., N\,. \tag{7.1}$$

Because we restrict ourselves to *local* field theories it is possible to write the Lagrange function as a volume integral over the density function $\mathcal{L}$

$$L = \int d^3x\, \mathcal{L}(\varphi^\mu(\boldsymbol{x},t); \nabla\varphi^\mu(\boldsymbol{x},t); \dot{\varphi}^\mu(\boldsymbol{x},t))\,. \tag{7.2}$$

The Lagrange density is a function depending on the fields and their spatial as well as time derivatives. We have to emphasize that in this section the index μ only counts the number of fields, it has *not* the meaning of a four-vector index. Hence, it makes no statement about the transformation properties of the fields. However, in the application of the formalism to the Maxwell field we specialize the fields to be components of a four-vector.

To pass into the Hamiltonian formulation one has to define a canonical conjugate momentum field as the *functional derivative* of the Lagrange function with respect to the time derivative of the fields

$$\pi_\mu = \frac{\delta L}{\delta \dot{\varphi}^\mu} = \frac{\delta L}{\delta(\partial_0 \varphi^\mu)} = \frac{\delta L}{\delta \varphi^\mu{}_{,0}}\,. \tag{7.3}$$

As in the Lagrangian case, the field theoretic Hamiltonian is a functional depending on the fields and their canonical momenta

$$H(t) = H[\varphi^\mu(\boldsymbol{x},t); \pi_\mu(\boldsymbol{x},t)]\,. \tag{7.4}$$

It can be obtained from the Lagrangian by the Legendre transformation

$$H = \int d^3x\,(\pi_\mu \dot{\varphi}^\mu) - L = \int d^3x\, \mathcal{H}(\pi_\mu; \varphi^\mu; \nabla\pi_\mu; \nabla\varphi^\mu)\,, \tag{7.5}$$

where we can read off the Hamilton density $\mathcal{H} = \pi_\mu \dot{\varphi}^\mu - \mathcal{L}$. The functional Hamilton equations of motion are given by minimizing the action

$$\mathcal{A} = \int d^4x\, \mathcal{L} = \int d^4x\,(\pi_\mu \dot{\varphi}^\mu - \mathcal{H})\,. \tag{7.6}$$

The standard procedure for the variation yields

$$\dot{\varphi}^\mu = \frac{\partial \mathcal{H}}{\partial \pi_\mu} - \partial_i \frac{\partial \mathcal{H}}{\partial \pi_{\mu,i}} = \frac{\delta H}{\delta \pi_\mu}\,, \tag{7.7a}$$

$$\dot{\pi}_\mu = -\frac{\partial \mathcal{H}}{\partial \varphi^\mu} + \partial_i \frac{\partial \mathcal{H}}{\partial \varphi^\mu{}_{,i}} = -\frac{\delta H}{\delta \varphi^\mu}\,, \tag{7.7b}$$

where we have introduced the functional derivative as

$$\frac{\delta}{\delta\varphi^\mu} := \frac{\partial}{\partial\varphi^\mu} - \partial_i \frac{\partial}{\partial\varphi^\mu{}_{,i}}\,. \tag{7.8}$$

The summation over i covers only the spatial derivatives. Using this definition of the functional derivative, we recognize that the equations of motion look like in point mechanics.

The Poisson bracket of two functionals $F[\varphi^\mu, \pi_\mu]$ and $G[\varphi^\mu, \pi_\mu]$ is defined as

$$\{F(\boldsymbol{x}), G(\boldsymbol{y})\} = \int d^3z \left(\frac{\delta F(\boldsymbol{x})}{\delta\varphi^\mu(\boldsymbol{z})} \frac{\delta G(\boldsymbol{y})}{\delta\pi_\mu(\boldsymbol{z})} - \frac{\delta G(\boldsymbol{y})}{\delta\varphi^\mu(\boldsymbol{z})} \frac{\delta F(\boldsymbol{x})}{\delta\pi_\mu(\boldsymbol{z})} \right). \tag{7.9}$$

It is understood that the Poisson bracket, as well as the Dirac bracket, are calculated at equal times so that we may drop the time dependence in the functionals and fields for brevity. Nevertheless, we have to keep in mind that the time dependence is *not* gone. Furthermore, the notation $F(\boldsymbol{x})$ means that the fields, which are in some kind the variables for the functionals, depend on $\boldsymbol{x}$. Because one can write a function as a functional depending on itself

$$\varphi^\mu(x,t) = \int d^3x'\, \varphi^\mu(\boldsymbol{x}',t)\delta^3(\boldsymbol{x}-\boldsymbol{x}')\,, \tag{7.10}$$

the functional derivative is defined according to

$$\frac{\delta\varphi^\mu(\boldsymbol{x},t)}{\delta\varphi^\nu(\boldsymbol{x}',t)} = \delta^\mu{}_\nu \delta^3(\boldsymbol{x}-\boldsymbol{x}')\,. \tag{7.11}$$

So the equations of motion can be written as

$$\dot{F}[\varphi^\mu,\pi_\mu] = \int d^3x \left(\frac{\delta F}{\delta\varphi^\mu}\dot{\varphi}^\mu + \frac{\delta F}{\delta\pi_\mu}\dot{\pi}_\mu \right) = \int d^3x \left(\frac{\delta F}{\delta\varphi^\mu}\frac{\delta H}{\delta\pi_\mu} - \frac{\delta H}{\delta\varphi^\mu}\frac{\delta F}{\delta\pi_\mu} \right) = \{F,H\}\,. \tag{7.12}$$

So far nothing changed in the transfer from finite to infinite degrees of freedom. Consequently, we may hope that our formalism will be applicable to field theories.

From now on constraints are taken into account. Again we have to face the fact that the constraints are not any longer algebraic relations. In the transition to field theory they become functionals of the fields, their canonical conjugate momenta and, additionally, the multiplier fields

$$\Phi(t) = \Phi[\varphi^\mu, \pi_\mu, \lambda] = \int d^3x\, \lambda(\boldsymbol{x},t)\phi(\boldsymbol{x},t)\,. \tag{7.13}$$

The actual constraints we are dealing with are now density functions, depending on the fields, their canonical conjugates and spatial derivatives. These constraint densities are coupled to the canonical Hamiltonian via the multiplier fields

$$H_P = H + \Phi[\varphi^\mu, \pi_\mu, \lambda^r] = H + \int d^3x\, \lambda^r(\boldsymbol{x},t)\phi_r(\boldsymbol{x},t)\,. \tag{7.14}$$

