

Bogoliubov Theory for Bose Gases in Random Potentials

Bachelor Thesis by
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1 Introduction

Bose-Einstein condensation is a phase transition that occurs in systems of bosons at very low temperatures if the thermal de Broglie wavelength approaches the mean particle distance. This phase transition is based on the quantum mechanical symmetry properties of bosons and describes the transition from the gaseous phase into a condensate phase where the ground state of the system is macroscopically occupied, that means the total occupation number of the ground state scales with the volume of the system. It was theoretically predicted in 1924 by Bose and Einstein [1–3] and experimentally realised for ultra-cold diluted gases of atoms in 1995 [4, 5].

In 1988 and 1990, experiments with liquid ^4He in porous media [6, 7] realized a system of interacting bosons in a random potential and motivated among others the theoretical work of Huang and Meng in 1992 [8] which is further discussed below. In the following years different concepts for creating an artificial and controllable disorder potential were theoretically discussed and experimentally implemented as e. g. reviewed in Ref. [9]. The main approaches are laser-speckle potentials created by laser beams reflected on rough surfaces or transmitted through diffuse media and optical lattices built up of two laser beams with incommensurate wave lengths. In 2008, it was possible in both setups to observe directly the for disordered systems characteristic Anderson localisation [10–12]. Further concepts are the usage of wire potentials which are created by current-carrying wires with induced disorder through the roughness and impurity of the conductor as additionally reviewed in Ref. [13] or of optical lattices of which a certain number of sites is randomly occupied by a first species of atoms which represents a random frozen in random potential for a second species of atoms as suggested in Ref. [14].

Huang and Meng calculated in Ref. [8] the condensate density n_0 in terms of the particle density n of a Bose gas with contact interaction at low temperatures in a random potential within a Bogoliubov theory and found perturbatively two different depletions of the condensate density which are on the one hand the known depletion due to the particle interaction

$$\frac{8}{3\sqrt{\pi}}(n_0 a)^{3/2} \quad (1)$$

and on the other hand a depletion due to the occurrence of the disorder potential

$$\frac{m^2}{8\sqrt{\pi}^3} \sqrt{\frac{n_0}{a}} R, \quad (2)$$

where a denotes the s-wave scattering length of the particles, R controls the strength of the disorder potential and \hbar was set to one. Note that (1) in [8] contains a typo as e. g. Ref. [15] confirms. The second term denotes physically a depletion of the condensate, which we will also refer to as global condensate in the following, due to the formation of local fragmented condensates in the valleys of the random potential. The result for the disorder depletion (2) was confirmed by several following works [16–18]. As the terms in (1) and (2) are in general small, most of the implemented experiments are not precise enough to measure their influence. Nevertheless there are experimental techniques to increase the depletion due to the particle interaction in not disordered three dimensional systems [19] by using a deep optical lattice which increases the particle density and modifies the dispersion relation. The results for the measured depletion in these experiments agree reasonable with the predictions of a Bogoliubov theoretical treatment.

Recently published works [20, 21] have shown that the result (2) of Huang and Meng can be reproduced within a mean field Gross-Pitaevskii theory and therefore in general does not represent the complete result of a Bogoliubov theory. In the following of this work we reproduce and extend the results (1) and (2) of Huang and Meng within a Bogoliubov theory. Therefore we discuss in Section 2 the properties of a Bose gas in a random potential within a perturbative approach and, in particular, the respective expressions for the particle and global condensate density and the properties of the random potential itself. In order to calculate the global condensate density we introduce a general Bogoliubov transformation in Section 3 and diagonalize the not disordered system as well as develop and apply in Section 4 a formalism to calculate the expansion referring to the disorder potential of correlations of creation and annihilation operators of the system. In Section 5 we conflate the results of the Sections 2–4 in order to calculate the depletion of the condensate explicitly. Finally, we summarize our results in Section 6 and present a short outlook of possible further calculations.

2 Bose Gases in Random Potentials

We describe a diluted homogeneous Bose gas in a random potential $U(\mathbf{x})$ with the property

$$U^*(\mathbf{x}) = U(\mathbf{x}) \quad (3)$$

and with contact interaction by a grand-canonical Hamiltonian in second quantized form

$$\hat{C} = \hat{H} - \mu\hat{N} = \int d^3x \left\{ \hat{\psi}^\dagger(\mathbf{x}) \left[-\frac{\hbar^2}{2m}\nabla^2 - \mu + U(\mathbf{x}) \right] \hat{\psi}(\mathbf{x}) + \frac{g}{2} \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{x}) \right\}, \quad (4)$$

where m denotes the particle mass, μ is the chemical potential and g controls the strength of the particle interaction which is given by

$$g = \frac{4\pi\hbar^2}{m}a, \quad (5)$$

with the s-wave scattering length a [15]. Due to the fact that we consider a diluted gas, we can neglect all higher terms for the interaction, e. g. three-particle interaction terms.

2.1 Equation of Motion

In the following we apply the field-theoretic background method [22–25] in order to determine the dependency of the grand-canonical partition function

$$Z = \text{Tr} \left[\exp \left(-\beta \hat{C} \right) \right] \quad (6)$$

on the condensate wave functions $\psi(\mathbf{x})$ and $\psi^*(\mathbf{x})$ which describe the macroscopic occupation of the ground state and therefore represent the order parameters of Bose-Einstein condensation. With the usual definition of quantum averages

$$\langle \cdot \rangle = \frac{1}{Z} \text{Tr} \left[\cdot \exp \left(-\beta \hat{C} \right) \right], \quad (7)$$

the condensate wave functions correspond to the expectation values of the field operators

$$\psi(\mathbf{x}) = \langle \hat{\psi}(\mathbf{x}) \rangle, \quad (8)$$

$$\psi^*(\mathbf{x}) = \langle \hat{\psi}^\dagger(\mathbf{x}) \rangle. \quad (9)$$

According to the background method we decompose the field operators

$$\hat{\psi}(\mathbf{x}) = \psi(\mathbf{x}) + \delta\hat{\psi}(\mathbf{x}), \quad (10)$$

$$\hat{\psi}^\dagger(\mathbf{x}) = \psi^*(\mathbf{x}) + \delta\hat{\psi}^\dagger(\mathbf{x}), \quad (11)$$

where the operators

$$\delta\hat{\psi}(\mathbf{x}) = \hat{\psi}(\mathbf{x}) - \langle \hat{\psi}(\mathbf{x}) \rangle, \quad (12)$$

$$\delta\hat{\psi}^\dagger(\mathbf{x}) = \hat{\psi}^\dagger(\mathbf{x}) - \langle \hat{\psi}^\dagger(\mathbf{x}) \rangle, \quad (13)$$

characterize the fluctuations around the respective mean value $\psi(\mathbf{x})$ or $\psi^*(\mathbf{x})$, so they describe the occupation of the excited states. The definition of the fluctuation operators $\delta\hat{\psi}(\mathbf{x})$ and $\delta\hat{\psi}^\dagger(\mathbf{x})$ in (12) and (13) directly yields that their expectation values vanish:

$$\langle \delta\hat{\psi}(\mathbf{x}) \rangle = \langle \delta\hat{\psi}^\dagger(\mathbf{x}) \rangle = 0. \quad (14)$$

Inserting (10) and (11) into (4) yields up to second order in $\delta\hat{\psi}^\dagger$ and $\delta\hat{\psi}$:

$$\begin{aligned} \hat{C} = & \int d^3x \left\{ \psi^*(\mathbf{x}) \left[-\frac{\hbar^2}{2m}\nabla^2 - \mu + U(\mathbf{x}) \right] \psi(\mathbf{x}) + \frac{g}{2} \psi^*(\mathbf{x})\psi^*(\mathbf{x})\psi(\mathbf{x})\psi(\mathbf{x}) \right. \\ & + \delta\hat{\psi}^\dagger(\mathbf{x}) \left[-\frac{\hbar^2}{2m}\nabla^2 - \mu + U(\mathbf{x}) \right] \psi(\mathbf{x}) + \psi^*(\mathbf{x}) \left[-\frac{\hbar^2}{2m}\nabla^2 - \mu + U(\mathbf{x}) \right] \delta\hat{\psi}(\mathbf{x}) \\ & + g \delta\hat{\psi}^\dagger(\mathbf{x})\psi^*(\mathbf{x})\psi(\mathbf{x})\psi(\mathbf{x}) + g \delta\hat{\psi}(\mathbf{x})\psi^*(\mathbf{x})\psi^*(\mathbf{x})\psi(\mathbf{x}) \\ & + \delta\hat{\psi}^\dagger(\mathbf{x}) \left[-\frac{\hbar^2}{2m}\nabla^2 - \mu + U(\mathbf{x}) + 2g\psi^*(\mathbf{x})\psi(\mathbf{x}) \right] \delta\hat{\psi}(\mathbf{x}) \\ & \left. + \frac{g}{2} \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}^\dagger(\mathbf{x})\psi(\mathbf{x})\psi(\mathbf{x}) + \frac{g}{2} \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x})\psi^*(\mathbf{x})\psi^*(\mathbf{x}) \right\}. \end{aligned} \quad (15)$$

The terms in (15) that do not depend on $\delta\hat{\psi}$ or $\delta\hat{\psi}^\dagger$ represent the usual Gross-Pitaevskii theory. Furthermore, it will turn out shortly that all terms in this expression which are linear to $\delta\hat{\psi}^\dagger$ or $\delta\hat{\psi}$ vanish. The terms in (15) which are proportional to $\delta\hat{\psi}^\dagger\delta\hat{\psi}$ represent the Hartree and Fock contributions which are degenerated due to the contact interaction, those which contain $\delta\hat{\psi}^\dagger\delta\hat{\psi}^\dagger$ or $\delta\hat{\psi}\delta\hat{\psi}$, correspond to the Bogoliubov contribution. Terms of third or fourth order in $\delta\hat{\psi}$ or $\delta\hat{\psi}^\dagger$ would correspond to corrections to the Bogoliubov theory. A consideration of these terms would go beyond the scope of this work, so we have to assume that the fluctuations in (10) and (11) are small.

Considering the grand-canonical partition function Z in (6), we define the grand-canonical potential Γ as follows:

$$Z = \exp \{-\beta \Gamma [\psi(\mathbf{x}), \psi^*(\mathbf{x})]\}. \quad (16)$$

By analogy with quantum statistics we have to minimize Γ with respect to ψ^* . This gives with (6) and (15) the condition

$$\begin{aligned} \frac{\delta\Gamma [\psi(\mathbf{x}), \psi^*(\mathbf{x})]}{\delta\psi^*(\mathbf{x})} &= \left[-\frac{\hbar^2}{2m} \nabla^2 - \mu + U(\mathbf{x}) + g |\psi(\mathbf{x})|^2 \right] \psi(\mathbf{x}) \\ &+ \frac{1}{Z} \text{Tr} \left(\left\{ \left[-\frac{\hbar^2}{2m} \nabla^2 - \mu + U(\mathbf{x}) + 2g |\psi(\mathbf{x})|^2 \right] \delta\hat{\psi}(\mathbf{x}) + g \psi(\mathbf{x}) \psi(\mathbf{x}) \delta\hat{\psi}^\dagger(\mathbf{x}) \right. \right. \\ &\left. \left. + 2g \psi(\mathbf{x}) \delta\hat{\psi}^\dagger(\mathbf{x}) \delta\hat{\psi}(\mathbf{x}) + g \psi^*(\mathbf{x}) \delta\hat{\psi}(\mathbf{x}) \delta\hat{\psi}(\mathbf{x}) \right\} \exp [-\beta \hat{C}] \right) \\ &= 0. \end{aligned} \quad (17)$$