The equations of motion with the coupled constraint densities (7.7a) and (7.7b) can be obtained from the primary action

$$\mathcal{A} = \int d^4x\, (\pi_\mu\dot{\varphi}^\mu - \mathcal{H} - \lambda^r\phi_r)\,. \tag{7.15}$$

The results of the variation are

$$\begin{aligned}\dot{\varphi}^\mu = & \frac{\delta H}{\delta \pi_\mu} + \frac{\delta \Phi[\varphi^\mu, \pi_\mu, \lambda^r]}{\delta \pi_\mu} = \{\varphi^\mu, H\} + \{\varphi^\mu, \Phi[\varphi^\mu, \pi_\mu, \lambda^r]\} \\ & = \{\varphi^\mu, H\} + \int d^3y\, \lambda^r(\boldsymbol{y}, t)\{\varphi^\mu(\boldsymbol{x}), \phi_r(\boldsymbol{y})\}\,, \end{aligned} \tag{7.16a}$$

$$\begin{aligned}\dot{\pi}_\mu = & -\frac{\delta H}{\delta \varphi^\mu} - \frac{\delta \Phi[\varphi^\mu, \pi_\mu, \lambda^r]}{\delta \varphi^\mu} = \{\pi_\mu, H\} + \{\pi_\mu, \Phi[\varphi^\mu, \pi_\mu, \lambda^r]\} \\ & = \{\pi_\mu, H\} + \int d^3y\, \lambda^r(\boldsymbol{y}, t)\{\pi_\mu(\boldsymbol{x}), \phi_r(\boldsymbol{y})\}\,, \end{aligned} \tag{7.16b}$$

or for a general phase space functional $F[\varphi^\mu, \pi_\mu]$

$$\dot{F} = \{F, H\} + \{F, \Phi[\varphi^\mu, \pi_\mu, \lambda^r]\} = \{F(\boldsymbol{x}), H(\boldsymbol{y})\} + \int d^3y\, \lambda^r(\boldsymbol{y}, t)\{F(\boldsymbol{x}), \phi_r(\boldsymbol{y})\}\,. \tag{7.17}$$

The Dirac-Bergmann algorithm to identify secondary constraints, as well as the classification to first and second class will not change during the transition to field theory, so that the resulting formulae are quite similar. Therefore, we do not write down all the results explicitly, but just mention the most important equations to be used in the next section.

The above discussion serves as a guide for the transition rules to field theory:

(i) The phase space variables and the Lagrange multipliers become fields depending on the coordinates.

(ii) A function of the phase space variables becomes a functional of the fields and their conjugate momentum fields: $F(q^i, p_i) \longrightarrow F[\varphi^\eta, \pi_\mu]$.

(iii) Whenever there is a summation over the canonical variables or the multipliers, there has to be a three-dimensional volume integral over the sum of the fields.

With these transition rules we can immediately read off the field-theoretic expression for the gauge transformations generated by first class constraints

$$\delta_\epsilon F = \{F, \Phi[\varphi^\mu, \pi_\mu, \epsilon]\} = \int d^3y\, \epsilon(\boldsymbol{y})\{F(\boldsymbol{x}), \phi(\boldsymbol{y})\}\,, \tag{7.18}$$

and the Dirac bracket of two functionals

$$\{F(\boldsymbol{x}), G(\boldsymbol{y})\}_{D(G)} = \{F(\boldsymbol{x}), G(\boldsymbol{y})\} - \iint d^3u\, d^3v\, \{F(\boldsymbol{x}), \sigma_\kappa(\boldsymbol{u})\}G^{\kappa\omega}(\boldsymbol{u}, \boldsymbol{v})\{\sigma_\omega(\boldsymbol{v}), G(\boldsymbol{y})\}\,. \tag{7.19}$$

We see that the second term is nothing but a matrix multiplication for discrete and continuous variables. The matrix $G^{\kappa\omega}(\boldsymbol{x}, \boldsymbol{y})$ is the inverse of

$$G_{\kappa\omega}(\boldsymbol{x}, \boldsymbol{y}) = \{\sigma_\kappa(\boldsymbol{x}), \sigma_\omega(\boldsymbol{y})\}\,, \tag{7.20}$$

whereas the inverse is defined by the integral relation

$$\int d^3z\, G^{\kappa\omega'}(\boldsymbol{x}, \boldsymbol{z})G_{\omega'\omega}(\boldsymbol{z}, \boldsymbol{y}) = \delta^\kappa{}_\omega \delta^3(\boldsymbol{x} - \boldsymbol{y})\,. \tag{7.21}$$

One could be tempted to use these naive transition rules for granted and apply all the results achieved in point mechanics. Unfortunately, we encounter several problems due to this limiting process. A rigorous treatment would go to far and we refer to Ref. [24] for further reading.

8 Application to the Maxwell Field

In this section we finally apply the developed theory to a physical system, the Maxwell field. The fields, we are dealing with, form a four vector A_μ and the index μ runs from 0 to 3.

8.1 Maxwell Field as Singular System

At first we have to point out that the Maxwell field is a singular system. Its Lagrangian in Heaviside units is given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j^\mu A_\mu = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu) - j^\mu A_\mu \,, \tag{8.1}$$

with the electromagnetic field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \,, \tag{8.2}$$

which is antisymmetric in the permutation of the two indices. The components of the field strength tensor are related to the components of the electric and magnetic fields by

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = \partial_0 A^i + \partial_i A^0 = \dot{A}^i + \partial_i A^0 = -E^i \,, \tag{8.3a}$$

$$F^{ij} = \partial^i A^j - \partial^j A^i = -\partial_i A^j + \partial_j A^i = -\epsilon^{ijk} B_k \,, \tag{8.3b}$$

where ϵ^{ijk} is the usual Levi-Civita-tensor. We see that the definition of the field strength tensor (8.2) immediately leads us to the *homogeneous* group of Maxwell's equations. The variation of the action integral with the above Lagrange density (8.1) yields the *inhomogeneous* group of Maxwell's equation

$$\partial_\mu F^{\mu\nu} = j^\nu \,. \tag{8.4}$$

The singular nature of electrodynamics becomes apparent by the definition of the canonical momentum fields

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = -F^{0\mu} \,. \tag{8.5}$$

We immediately recognize, due to the antisymmetry of the field strength tensor, that there is no momentum field for A_0

$$\pi^0 = 0 \,. \tag{8.6}$$

This is the primary constraint of the theory and the proof that the Maxwell field is, indeed, a constrained system. Additionally, we identify the canonical momentum field with the electric field

$$\pi^i = E^i \,, \tag{8.7}$$

by comparing (8.3a) with (8.5).