With the quantum average defined in (7) we can rewrite (17) more concisely as

$$\begin{aligned} 0 &= \left[-\frac{\hbar^2}{2m} \nabla^2 - \mu + U(\mathbf{x}) + g |\psi(\mathbf{x})|^2 \right] \psi(\mathbf{x}) + \left[-\frac{\hbar^2}{2m} \nabla^2 - \mu + U(\mathbf{x}) + 2g |\psi(\mathbf{x})|^2 \right] \langle \delta\hat{\psi}(\mathbf{x}) \rangle \\ &+ g \psi(\mathbf{x}) \psi(\mathbf{x}) \langle \delta\hat{\psi}(\mathbf{x}) \rangle + 2g \psi(\mathbf{x}) \langle \delta\hat{\psi}^\dagger(\mathbf{x}) \delta\hat{\psi}(\mathbf{x}) \rangle + g \psi^*(\mathbf{x}) \langle \delta\hat{\psi}(\mathbf{x}) \delta\hat{\psi}(\mathbf{x}) \rangle. \end{aligned} \quad (18)$$

Taking into account (14) eq. (18) simplifies to

$$\left[-\frac{\hbar^2}{2m} \nabla^2 - \mu + U(\mathbf{x}) + g |\psi(\mathbf{x})|^2 \right] \psi(\mathbf{x}) + 2g \psi(\mathbf{x}) \langle \delta\hat{\psi}^\dagger(\mathbf{x}) \delta\hat{\psi}(\mathbf{x}) \rangle + g \psi^*(\mathbf{x}) \langle \delta\hat{\psi}(\mathbf{x}) \delta\hat{\psi}(\mathbf{x}) \rangle = 0. \quad (19)$$

Thus, we conclude that the usual Gross-Pitaevskii equation for the condensate wave function is modified by a normal correlation $\langle \delta\hat{\psi}^\dagger(\mathbf{x}) \delta\hat{\psi}(\mathbf{x}) \rangle$, which corresponds to the Hartree and Fock contribution, as well as an anomalous correlation $\langle \delta\hat{\psi}(\mathbf{x}) \delta\hat{\psi}(\mathbf{x}) \rangle$ which originates from the Bogoliubov contribution.

Due to the hermiticity of the one-particle Hamiltonian the terms of (15) which are linear in $\delta\hat{\psi}(\mathbf{x})$ read

$$\delta\hat{\psi}(\mathbf{x}) \left[-\frac{\hbar^2}{2m} \nabla^2 - \mu + U(\mathbf{x}) + g |\psi(\mathbf{x})|^2 \right] \psi^*(\mathbf{x}), \quad (20)$$

where the terms of (15) which are linear in $\delta\hat{\psi}(\mathbf{x})$ directly take the form

$$\delta\hat{\psi}^\dagger(\mathbf{x}) \left[-\frac{\hbar^2}{2m} \nabla^2 - \mu + U(\mathbf{x}) + g |\psi(\mathbf{x})|^2 \right] \psi(\mathbf{x}). \quad (21)$$

We identify the prefactor of $\delta\hat{\psi}$ in (20) as the complex conjugate of the zeroth order in the fluctuation of (19) and the prefactor of $\delta\hat{\psi}^\dagger$ in (21) as the zeroth order of (19) itself. By realizing that we can neglect terms of the structure $\delta\hat{\psi}(\mathbf{x}) \langle \delta\hat{\psi}(\mathbf{x}) \delta\hat{\psi}(\mathbf{x}) \rangle$ because they correspond to higher than second order in $\delta\hat{\psi}(\mathbf{x})$ we deduce that (20) and (21) vanish in our approximation due to (19). Therefore we can simplify (15) to

$$\begin{aligned} \hat{C} &= \int d^3x \left\{ \psi^*(\mathbf{x}) \left[-\frac{\hbar^2}{2m} \nabla^2 - \mu + U(\mathbf{x}) \right] \psi(\mathbf{x}) + \frac{g}{2} \psi^*(\mathbf{x}) \psi^*(\mathbf{x}) \psi(\mathbf{x}) \psi(\mathbf{x}) \right. \\ &+ \delta\hat{\psi}^\dagger(\mathbf{x}) \left[-\frac{\hbar^2}{2m} \nabla^2 - \mu + U(\mathbf{x}) + 2g \psi^*(\mathbf{x}) \psi(\mathbf{x}) \right] \delta\hat{\psi}(\mathbf{x}) \\ &\left. + \frac{g}{2} \delta\hat{\psi}^\dagger(\mathbf{x}) \delta\hat{\psi}^\dagger(\mathbf{x}) \psi(\mathbf{x}) \psi(\mathbf{x}) + \frac{g}{2} \delta\hat{\psi}(\mathbf{x}) \delta\hat{\psi}(\mathbf{x}) \psi^*(\mathbf{x}) \psi^*(\mathbf{x}) \right\}. \end{aligned} \quad (22)$$

Thus, in the following every calculation of a quantum average is meant to be evaluated by (7) with the Hamiltonian given in (22).

2.2 Perturbative Approach

We expand the condensate wave function ψ in both the quantum fluctuations $\delta\hat{\psi}$ and the disorder potential U . Therefore we choose the ansatz

$$\psi(\mathbf{x}) = \psi_{00}(\mathbf{x}) + \psi_{01}(\mathbf{x}) + \psi_{02}(\mathbf{x}) + \psi_{10}(\mathbf{x}) + \psi_{11}(\mathbf{x}) + \psi_{12}(\mathbf{x}) + \dots, \quad (23)$$

where the first index denotes the order of $\delta\hat{\psi}^\dagger\delta\hat{\psi}$ or $\delta\hat{\psi}\delta\hat{\psi}$ and the second index contains the order of U .

In order to simplify the perturbative calculation, we can take into account the following symmetry considerations. Because of the spatial homogeneity of the problem, we know that all terms in this expansion which are independent of U will be spatially constant i.e.

$$\psi_{00}(\mathbf{x}) = \psi_{00}, \quad (24)$$

$$\psi_{10}(\mathbf{x}) = \psi_{10}. \quad (25)$$

Furthermore we use the relation

$$\psi^*(\mathbf{x}) = \psi(\mathbf{x}) \quad (26)$$

which is apparent due to the fact that ψ describes the ground state of the system which is a condensate at rest and the random potential U is assumed to be real as stated in (3).

Inserting (23) into (19) with our preliminary considerations (24)–(26) in mind gives

$$0 = [-\mu + g\psi_{00}^2] \psi_{00}, \quad (27)$$

$$0 = \left[-\frac{\hbar^2}{2m}\nabla^2 - \mu + 3g\psi_{00}^2\right] \psi_{01}(\mathbf{x}) + \psi_{00}U(\mathbf{x}), \quad (28)$$

$$0 = \left[-\frac{\hbar^2}{2m}\nabla^2 - \mu + 3g\psi_{00}^2\right] \psi_{02}(\mathbf{x}) + \psi_{01}(\mathbf{x})U(\mathbf{x}) + 3g\psi_{00}\psi_{01}^2(\mathbf{x}), \quad (29)$$

$$0 = [-\mu + 3g\psi_{00}^2] \psi_{10} + g\psi_{00} \left\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_0 + 2g\psi_{00} \left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_0, \quad (30)$$

$$0 = \left[-\frac{\hbar^2}{2m}\nabla^2 - \mu + 3g\psi_{00}^2\right] \psi_{11}(\mathbf{x}) + U(\mathbf{x})\psi_{10} + 6g\psi_{00}\psi_{10}\psi_{01}(\mathbf{x}) + g\psi_{01}(\mathbf{x}) \left\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_0 \\ + 2g\psi_{01}(\mathbf{x}) \left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_0 + g\psi_{00} \left\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_U + 2g\psi_{00} \left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_U, \quad (31)$$

$$0 = \left[-\frac{\hbar^2}{2m}\nabla^2 - \mu + 3g\psi_{00}^2\right] \psi_{12}(\mathbf{x}) + U(\mathbf{x})\psi_{11}(\mathbf{x}) + 6g\psi_{00}\psi_{10}\psi_{02}(\mathbf{x}) + 6g\psi_{00}\psi_{01}(\mathbf{x})\psi_{11}(\mathbf{x}) \\ + 3g\psi_{10}\psi_{01}^2(\mathbf{x}) + g\psi_{02}(\mathbf{x}) \left\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_0 + 2g\psi_{02}(\mathbf{x}) \left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_0 + g\psi_{01}(\mathbf{x}) \left\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_U \\ + 2g\psi_{01}(\mathbf{x}) \left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_U + g\psi_{00} \left\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_{U^2} + 2g\psi_{00} \left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_{U^2}, \quad (32)$$

where $\langle \cdot \rangle_0$, $\langle \cdot \rangle_U$ and $\langle \cdot \rangle_{U^2}$ denote the corresponding terms in an expansion referring to U of $\langle \cdot \rangle$.

2.3 Properties of Random Potentials

A random potential $U(\mathbf{x})$ is defined by its statistical properties which can be expressed in terms of its cumulants. As we expanded in Section 2.2 the condensate wave function up to second order of U , we only need to take care of the first and second cumulant of U .

We define the disorder average

$$\overline{\circ} \quad (33)$$

as the average of the quantity \circ over all possible configurations of the random potential.

By demanding homogeneity we can write immediately for the first cumulant, which corresponds to the first moment,

$$\overline{U(\mathbf{x})} = \overline{U(\mathbf{x}_0)} = U_0 = \text{const.} \quad (34)$$

Due to the structure of the Hamiltonian in (4), this constant can always be pulled into the chemical potential μ , so that we can assume without loss of generality

$$\overline{U(\mathbf{x})} = 0. \quad (35)$$

With this we can immediately conclude that the disorder average of any quantity which is linear in U vanishes, i.e. we obtain

$$\overline{\psi_{10}(\mathbf{x})} = \overline{\psi_{11}(\mathbf{x})} = \overline{\left\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_U} = \overline{\left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_U} = 0. \quad (36)$$

Because of (35), the second cumulant of U corresponds to its second moment. Due to homogeneity this cumulant can only depend on the difference of the respective coordinates:

$$\overline{U(\mathbf{x})U(\mathbf{x}')} = R(\mathbf{x} - \mathbf{x}'). \quad (37)$$

2.4 Particle Densities

For each realisation of the disorder potential U the quantity

$$n[U] = \langle \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x}) \rangle \quad (38)$$

represents the particle density.

Correspondingly the density of the globally condensed particles n_0 is defined according to

$$\sqrt{n_0}[U] = \langle \hat{\psi}(\mathbf{x}) \rangle. \quad (39)$$

Using the disorder average introduced in Section 2.3 we find the following equations for the disorder averaged particle and condensate density:

$$n = \overline{\langle \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x}) \rangle}, \quad (40)$$

$$n_0 = \overline{\langle \hat{\psi}^\dagger(\mathbf{x}) \rangle} \overline{\langle \hat{\psi}(\mathbf{x}) \rangle}. \quad (41)$$

Note that for the density of the condensed particles n_0 we have to perform the disorder average (33) before taking the square of the expression because in general these two operations will not commute.