8.2 Dirac-Bergmann Algorithm for the Maxwell Field

At first let us perform a (3+1)-decomposition of the Lagrangian (8.1)

$$\mathcal{L} = -\frac{1}{4}F^{ij}F_{ij} - \frac{1}{4}F^{i0}F_{i0} - \frac{1}{4}F^{0j}F_{0j} - j^0 A_0 - j^i A_i \tag{8.8}$$

Inserting (8.5) reduces this to

$$\mathcal{L} = -\frac{1}{4}F^{ij}F_{ij} - \frac{1}{2}\pi^i \pi_i - j^0 A_0 - j^i A_i \,. \tag{8.9}$$

The Legendre transformation $\mathcal{H} = \pi^\mu \dot{A}_\mu - \mathcal{L}$, yields the Hamilton density

$$\mathcal{H} = \pi^i \dot{A}_i + \frac{1}{4} F^{ij} F_{ij} + \frac{1}{2} \pi^i \pi_i + j^0 A_0 + j^i A_i \,. \tag{8.10}$$

With the help of (8.2) the "velocities" $\dot{A}_i$ can be expressed in terms of the momenta as

$$\begin{aligned} F_{0i} = \partial_0 A_i - \partial_i A_0 &= -\pi_i \\ \partial_0 A_i = \dot{A}_i &= -\pi_i + \partial_i A_0 \,, \end{aligned} \tag{8.11}$$

and the Hamilton density takes the form

$$\mathcal{H} = \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi^i \pi_i - \partial_i \pi^i A_0 + j^0 A_0 + j^i A_i \,. \tag{8.12}$$

The primary Hamilton density is obtained by coupling the only primary constraint (8.6) via a Lagrange multiplier field λ_1

$$\mathcal{H}_P = \mathcal{H} + \lambda_1 \pi^0 \,, \tag{8.13}$$

and the Hamiltonian is just the spatial volume integral

$$H_P = \int d^3x \, \mathcal{H}_P = H + \int d^3x \lambda_1 \pi^0 \,. \tag{8.14}$$

The calculation of the fundamental Poisson brackets without worrying about the constraint

$$\begin{aligned} \{A_\mu(\boldsymbol{x}), \pi^\nu(\boldsymbol{y})\} &= \int d^3z \left(\frac{\delta A_\mu(\boldsymbol{x})}{\delta A_\lambda(\boldsymbol{z})} \frac{\delta \pi^\nu(\boldsymbol{y})}{\delta \pi^\lambda(\boldsymbol{z})} - \frac{\delta \pi^\nu(\boldsymbol{y})}{\delta A_\lambda(\boldsymbol{z})} \frac{\delta A_\mu(\boldsymbol{x})}{\delta \pi^\lambda(\boldsymbol{z})} \right) \\ &= \int d^3z \, \delta_\mu{}^\nu \delta^3(\boldsymbol{x} - \boldsymbol{z}) \delta^3(\boldsymbol{y} - \boldsymbol{z}) \\ &= \delta_\mu{}^\nu \delta^3(\boldsymbol{x} - \boldsymbol{y}) \,, \end{aligned} \tag{8.15}$$

shows their incompatibility with the primary constraint (8.6). And here is the need for the Dirac brackets, if we want to go over to quantum theory.

The demand for consistency yields

$$\begin{aligned} \{\pi^0(\boldsymbol{x}), H_P(\boldsymbol{y})\} &= \int d^3z \left(\frac{\delta \pi^0(\boldsymbol{x})}{\delta A_\lambda(\boldsymbol{z})} \frac{\delta H_P(\boldsymbol{y})}{\delta \pi^\lambda(\boldsymbol{z})} - \frac{\delta H_P(\boldsymbol{y})}{\delta A_\lambda(\boldsymbol{z})} \frac{\delta \pi^0(\boldsymbol{x})}{\delta \pi^\lambda(\boldsymbol{z})} \right) \\ &= - \int d^3z \, \frac{\delta H_P(\boldsymbol{y})}{\delta A_\lambda(\boldsymbol{z})} \delta^0_\lambda \delta^3(\boldsymbol{x} - \boldsymbol{z}) \\ &= - \frac{\delta H_P(\boldsymbol{y})}{\delta A_0(\boldsymbol{x})} \\ &= (\partial_i \pi^i - j^0)(\boldsymbol{x}) \overset{!}{\approx} 0 \,. \end{aligned} \tag{8.16}$$

This is our secondary constraint, which is nothing else than Gauss' law in terms of the momentum fields. A short calculation shows that the consistency requirement of this secondary constraint

$$\{(\partial_i \pi^i - j^0)(\boldsymbol{x}), H_P(\boldsymbol{y})\} \overset{!}{\approx} 0 \,, \tag{8.17}$$

yields no further independent constraints. Coupling both constraints to the canonical Hamiltonian yields

$$\mathcal{H}_E = \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi^i \pi_i - \partial_i \pi^i A_0 + j^0 A_0 + j^i A_i + \lambda_1 \pi^0 + \lambda_2 \partial_i \pi^i \,. \tag{8.18}$$

This function is the extended Hamilton density in Maxwell theory.

8.3 Gauge Fixing and Dirac Bracket

The Dirac-Bergmann algorithm yields the following two independent constraints

$$\pi^0 \approx 0\,, \tag{8.19a}$$
$$\partial_i \pi^i - j^0 \approx 0\,. \tag{8.19b}$$

Both constraints are first class, since

$$\begin{aligned}\{\pi^0(\boldsymbol{x}), (\partial_i\pi^i - j^0)(\boldsymbol{y})\} &= \int d^3z \left(\frac{\delta\pi^0(\boldsymbol{x})}{\delta A_\lambda(\boldsymbol{z})}\frac{\delta(\partial_i\pi^i - j^0)(\boldsymbol{y})}{\delta\pi^\lambda(\boldsymbol{z})} - \frac{\delta(\partial_i\pi^i - j^0)(\boldsymbol{y})}{\delta A_\lambda(\boldsymbol{z})}\frac{\delta\pi^0(\boldsymbol{x})}{\delta\pi^\lambda(\boldsymbol{z})}\right)\\ &= 0\end{aligned} \tag{8.20}$$

vanishes *identically*. As expected we have to deal with gauge transformations generated by these first class constraints. But how are they related to the well known gauge transformation

$$A^\mu \longrightarrow A^\mu + \partial^\mu\epsilon\,, \tag{8.21}$$

of the four potential? To answer that question we have to have a closer look at the group properties of the theory.

As mentioned above, the Poisson bracket (8.20) is essentially not just a linear combination of the constraints, it vanishes identically. Consequently, all structure factors $f_{mn}{}^{\ell}$ in the Lie algebra (5.13) vanish and the commutators of the gauge generating operators (5.15) fulfill

$$\left[\widehat{T}_m, \widehat{T}_n\right] = 0\,. \tag{8.22}$$

In other words, the symmetry group induced by the two constraints is abelian. This is a necessary condition that this group can be identified with the symmetry group of the Maxwell field, which is the most simple unitary, abelian group $U(1)$. The gauge transformatons generated by (8.19a) read

$$\delta_{\epsilon_1} A_\mu = \int d^3y\, \epsilon_1(\boldsymbol{y})\{A_\mu(\boldsymbol{x}), \pi^0(\boldsymbol{y})\} = \delta_\mu{}^0\epsilon_1(\boldsymbol{x}) \tag{8.23a}$$
$$\delta_{\epsilon_1}\pi^\mu = \int d^3y\, \epsilon_1(\boldsymbol{y})\{\pi^\mu(\boldsymbol{x}), \pi^0(\boldsymbol{y})\} = 0\,. \tag{8.23b}$$

and correspondingly we obtain for (8.19b)

$$\delta_{\epsilon_2} A_\mu = \int d^3y\, \epsilon_2(\boldsymbol{y})\{A_\mu(\boldsymbol{x}), (\partial_i\pi^i - j^0)(\boldsymbol{y})\} = \delta_\mu{}^i\partial_i\epsilon_2(\boldsymbol{x}) \tag{8.24a}$$
$$\delta_{\epsilon_2}\pi^\mu = \int d^3y\, \epsilon_2(\boldsymbol{y})\{\pi^\mu(\boldsymbol{x}), (\partial_i\pi^i - j^0)(\boldsymbol{y})\} = 0\,. \tag{8.24b}$$

With the identifications

$$\delta_{\gamma_1} A_0 = \epsilon_1(\boldsymbol{x}) = \delta A_0\,, \qquad \delta_{\gamma_1} A_i = 0\,, \tag{8.25a}$$
$$\delta_{\gamma_2} A_i = -\partial_i\epsilon_2(\boldsymbol{x}) = \delta A_i\,, \qquad \delta_{\gamma_2} A_0 = 0\,, \tag{8.25b}$$

we, indeed, recognize the gauge transformations of the unitary group $U(1)$. As a result, the group, which is generated by the constraints, is in this case an isomorphism of the symmetry group of the Maxwell field.

In order to arrive at a quantum theory we have to remove all the gauge freedom. This means that we have to impose as many gauge constraints "by hand" as there are first class constraints. These gauge constraints convert the first class constraints γ_m into second class and we can quantize the system according to (6.29). There are several common gauges available in the Maxwell theory. The most popular choices are the Lorentz gauge

$$\partial_\mu A^\mu = 0\,, \tag{8.26}$$

being manifestly Lorentz-invariant, and the Coulomb gauge

$$\partial_i A^i = 0\,, \tag{8.27}$$

breaking the Lorentz-invariance. Unfortunately, we cannot handle the Lorentz gauge with the formalism developed in this form. This circumstance is due to the fact, that the Lorentz gauge contains the time derivative of A_0. It is possible to extend the formalism to handle such relativistic constraints, but this would be too extensive to be presented here, we just refer to [11] for further reading.

The Coulomb gauge is much more suited for our formalism of gauge fixing. However, the gauge condition (8.27) does not remove all the gauge freedom. There is one condition to impose left, since we need as many gauge conditions as there are first class constraints. We choose A_0 to vanish and arrive at the radiation gauge

$$\Omega_1 = \quad A_0 \approx 0\,, \tag{8.28a}$$
$$\Omega_2 = \partial_i A^i \approx 0\,. \tag{8.28b}$$

As we will see soon, these two equations, indeed, fix the gauge completely. Additionally, this gauge choice implicitly means that the zeroth component of the current four-vector vanishes $j^0 \approx 0$. This can be seen by imposing the Coulomb gauge constraint (8.27) to Maxwell's equations (8.4), which yields

$$\partial_i \partial^i A^0 \approx j^0\,. \tag{8.29}$$

With these additional gauge constraints we are able to calculate the G-matrix

$$G(\boldsymbol{x},\boldsymbol{y}) = \begin{pmatrix} \{\gamma_1,\gamma_1\} & \{\gamma_1,\gamma_2\} & \{\gamma_1,\Omega_1\} & \{\gamma_1,\Omega_2\} \\ \{\gamma_2,\gamma_1\} & \{\gamma_2,\gamma_2\} & \{\gamma_2,\Omega_1\} & \{\gamma_2,\Omega_2\} \\ \{\Omega_1,\gamma_1\} & \{\Omega_1,\gamma_2\} & \{\Omega_1,\Omega_1\} & \{\Omega_1,\Omega_2\} \\ \{\Omega_2,\gamma_1\} & \{\Omega_2,\gamma_2\} & \{\Omega_2,\Omega_1\} & \{\Omega_2,\Omega_2\} \end{pmatrix}. \tag{8.30}$$

Because of the first class property of the γ's we have

$$\{\gamma_m,\gamma_n\} = 0\,. \tag{8.31}$$

It is easy to verify that the gauge constraints are also first class

$$\{\Omega_m,\Omega_n\} = 0\,, \tag{8.32}$$

but in order to fulfill the invertibility condition (6.16) at least some of the "mixed" Poisson brackets must not vanish. A straightforward calculation shows that this is, indeed, the case

$$\{\gamma_1(\boldsymbol{x}),\Omega_1(\boldsymbol{y})\} = -\delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \tag{8.33a}$$
$$\{\gamma_2(\boldsymbol{x}),\Omega_1(\boldsymbol{y})\} = \nabla^2\delta^3(\boldsymbol{x}-\boldsymbol{x})\,, \tag{8.33b}$$
$$\{\gamma_1(\boldsymbol{x}),\Omega_2(\boldsymbol{y})\} = 0\,, \tag{8.33c}$$
$$\{\gamma_2(\boldsymbol{x}),\Omega_2(\boldsymbol{y})\} = 0\,. \tag{8.33d}$$

The resulting G-matrix reads

$$G(\boldsymbol{x},\boldsymbol{y}) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \nabla^2 \\ 1 & 0 & 0 & 0 \\ 0 & -\nabla^2 & 0 & 0 \end{pmatrix} \delta^3(\boldsymbol{x}-\boldsymbol{y}) = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}, \tag{8.34}$$

where A is a (2×2)-matrix. Because of this block structure we can immediately read off the inverse as

$$G^{-1}(\boldsymbol{x},\boldsymbol{y}) = \begin{pmatrix} 0 & -A^{-1} \\ A^{-1} & 0 \end{pmatrix}. \tag{8.35}$$