Inserting the decomposition of the field operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ (10) and (11) and the expansion of the condensate wave function ψ (23) by taking into account (14), (24)–(26) and (36) leads to

$$\begin{aligned} n &= \psi_{00}\psi_{00} + \overline{\psi_{01}(\mathbf{x})\psi_{01}(\mathbf{x})} + 2\overline{\psi_{00}\psi_{02}(\mathbf{x})} + 2\overline{\psi_{00}\psi_{10}} + 2\overline{\psi_{01}(\mathbf{x})\psi_{11}(\mathbf{x})} + 2\overline{\psi_{10}\psi_{02}(\mathbf{x})} \\ &\quad + 2\overline{\psi_{00}\psi_{12}(\mathbf{x})} + \left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_0 + \overline{\left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_{U^2}} \end{aligned} \quad (42)$$

$$n_0 = \psi_{00}\psi_{00} + 2\overline{\psi_{00}\psi_{02}(\mathbf{x})} + 2\overline{\psi_{00}\psi_{10}} + 2\overline{\psi_{10}\psi_{02}(\mathbf{x})} + 2\overline{\psi_{00}\psi_{12}(\mathbf{x})}. \quad (43)$$

By inserting (43) into (42) we end up with

$$n = n_0 + \left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_0 + \overline{\psi_{01}(\mathbf{x})\psi_{01}(\mathbf{x})} + 2\overline{\psi_{01}(\mathbf{x})\psi_{11}(\mathbf{x})} + \overline{\left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_{U^2}}. \quad (44)$$

With (42) and (43) we are able to calculate the particle and condensate density n and n_0 separately as functions of the chemical potential μ . This involves, in particular, to compute $\psi_{12}(\mathbf{x})$ from (32). However, as long as we are only interested in the relation between the particle density and the condensate density, we have to invert the relation $n_0 = n_0(\mu)$ and insert the result into $n = n(\mu)$. On the other hand, with (44) we can directly calculate the particle density n as a function of the condensate density n_0 without taking into account $\psi_{12}(\mathbf{x})$.

This means that the particle density n decomposes into the occupation of the ground state n_0 and additional depletion terms, where $\left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_0$ represents the Bogoliubov depletion due to the excitation of particles irrespective of the random potential, whereas $\overline{\psi_{01}(\mathbf{x})\psi_{01}(\mathbf{x})}$ describes the depletion because of the appearance of the disorder potential on the level of the Gross-Pitaevskii-equation. Finally, the term $2\overline{\psi_{01}(\mathbf{x})\psi_{11}(\mathbf{x})} + \overline{\left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_{U^2}}$ represents again a depletion owing to the disorder potential but this time on the level of the Bogoliubov theory. But in all those terms we still have to replace the chemical potential μ by an expansion analogous to (23)

$$\mu = \mu_{00} + \mu_{01} + \mu_{02} + \mu_{10} + \dots \quad (45)$$

up to the respective order, where the zeroth order is given by

$$\mu_{00} = gn, \quad (46)$$

as follows from considering (27) and (42) and the first order in the disorder potential U μ_{01} vanishes due to (35). By inserting the expansion of the chemical potential into the different depletion terms, we obtain in both terms $\overline{\psi_{01}(\mathbf{x})\psi_{01}(\mathbf{x})}$ and $\overline{\left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_{U^2}}$ additional terms that are of first order in the fluctuations and of second order in the disorder potential and therefore they correspond to depletion terms that arise from the random potential on the level of the Bogoliubov theory.

3 Bogoliubov Transformation

In order to calculate the correlation function $\langle \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x}) \rangle$ and $\langle \hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{x}) \rangle$ we have to evaluate traces as given in (7). To allow the calculation of these traces we diagonalize the terms of the Hamiltonian (22) which do not depend on the disorder potential and regard every term that depends on the random potential as a small perturbation.

3.1 Generalized Bogoliubov Transformation

As introduced in Ref. [26] we consider a general Hamiltonian

$$\hat{H} = \sum_{\mathbf{k}_1, \mathbf{k}_2} \left(A_{\mathbf{k}_1 \mathbf{k}_2} \hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2} + \frac{1}{2} B_{\mathbf{k}_1 \mathbf{k}_2} \hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2}^\dagger + \frac{1}{2} B_{\mathbf{k}_1 \mathbf{k}_2}^* \hat{a}_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2} \right), \quad (47)$$

that is quadratic in the bosonic annihilation and creation operators \hat{a} and \hat{a}^\dagger which fulfill the commutation relations

$$[\hat{a}_{\mathbf{k}_1}^\dagger, \hat{a}_{\mathbf{k}_2}^\dagger] = [\hat{a}_{\mathbf{k}_1}, \hat{a}_{\mathbf{k}_2}] = 0 \quad (48)$$

$$[\hat{a}_{\mathbf{k}_1}, \hat{a}_{\mathbf{k}_2}^\dagger] = \delta_{\mathbf{k}_1, \mathbf{k}_2} \quad (49)$$

and A and B are coefficient matrices with the property

$$A = A^\dagger, \quad (50)$$

$$B = B^T. \quad (51)$$

Using a convenient matrix-vector notation with

$$\hat{\mathbf{a}}^T = (\hat{a}_1, \hat{a}_2, \hat{a}_3, \dots) \quad (52)$$

$$\hat{\mathbf{a}}^{\dagger T} = (\hat{a}_1^\dagger, \hat{a}_2^\dagger, \hat{a}_3^\dagger, \dots) \quad (53)$$

we can rewrite (47) as

$$\hat{H} = \hat{\mathbf{a}}^{\dagger T} A \hat{\mathbf{a}} + \frac{1}{2} \hat{\mathbf{a}}^{\dagger T} B \hat{\mathbf{a}}^\dagger + \frac{1}{2} \hat{\mathbf{a}}^T B^* \hat{\mathbf{a}} \quad (54)$$

and reformulate (48) and (49) as

$$\hat{\mathbf{a}} \hat{\mathbf{a}}^T - (\hat{\mathbf{a}} \hat{\mathbf{a}}^T)^T = \hat{\mathbf{a}}^\dagger \hat{\mathbf{a}}^{\dagger T} - (\hat{\mathbf{a}}^\dagger \hat{\mathbf{a}}^{\dagger T})^T = 0, \quad (55)$$

$$\hat{\mathbf{a}} \hat{\mathbf{a}}^{\dagger T} - (\hat{\mathbf{a}}^\dagger \hat{\mathbf{a}}^T)^T = I, \quad (56)$$

where I denotes the identity.