The problem of finding the inverse of the G-matrix reduces to find the inverse of the subblock A. Fortunately, it is diagonal and so we just have to invert the diagonal elements separately. The inverse of the delta function is simply the delta function itself and the inverse of the Laplacian ∇^2 is $-\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}$. So the inverse of the G-matrix becomes

$$G^{-1}(\boldsymbol{x},\boldsymbol{y}) = \begin{pmatrix} 0 & 0 & \delta^3(\boldsymbol{x}-\boldsymbol{y}) & 0 \\ 0 & 0 & 0 & \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \\ -\delta^3(\boldsymbol{x}-\boldsymbol{y}) & 0 & 0 & 0 \\ 0 & -\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} & 0 & 0 \end{pmatrix}. \tag{8.36}$$

The existence of the inverse matrix proves that the gauge conditions really fix the gauge completely. Having calculated the inverse, we are now able to write down the Dirac brackets using (7.19)

$$\{F(\boldsymbol{x}),G(\boldsymbol{y})\}_{D(G)} = \{F(\boldsymbol{x}),G(\boldsymbol{y})\} - \iint d^3u\, d^3v\, \{F(\boldsymbol{x}),\sigma_\kappa(\boldsymbol{u})\}G^{\kappa\omega}(\boldsymbol{u},\boldsymbol{v})\{\sigma_\omega(\boldsymbol{v}),G(\boldsymbol{y})\}\,.$$

The first two fundamental Dirac brackets vanish identically

$$\{A_\mu(\boldsymbol{x}),A_\nu(\boldsymbol{y})\}_{D(G)} = 0\,, \tag{8.37}$$

$$\{\pi^\mu(\boldsymbol{x}),\pi^\nu(\boldsymbol{y})\}_{D(G)} = 0\,, \tag{8.38}$$

and, therefore, do not differ from the Poisson bracket. The last one needs a little more work to calculate

$$\begin{aligned}\{A_\mu(\boldsymbol{x}),\pi^\nu(\boldsymbol{y})\}_{D(G)} &= \{A_\mu(\boldsymbol{x}),\pi^\nu(\boldsymbol{y})\} \\ &\quad - \iint d^3u\, d^3v\, \{A_\mu(\boldsymbol{x}),\sigma_\kappa(\boldsymbol{u})\}G^{\kappa\omega}(\boldsymbol{u},\boldsymbol{v})\{\sigma_\omega(\boldsymbol{v}),\pi^\nu(\boldsymbol{y})\}\,.\end{aligned} \tag{8.39}$$

Inserting (8.36) yields at first

$$\begin{aligned}\{A_\mu(\boldsymbol{x}),\pi^\nu(\boldsymbol{y})\}_{D(G)} &= \{A_\mu(\boldsymbol{x}),\pi^\nu(\boldsymbol{y})\} \\ &\quad - \iint d^3u\, d^3v\, \{A_\mu(\boldsymbol{x}),\gamma_1(\boldsymbol{u})\}\delta^3(\boldsymbol{u}-\boldsymbol{v})\{\Omega_1(\boldsymbol{v}),\pi^\nu(\boldsymbol{y})\} \\ &\quad - \iint d^3u\, d^3v\, \{A_\mu(\boldsymbol{x}),\gamma_2(\boldsymbol{u})\}\frac{1}{4\pi|\boldsymbol{u}-\boldsymbol{v}|}\{\Omega_2(\boldsymbol{v}),\pi^\nu(\boldsymbol{y})\} \\ &\quad + \iint d^3u\, d^3v\, \{A_\mu(\boldsymbol{x}),\Omega_1(\boldsymbol{u})\}\delta^3(\boldsymbol{u}-\boldsymbol{v})\{\gamma_1(\boldsymbol{v}),\pi^\nu(\boldsymbol{y})\} \\ &\quad + \iint d^3u\, d^3v\, \{A_\mu(\boldsymbol{x}),\Omega_2(\boldsymbol{u})\}\frac{1}{4\pi|\boldsymbol{u}-\boldsymbol{v}|}\{\gamma_2(\boldsymbol{v}),\pi^\nu(\boldsymbol{y})\}\,,\end{aligned} \tag{8.40}$$

which reduces to

$$\begin{aligned}\{A_\mu(\boldsymbol{x}),\pi^\nu(\boldsymbol{y})\}_{D(G)} &= \delta_\mu{}^\nu\delta^3(\boldsymbol{x}-\boldsymbol{y}) - \int d^3v\,\delta_\mu{}^0\delta^3(\boldsymbol{x}-\boldsymbol{v})\delta_0{}^\nu\delta^3(\boldsymbol{v}-\boldsymbol{y})\\ &\quad - \iint d^3u\,d^3v\,\delta_\mu{}^i\partial_i\delta^3(\boldsymbol{u}-\boldsymbol{x})\frac{1}{4\pi|\boldsymbol{u}-\boldsymbol{v}|}\eta^{\nu j}\partial_j\delta^3(\boldsymbol{v}-\boldsymbol{y})\\ &= (\delta_\mu{}^\nu - \delta_\mu{}^0\delta_0{}^\nu)\delta^3(\boldsymbol{x}-\boldsymbol{y}) - \delta_\mu{}^i\eta^{\nu j}\partial_i\partial_j\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,. \end{aligned} \tag{8.41}$$

The important conclusion is that the only Dirac brackets deviating from the Poisson brackets are the following

$$\{A_0(\boldsymbol{x}),\pi^\mu(\boldsymbol{y})\}_{D(G)} = 0 \tag{8.42a}$$

$$\{A_\mu(\boldsymbol{x}),\pi^0(\boldsymbol{y})\}_{D(G)} = 0 \tag{8.42b}$$

$$\{A_i(\boldsymbol{x}),\pi^j(\boldsymbol{y})\}_{D(G)} = \delta_i{}^j\delta^3(\boldsymbol{x}-\boldsymbol{y}) + \partial_i\partial^j\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,. \tag{8.42c}$$

These brackets are by construction compatible with *all* the constraints. Especially the last one is interesting, because it is nothing but a representation of the transverse delta function in real space

$$\delta^3_{\perp}{}_i{}^j(\boldsymbol{x}-\boldsymbol{y}) \equiv \delta_i{}^j\delta^3(\boldsymbol{x}-\boldsymbol{y}) + \partial_i\partial^j\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,. \tag{8.43}$$

In the Appendix B we discuss some properties of that function or rather distribution. Now that we know the fundamental Dirac brackets, we can go over to quantum theory. Using the substitution rule (6.29) yields

$$\left[\widehat{A}_i(\boldsymbol{x}),\widehat{\pi}^j(\boldsymbol{y})\right] = i\hbar\,\delta^3_{\perp}{}_i{}^j(\boldsymbol{x}-\boldsymbol{y})\,. \tag{8.44}$$