3.1.1 Bogoliubov Quasi-Particle Operators

In order to diagonalize the Hamiltonian (47) we choose a new set of operators $\hat{\mathbf{b}}$ and $\hat{\mathbf{b}}^\dagger$ which depend on the old operators $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}^\dagger$ according to

$$\hat{\mathbf{b}} = U^\dagger \hat{\mathbf{a}} - V^\dagger \hat{\mathbf{a}}^\dagger, \quad (57)$$

$$\hat{\mathbf{b}}^\dagger = U^T \hat{\mathbf{a}}^\dagger - V^T \hat{\mathbf{a}}. \quad (58)$$

The yet unknown transformation matrices U and V can then be fixed by the condition that the Hamiltonian (47) takes the form

$$\hat{H} = \hat{\mathbf{b}}^{\dagger T} \Lambda \hat{\mathbf{b}} + \kappa I, \quad (59)$$

where κ is a constant and Λ a diagonal matrix. By taking the hermitian conjugate of (59) and realizing that \hat{H} is an hermitian operator, we immediately obtain the conditions

$$\Lambda^* = \Lambda, \quad (60)$$

$$\kappa^* = \kappa. \quad (61)$$

By demanding that $\hat{\mathbf{b}}$ and $\hat{\mathbf{b}}^\dagger$ fulfill the same bosonic commutation relations as $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}^\dagger$ in (55) and (56) and inserting the definition of $\hat{\mathbf{b}}$ and $\hat{\mathbf{b}}^\dagger$ from (57) and (58) we effectively end up with the conditions

$$U^\dagger U - V^\dagger V = I \quad (62)$$

$$V^\dagger U^* - U^\dagger V^* = 0, \quad (63)$$

where we used (55) and (56) several times.

We can express (62) and (63) and their respective complex conjugate in one matrix equation by

$$\begin{pmatrix} U^\dagger & V^\dagger \\ V^T & U^T \end{pmatrix} \begin{pmatrix} U & -V^* \\ -V & U^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad (64)$$

where $(\circ \cdot)$ denotes a matrix with the dimension $2 \dim(\circ)$ and the corresponding blocks.

Interpreting (64) as the product of a matrix and its inverse we can immediately conclude

$$\begin{pmatrix} U & -V^* \\ -V & U^* \end{pmatrix} \begin{pmatrix} U^\dagger & V^\dagger \\ V^T & U^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \quad (65)$$

With the relations contained in (65) and the definitions (57) and (58) the inverse transformation between old and new operators reads

$$\hat{\mathbf{a}} = U\hat{\mathbf{b}} + V^*\hat{\mathbf{b}}^\dagger \quad (66)$$

$$\hat{\mathbf{a}}^\dagger = U^*\hat{\mathbf{b}}^\dagger + V\hat{\mathbf{b}}. \quad (67)$$

3.1.2 Determination of the Bogoliubov Dispersion and Transformation Matrices

Considering the commutator of \hat{b}_j , \hat{b}_j^\dagger and \hat{a}_j with the respective representation of \hat{H} in (47) and (59) yields

$$[\hat{b}_j, \hat{H}] = (\Lambda\hat{\mathbf{b}})_j \quad (68)$$

$$[\hat{b}_j^\dagger, \hat{H}] = -(\Lambda\hat{\mathbf{b}}^\dagger)_j \quad (69)$$

$$[\hat{a}_j, \hat{H}] = (A\hat{\mathbf{a}})_j + (B\hat{\mathbf{a}}^\dagger)_j \quad (70)$$

where we used (51) for (70) and $(\circ)_j$ denotes the j -th component of \circ .

By using (66), (68) and (69) we obtain

$$[\hat{a}_j, \hat{H}] = (U\Lambda\hat{\mathbf{b}})_j - (V^*\Lambda\hat{\mathbf{b}}^\dagger)_j. \quad (71)$$

Inserting (66) and (67) directly into (70) yields

$$[\hat{a}_j, \hat{H}] = ([AU + BV]\hat{\mathbf{b}})_j + ([AV^* + BU^*]\hat{\mathbf{b}}^\dagger)_j \quad (72)$$

and we obtain by evaluating $[[\hat{a}_j, \hat{H}], \hat{b}_j]$ and $[[\hat{a}_j, \hat{H}], \hat{b}_j^\dagger]$ with (71) and (72)

$$U\Lambda = AU + BV \quad (73)$$

$$-V^*\Lambda = AV^* + BU^*. \quad (74)$$

Applying the same argumentation to $[\hat{\mathbf{a}}^\dagger, \hat{H}]$ leads with (60) only to the complex conjugate of (73) and (74).

We can now determine κ by inserting (66) and (67) into (47) and taking into account the commutation relations of $\hat{\mathbf{b}}$ and $\hat{\mathbf{b}}^\dagger$ and the relations (73) and (74) which yields

$$\kappa = -\text{Tr}(\Lambda V^T V^*). \quad (75)$$

In order to specify Λ , U and V further, we express (73) and the complex conjugate of (74) in one matrix equation

$$\begin{pmatrix} A & B^* \\ -B^* & -A^* \end{pmatrix} \begin{pmatrix} U & 0 \\ V & 0 \end{pmatrix} = \begin{pmatrix} U & 0 \\ V & 0 \end{pmatrix} \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}, \quad (76)$$

where we used the same notation as e.g. in (64). Considering the j -th column of this matrix equation (76) yields

$$\begin{pmatrix} A & B \\ -B^* & -A^* \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}_j = \lambda_j \begin{pmatrix} U \\ V \end{pmatrix}_j, \quad (77)$$

with

$$\begin{pmatrix} U \\ V \end{pmatrix}_j^T = (u_{1j}, \dots, u_{\dim(U)j}, v_{1j}, \dots, v_{\dim(V)j}). \quad (78)$$

We can interpret (77) as an eigenvalue equation of the dimensionality $2 \dim(A)$ which is twice of the dimensionality of our original problem (47). By taking the complex conjugate of (77), multiplying both sides by -1 and interchanging the upper $\dim(A)$ rows of the matrix with the lower $\dim(A)$ rows, one can show that if λ_j is the eigenvalue with the eigenvector $(U, V)_j^T$ then $-\lambda_j$ is the eigenvalue with the eigenvector $(V^*, U^*)_j^T$. In the following we will apply this formalism to a positive definite Hamiltonian, therefore Λ consists only of positive eigenvalues λ_j .

The diagonalisation of a general Hamiltonian as given in (47) can be performed explicitly by solving the eigenvalue problem given in (77), with the respective coefficient matrices of the Hamiltonian, which leads to the Bogoliubov dispersion as the resulting eigenvalues. In order to fix the transformation matrices we have to take into account in addition the relations given in (64).

Therefore we finally fixed Λ , U and V due to the choice of A and B and can elaborate the explicit diagonal form (59) of (47) by evaluating κ according to (75).

3.2 Application of Bogoliubov Transformation

Using the fluctuation operators $\delta\hat{\psi}(\mathbf{x})$ and $\delta\hat{\psi}^\dagger(\mathbf{x})$, the disorder potential $U(\mathbf{x})$ and the fields $\psi(\mathbf{x})$ and $\psi^*(\mathbf{x})$ in their \mathbf{k} -representation which is defined via

$$f(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} f(\mathbf{k}) \exp(i\mathbf{k}\mathbf{x}), \quad (79)$$

$$f^\dagger(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} f^\dagger(-\mathbf{k}) \exp(i\mathbf{k}\mathbf{x}), \quad (80)$$

where we substitute

$$\delta\hat{\psi}(\mathbf{k}) = \hat{a}_{\mathbf{k}}, \quad (81)$$

$$\delta\hat{\psi}^\dagger(\mathbf{k}) = \hat{a}_{\mathbf{k}}^\dagger, \quad (82)$$

we can write the Hartree, Fock and Bogoliubov terms of (22) in \mathbf{k} -space

$$\begin{aligned} \hat{C}_{\text{Bog}} = & \sum_{\mathbf{k}_1, \mathbf{k}_2} \left[\left(\frac{\hbar^2 \mathbf{k}_2^2}{2m} - \mu + 2g\psi_{00}^2 \right) \hat{a}_{-\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2} + \frac{g}{2} \psi_{00}^2 \left(\hat{a}_{-\mathbf{k}_1}^\dagger \hat{a}_{-\mathbf{k}_2}^\dagger + \hat{a}_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2} \right) \right] \delta_{\mathbf{k}_1 + \mathbf{k}_2, 0} \\ & + \frac{1}{\sqrt{V}} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3} \left(\left\{ U(\mathbf{k}_3) + 4g\psi_{00} [\psi_{01}(\mathbf{k}_3) + \psi_{02}(\mathbf{k}_3)] \right\} \hat{a}_{-\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2} + g\psi_{00} [\psi_{01}(\mathbf{k}_3) + \psi_{02}(\mathbf{k}_3)] \right. \\ & \times \left. \left(\hat{a}_{-\mathbf{k}_1}^\dagger \hat{a}_{-\mathbf{k}_2}^\dagger + \hat{a}_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2} \right) \right) \delta_{\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3, 0} \\ & + \frac{1}{V} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4} \frac{g}{2} \psi_{01}(\mathbf{k}_3) \psi_{01}(\mathbf{k}_4) \left(4 \hat{a}_{-\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2} + \hat{a}_{-\mathbf{k}_1}^\dagger \hat{a}_{-\mathbf{k}_2}^\dagger + \hat{a}_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2} \right) \delta_{\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4, 0}. \end{aligned} \quad (83)$$

Note that the sum over \mathbf{k}_1 and \mathbf{k}_2 only runs over all values with

$$\mathbf{k}_1 \neq 0 \wedge \mathbf{k}_2 \neq 0, \quad (84)$$

because $\mathbf{k}_1 = 0$ or $\mathbf{k}_2 = 0$ corresponds to creation or annihilation operators for the condensate which were explicitly taken out by the ansatz (10) and (11).

The first sum of this expression is of zeroth order in U ; all the other parts are at least linear or quadratic in U . Therefore, as described in the introduction of Section 3, we diagonalize

$$\hat{C}_{\text{Bog}}^0 = \sum_{\mathbf{k}} \left[\left(\frac{\hbar^2 \mathbf{k}^2}{2m} - \mu + 2g\psi_{00}^2 \right) \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \frac{g}{2} \psi_{00}^2 \left(\hat{a}_{-\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}^\dagger + \hat{a}_{-\mathbf{k}} \hat{a}_{\mathbf{k}} \right) \right] \quad (85)$$

and consider every additional term in (83) as a small perturbation. Due to the fact that (85) characterizes the excitation out of the ground state of the system in the absence of the random potential U , we can express the grand-canonical energy of the excited quasi-particles in zeroth order in U with (59) as

$$C_{\text{ex. part.}}^0 \propto \sum_{\mathbf{k}} n_{\mathbf{k}} \lambda_{\mathbf{k}}. \quad (86)$$

If one of the $\lambda_{\mathbf{k}}$ is smaller than 0, the minimal grand-canonical energy would diverge which corresponds to an unphysical behavior. Therefore we conclude

$$\lambda_{\mathbf{k}} > 0, \quad (87)$$

which means that the grand-canonical Hamiltonian (85) is positive definite. Note that the case \mathbf{k} with $\lambda_{\mathbf{k}} = 0$, which could lead to ambiguities, does not appear due to (84) as we will see below in (96).