Writing this in a more familiar form we get

$$\left[\widehat{A}^i(\boldsymbol{x}),\widehat{\pi}^j(\boldsymbol{y})\right] = \left[\widehat{A}^i(\boldsymbol{x}),\widehat{E}^j(\boldsymbol{y})\right] = -i\hbar\,\delta^3_{\perp}{}^{ji}(\boldsymbol{x}-\boldsymbol{y})\,, \tag{8.45}$$

which is precisely the commutation relation extensively used in quantum optics [5]. Note that the minus sign is due to the transition to contravariant variables $\widehat{A}_i = -\widehat{A}^i$. After all the calculations we see that the Dirac-Bergmann algorithm gives us very naturally a consistent commutation relation. Indeed, this commutation relation gives us already a hint about the true degrees of freedom. As pointed out in the Appendix B the transverse delta function acts as the unity operator in the space of transverse vector fields. Since the Dirac bracket equals the Poisson bracket in the reduced phase space, we can conclude that the true degrees of freedom for the Maxwell field are just the transverse components. A detailed discussion would show that this is, indeed, the case.

9 Conclusion and Outlook

In this thesis we investigated the properties of singular physical systems. We developed a consistent theory for the dynamical evolution of such systems within the Hamiltonian formulation of point mechanics. The transition to the Hamiltonian treatment required a Legendre transformation, whereat the canonical momenta had to be introduced. Due to the singular nature of the Lagrangian, this gave rise to certain relations between the phase space variables. These relations could be interpreted as conditions on the physical phase space and are therefore constraint functions. The formalism gave us a natural distinction between these constraints into first and second class.

We were able to show that first class constraints generate gauge transformations, since the Lagrange multipliers coupling these functions cannot be fixed with consistency requirements. This is due to the fact that they form a closed Lie algebra with the Poisson bracket as the underlying operation. Because of this property, that the Poisson brackets of two first class constraints is again a linear combination of first class constraints, we defined linear operators containing first class constraints. These operators also satisfy a Lie algebra with the commutator as underlying operation. According to the theory of Lie groups, shortly presented in Appendix A, the operators act as infinitesimal generators. Hence, each physical system involving first class constraints possesses gauge symmetries. In contrast to first class constraints, it is possible to fix the multipliers coupling second class constraints. This led us to the definition of the Dirac bracket, which is a generalization of the Poisson bracket for systems with second class constraints.

After investigating the two classes of constraints, we attended to the quantization of the developed theory. A general discussion containing path integrals showed that it was necessary to get rid of the first class constraints. We demonstrated how to remove the first class constraints from the theory by imposing new, so called gauge constraints, and therewith all the gauge freedom. Using the Dirac bracket we were able to handle all remaining second class constraints, and quantize the gauged system. The "justification" of this procedure is given by the important result of Maskawa and Nakajima [21]. They proved that the Dirac bracket is nothing but the Poisson bracket calculated with the variables of the reduced phase space. Consequently, it is reasonable to quantize a singular system by the quntization rules as described in Section 6.

Since most of all physically interesting examples lie in the branch of field theory, our next step was to generalize the theory developed so far to field theory. We could indicate "naive" translation rules so that the limiting process to infinite degrees of freedom, as it is the case for field theory, could be performed for each formula. However, it must be emphasized that this limiting process is problematic.

Finally, we applied the whole apparatus to a physically interesting example: the Maxwell field. We identified two first class constraints generating gauge transformations. The latter could be identified with the known $U(1)$ gauge transformations as expected. After removing the gauge freedom by imposing gauge constraints, in our case the radiation gauge, we calculated the fundamental Dirac brackets. In the transition to quantum theory it turned out, that the Dirac brackets yield a consistent commutation relation for the field operators, which is precisely the starting point for quantum optics.

Despite the problems of the operator formalism, there are several possibilities for further investigations in this formalism itself as well as applications to real physical systems. An important example for the latter is linearized quantum gravity. Consider the action of the Einstein field

$$\mathcal{A} = \int d^4x \, \sqrt{-g} R \,, \tag{9.1}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$ and R the Ricci scalar. Expanding the metric for small perturbations $h_{\mu\nu}$ around the Minkowski metric $\eta_{\mu\nu}$ up to second order, yields the linearized Lagrange density

$$\mathcal{L}_{\text{lin}} = h^{\mu\nu} G_{\mu\nu} \,, \tag{9.2}$$

with the linearized Einstein tensor

$$G_{\mu\nu} \simeq -\frac{1}{2}(h_{\mu\nu,}{}^{\lambda}{}_{\lambda} + h^{\lambda}{}_{\lambda,\mu\nu} - h^{\lambda}{}_{\mu,\nu\lambda} - h^{\lambda}{}_{\nu,\mu\lambda}) + \frac{1}{2}\eta_{\mu\nu}(h^{\lambda}{}_{\lambda,}{}^{\kappa}{}_{\kappa} - h^{\lambda\kappa,}{}_{\lambda\kappa}) \,. \tag{9.3}$$

This is the Fierz-Pauli Lagrangian for the linearized free gravitational field [10]. Introducing the fields

$$\phi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^{\lambda}{}_{\lambda} \,, \tag{9.4}$$

the linearized Lagrangian can be written as

$$\mathcal{L}_{\text{lin}} = -\frac{1}{2}\phi^{\mu\nu,\lambda}\phi_{\mu\nu,\lambda} + \frac{1}{2}\phi^{\mu\nu,\lambda}\phi_{\lambda\mu,\nu} + \frac{1}{2}\phi^{\mu\nu,\lambda}\phi_{\lambda\nu,\mu} + \frac{1}{4}\phi^{\mu}{}_{\nu,}{}^{\lambda}\phi_{\mu\nu,\lambda} \,. \tag{9.5}$$

At first sight this Lagrangian looks quite complicated. But a closer look reveals that this structure is very similar to the one for the electromagnetic field, apart from that we deal in Maxwell theory with tensors of first rank, whereas in linearized gravity with tensors of second rank. The introduction of these new fields is supposed to emphasize the analogy to the Maxwell field. In fact, linearized gravity allows the following gauge transformations

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \,, \tag{9.6}$$

which resemble the gauge transformations of the four potential in Maxwell theory. Therefore, one may expect that the formalism for quantization in a "Coulomb-like" gauge will be very similar to the Maxwell theory as developed in Section 8.

Additionally, we could generalize the developed theory in several aspects. In the transition to quantum theory we only considered the bosonic case, because we replaced the Poisson bracket and Dirac bracket, respectively, by a commutator of operators. Nevertheless, an application to fermionic particles and fields would be interesting as well. In fact such a generalization is not excluded and requires the theory of Grassmann algebras. Important works in that regime have been done by Belinfante et al. [2] and Casalbuoni [4]. Furthermore it would be nice if the formalism could be applied to relativistic constraints. In the application of the formalism in Section 8.3 we mentioned that difficulties arise, when considering the relativistically covariant Lorentz gauge. A discussion how to include covariant constraints could be found in Ref. [24] and was originally worked out by Fradkin and Fradkina [11]. In addition, the limiting process to field theory naively done in Section 7 has to be revised more carefully. For instance, it is dubious whether the summations for a finite number of degrees of freedom could be replaced by integrals in the infinite case and, therefore, wether our constrained field theory is based on a consistent mathematical footing.

Concluding all these considerations we can state, that there are several topics worth to be investigated further as it was possible within this thesis.

A Lie Groups and Lie Algebras

Of course, the topic of Lie groups and Lie algebras is way to broad to be presented here in all detail. The reader is assumed to be familiar with the general concepts of groups and manifolds since we cannot introduce each term used in the following discussion. Nevertheless we want to mention some important properties to clarify the connection between Lie algebras, Lie groups and gauge transformations. Therefore, the following section is quite mathematical, since we may introduce abstractly the concepts of Lie algebras and Lie groups. Anyhow, we do not claim for mathematical correctness. In our discussion we follow the treatment of W. Miller [22], M. Hausner/J. T. Schwartz [15] and T. Frankel [12].

An abstract algebra $(*, \mathcal{V})$ is a vector space $\mathcal{V}$ over a field $\mathbb{K}$ with an operation $*$. The field can be either $\mathbb{R}$ or $\mathbb{C}$. The operation assigns to each pair of vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$ another vector $\boldsymbol{z} \in \mathcal{V}$ via the multiplication mapping

$$* : \mathcal{V} \times \mathcal{V} \to \mathcal{V}\,, \quad (\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{x} * \boldsymbol{y} = \boldsymbol{z}\,. \tag{A.1}$$

This mapping has to be bilinear for vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{V}$ and scalars $\alpha, \beta \in \mathbb{K}$

(*i*) $(\alpha \boldsymbol{x} + \beta \boldsymbol{y}) * \boldsymbol{z} = \alpha \boldsymbol{x} * \boldsymbol{z} + \beta \boldsymbol{y} * \boldsymbol{z}$

(*ii*) $\boldsymbol{z} * (\alpha \boldsymbol{x} + \beta \boldsymbol{y}) = \alpha \boldsymbol{z} * \boldsymbol{x} + \beta \boldsymbol{z} * \boldsymbol{y}$

A Lie algebra is an algebra with the Lie bracket as underlying operation. Additionally to the bilinearity it has to satisfy two more conditions

(*i*) Alternating: $[\boldsymbol{x}, \boldsymbol{x}] = 0$ for all $\boldsymbol{x} \in \mathcal{V}$,

(*ii*) Jacobi identity: $[\boldsymbol{x}, [\boldsymbol{y}, \boldsymbol{z}]] + [\boldsymbol{y}, [\boldsymbol{z}, \boldsymbol{x}]] + [\boldsymbol{z}, [\boldsymbol{x}, \boldsymbol{y}]] = 0$.

The second term to be introduced is the Lie group. An n-dimensional *global* Lie group $(*, \mathcal{G})$ is an n-dimensional differentiable manifold $\mathcal{G}$ occupied with a product operation $*$

$$* : \mathcal{G} \times \mathcal{G} \to \mathcal{G}\,, \quad (P, Q) \mapsto P * Q\,, \tag{A.2}$$

making $\mathcal{G}$ a group. Note that this product operation is not to be confused with the one in (A.1). The group multiplication and inversion have to be analytic with respect to the manifold structure or in other words the mappings have to be diffeomorphisms[2].

We arrive at the concept of *local* Lie groups by endowing the manifold $\mathcal{G}$ at some point $P \in \mathcal{G}$ with a definite "coordinate" mapping. The latter maps P and an open neighborhood U to $\mathbb{K}^n$. We denote these "coordinates" by

$$\boldsymbol{g} = \left(g^1, ..., g^n\right)\,. \tag{A.3}$$

In the following we will restrict ourselves to the study of such local Lie groups. At this stage, one may ask what is the relation between the above mentioned Lie algebras and Lie groups? It will be set out that any abstract Lie algebra corresponds to a local Lie group and vice versa.

To make this evident, we introduce the curve $\boldsymbol{g}(t)$ on a manifold $\mathcal{G}$. Since we are working in a local access, the curve can be parametrized in a certain coordinate system by

$$\boldsymbol{g}(t) = \left(g^1(t), ..., g^n(t)\right)\,. \tag{A.4}$$

[2]A diffeomorphism is a bijective, continuous differential mapping, whose inverse mapping is also continuous differentiable.

We have to distinguish between the curve itself and its trace. Two different parametrizations for t yield two different curves, but the trace of them may be the same. But this subtlety will not busy us since we are interested in an other concept independent of the specific parametrization. The vector

$$\boldsymbol{x} \equiv \boldsymbol{g}'(t_0) = \frac{d\boldsymbol{g}(t)}{dt}\bigg|_{t=t_0} \tag{A.5}$$

is called the tangent vector at the point $P = \boldsymbol{g}(t_0)$. The set of all tangent vectors in a point $P \in \mathcal{G}$ forms a vector space which will be denoted as $\mathcal{G}_P$. Our special interest lies in the tangent vector space at $P = \boldsymbol{e} = (0, ..., 0)$, which is the identity for the group operation in $\mathcal{G}$. We expand the product between two group elements $\boldsymbol{g}$ and $\boldsymbol{h}$ in a Taylor series around that point. The i-th component of this vector valued function becomes

$$(\boldsymbol{g} * \boldsymbol{h})^i = g^i + h^i + c^i{}_{jk} g^j h^k + \mathcal{O}(3)\,, \tag{A.6}$$

where the last term denotes higher orders in $\boldsymbol{g}$ and $\boldsymbol{h}$. The coefficients $c^i{}_{jk}$ are given by

$$c^i{}_{jk} = \frac{\partial^2 (\boldsymbol{g} * \boldsymbol{h})^i}{\partial g^j \partial h^k}\bigg|_{\boldsymbol{g}=\boldsymbol{h}=\boldsymbol{e}}\,. \tag{A.7}$$

It turns out that the properties of the whole group can be described in an infinitesimal neighborhood of $\boldsymbol{e}$. To be more specific, the *commutator* of two tangent vectors $\boldsymbol{x}$ and $\boldsymbol{y}$

$$([\boldsymbol{x}, \boldsymbol{y}])^i = (c^i{}_{jk} - c^i{}_{kj})\, x^j y^k = a^i{}_{jk}\, x^j y^k\,, \tag{A.8}$$

contains all the information of the group via the structure coefficients $a^i{}_{jk}$. The Lie algebra $L(\mathcal{G})$ of a local Lie group is the vector space formed by all tangent vectors at $\boldsymbol{e}$ equipped with the Lie bracket as group operation. This is precisely the connection we were after. We can identify the vector space $\mathcal{V}$ from (A.1) with the tangent vector space $\mathcal{G}_{\boldsymbol{e}}$ and the commutator with the abstract Lie bracket. Conversely, one can show that an abstract Lie algebra defines a local Lie group via a mapping called the exponential map.

Let $\mathcal{G}$ be a local Lie group and $L(\mathcal{G})$ its Lie algebra. The exponential map

$$\exp : L(\mathcal{G}) \to \mathcal{G}\,, \quad \boldsymbol{x} \mapsto \boldsymbol{g} = \exp(\boldsymbol{x})\,, \tag{A.9}$$

maps a neighborhood of $\mathbf{0} \in L(\mathcal{G})$ in a one-to-one manner to a neighborhood of the identity $\boldsymbol{e} \in \mathcal{G}$. So, any element of the local Lie group can be represented as the exponential map of an element of its generating algebra. In the case of a *linear* Lie group, which is simply a group of quadratical matrices, the exponential map is nothing but the exponential of the matrix. Indeed, linear Lie groups have numerous applications in gauge theories.

After revealing the relation between Lie algebras and Lie groups, we have to clarify the connection to gauge transformations. The underlying structure of a gauge transformation is a continuous gauge symmetry, which is precisely a Lie group. In fact, in modern field theoretical approaches to condensed matter or elementary particles one constructs Lagrangians by imposing several symmetries. Examples for the latter could be Lorentz-invariance or special transformation properties of the fields. Now it becomes apparent that the identification of an algebraic structure for first class constraints is fundamental for the understanding of their gauge generating character.

B Transverse Delta Function

In this section we investigate some properties of the transverse Delta function, following the description in [5]. Therefore, we work in the abstract Hilbert space of all three-vector fields. An element of this space is denoted in Dirac notation by $|\boldsymbol{A}\rangle$. We begin the discussion of the transverse delta function by introducing an operator which projects the vector $|\boldsymbol{A}\rangle$ to its transverse subspace

$$|\boldsymbol{A}_\perp\rangle = \mathcal{P}_\perp |\boldsymbol{A}\rangle \; . \tag{B.1}$$

The components of this vector in $|\boldsymbol{k}, \boldsymbol{e}_i\rangle$ basis are given by

$$\begin{aligned} \langle \boldsymbol{k}, \boldsymbol{e}_i | \boldsymbol{A}_\perp \rangle &= \langle \boldsymbol{k}, \boldsymbol{e}_i | \mathcal{P}_\perp |\boldsymbol{A}\rangle \\ &= \langle \boldsymbol{k}, \boldsymbol{e}_i | \mathcal{P}_\perp \sum_j \int d^3k' \; |\boldsymbol{k}', \boldsymbol{e}_j\rangle \langle \boldsymbol{k}', \boldsymbol{e}_j | \boldsymbol{A}\rangle \\ &= \sum_j \int d^3k' \; \langle \boldsymbol{k}, \boldsymbol{e}_i | \mathcal{P}_\perp |\boldsymbol{k}', \boldsymbol{e}_j\rangle \langle \boldsymbol{k}', \boldsymbol{e}_j | A\rangle \; , \end{aligned} \tag{B.2}$$

where we inserted in the second line the completeness relation. Executing these projections yields a representation of these abstract vectors and matrix elements in $\boldsymbol{k}$-space

$$A_{\perp i}(\boldsymbol{k}) = \sum_j \int d^3k' \, \mathcal{P}_{\perp ij}(\boldsymbol{k}, \boldsymbol{k}') A_j(\boldsymbol{k}') \, . \tag{B.3}$$

The matrix elements of the transverse projection operator in this basis are given by

$$\mathcal{P}_{\perp ij}(\boldsymbol{k}, \boldsymbol{k}') = \left(\delta_{ij} - \frac{k_i k_j}{\boldsymbol{k}^2} \right) \delta^3(\boldsymbol{k} - \boldsymbol{k}') \, . \tag{B.4}$$

This identity follows from a short calculation if we work in k-space using

$$\boldsymbol{A}_\perp(\boldsymbol{k}) = \sum_{\varepsilon} (\boldsymbol{A}(\boldsymbol{k}) \cdot \boldsymbol{\varepsilon}) \, \boldsymbol{\varepsilon} \, , \tag{B.5}$$

where $\boldsymbol{\varepsilon}$ are the two unit transverse vectors [5]. The change to the real space basis $|\boldsymbol{r}, \boldsymbol{e}_i\rangle$ is done by a Fourier transformation of the matrix elements

$$\mathcal{P}_{\perp ij}(\boldsymbol{r}, \boldsymbol{r}') = \int \frac{d^3k}{(2\pi)^{3/2}} \int \frac{d^3k'}{(2\pi)^{3/2}} \, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}'} \mathcal{P}_{\perp ij}(\boldsymbol{k}, \boldsymbol{k}') \, . \tag{B.6}$$

Note the minus sign in the second Fourier transformation. Inserting (B.4) yields

$$\begin{aligned} \mathcal{P}_{\perp ij}(\boldsymbol{r}, \boldsymbol{r}') &= \int \frac{d^3k}{(2\pi)^3} \, e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')} \left(\delta_{ij} - \frac{k_i k_j}{\boldsymbol{k}^2} \right) \\ &= \delta_{ij} \delta^3(\boldsymbol{r} - \boldsymbol{r}') + \partial_i \partial_j \frac{1}{4\pi |\boldsymbol{r} - \boldsymbol{r}'|} \, . \end{aligned} \tag{B.7}$$

We recognize that the transverse projection operator in $|\boldsymbol{r}, \boldsymbol{e}_i\rangle$ can be identified with the transverse delta function as introduced in (8.43). Therefore we have the interpretation that the transverse delta function really maps a three-vector field on its transverse components. Consequently, it acts as a unity operator in the space of transverse vector fields.

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Commitment

I declare that I wrote this thesis on my own and that the references contain all the used sources of information.

Berlin, May 7, 2011

Christian Fräßdorf

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