Following Section 3.1 we define the matrices

$$A_{\mathbf{k},\mathbf{k}'} = \varepsilon_{\mathbf{k}} \delta_{\mathbf{k},\mathbf{k}'} \quad (88)$$

$$B_{\mathbf{k},\mathbf{k}'} = g \psi_{00}^2 \delta_{\mathbf{k},-\mathbf{k}'}, \quad (89)$$

with the abbreviation

$$\varepsilon_{\mathbf{k}} = \frac{\hbar^2 \mathbf{k}^2}{2m} - \mu + 2g \psi_{00}^2. \quad (90)$$

Consequently we have to solve the eigenvalue problem given in (77) with (88) and (89) which takes the form

$$\begin{pmatrix} \vdots \\ (\varepsilon_{\mathbf{k}} - \lambda_{\mathbf{k}'})u_{\mathbf{k},\mathbf{k}'} + g \psi_{00}^2 v_{-\mathbf{k},\mathbf{k}'} \\ \vdots \\ -(\varepsilon_{-\mathbf{k}} + \lambda_{\mathbf{k}'})v_{-\mathbf{k},\mathbf{k}'} - g \psi_{00}^2 u_{\mathbf{k},\mathbf{k}'} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (91)$$

where we explicitly stated the \mathbf{k} -th row of the upper half and the $-\mathbf{k}$ -th row of the lower half of the resulting vector. Considering (91), only $u_{\mathbf{k},\mathbf{k}'}$ and $v_{-\mathbf{k},\mathbf{k}'}$ are coupled. Therefore we choose the following ansatz for the eigenvectors:

$$u_{\mathbf{k},\mathbf{k}'} = u_{\mathbf{k}'} \delta_{\mathbf{k},\mathbf{k}'} \quad (92)$$

$$v_{\mathbf{k},\mathbf{k}'} = v_{\mathbf{k}'} \delta_{-\mathbf{k},\mathbf{k}'}. \quad (93)$$

Inserting (92) and (93) into (91) yields the system of linear equations

$$\begin{pmatrix} \varepsilon_{\mathbf{k}} - \lambda_{\mathbf{k}} & g \psi_{00}^2 \\ -g \psi_{00}^2 & -\varepsilon_{\mathbf{k}} - \lambda_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} u_{\mathbf{k}} \\ v_{-\mathbf{k}} \end{pmatrix} = 0, \quad (94)$$

which has a non-trivial solution only if the condition

$$\det \begin{pmatrix} \varepsilon_{\mathbf{k}} - \lambda_{\mathbf{k}} & g \psi_{00}^2 \\ -g \psi_{00}^2 & -\varepsilon_{\mathbf{k}} - \lambda_{\mathbf{k}} \end{pmatrix} = 0 \quad (95)$$

is fulfilled. With (87) the condition (95) immediately leads to

$$\lambda_{\mathbf{k}} = \sqrt{\varepsilon_{\mathbf{k}}^2 - (g \psi_{00}^2)^2}, \quad (96)$$

$$u_{\mathbf{k}} = -v_{-\mathbf{k}} \frac{g \psi_{00}^2}{\varepsilon_{\mathbf{k}} - \lambda_{\mathbf{k}}}. \quad (97)$$

The transformation matrices entries $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ will only depend on $\varepsilon_{\mathbf{k}}$ and $\lambda_{\mathbf{k}}$ which are both symmetric in \mathbf{k} . This yields

$$u_{-\mathbf{k}} = u_{\mathbf{k}}, \quad (98)$$

$$v_{-\mathbf{k}} = v_{\mathbf{k}}. \quad (99)$$

With the relation given in (64) we conclude

$$u_{\mathbf{k}} = -\sqrt{\frac{\varepsilon_{\mathbf{k}} + \lambda_{\mathbf{k}}}{2\lambda_{\mathbf{k}}}}, \quad (100)$$

$$v_{\mathbf{k}} = \sqrt{\frac{\varepsilon_{\mathbf{k}} - \lambda_{\mathbf{k}}}{2\lambda_{\mathbf{k}}}}, \quad (101)$$

where $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are fixed up to the minus sign which can be interchanged between both terms.

With (59) and (75) we can state the explicit diagonalized form of the grand-canonical Hamiltonian (85):

$$\hat{C}_{\text{Bog}}^0 = \sum_{\mathbf{k}} \lambda_{\mathbf{k}} \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} + \sum_{\mathbf{k}} \frac{\lambda_{\mathbf{k}} - \varepsilon_{\mathbf{k}}}{2}. \quad (102)$$

Note that the usual Bogoliubov transformation as written in textbooks (e. g. [15]) directly introduces the ansatz given in (92) and (93) and derive the Bogoliubov-de Gennes equation in (94) by demanding

the conservation of the commutation relations. In Section 3.1 we derived a more general formalism which justifies (92) and (93) and can be used for diagonalizing explicitly known quadratic Hamiltonians.

With the structure of the transformation matrices (92) and (93) and the relations in (99)–(101) we can write the old operators \hat{a} and \hat{a}^\dagger in terms of the new operators \hat{b} and \hat{b}^\dagger with (66) and (67) as follows:

$$\hat{a}_{\mathbf{k}} = u_{\mathbf{k}}\hat{b}_{\mathbf{k}} + v_{\mathbf{k}}\hat{b}_{-\mathbf{k}}^\dagger \quad (103)$$

$$\hat{a}_{\mathbf{k}}^\dagger = u_{\mathbf{k}}\hat{b}_{\mathbf{k}}^\dagger + v_{\mathbf{k}}\hat{b}_{-\mathbf{k}}. \quad (104)$$

Inserting (102), (103) and (104) into (83) leads to

$$\begin{aligned} \hat{C}_{\text{Bog}} = & \sum_{\mathbf{k}} \left(\lambda_{\mathbf{k}} \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} - \frac{\varepsilon_{\mathbf{k}} - \lambda_{\mathbf{k}}}{2} \right) + \sum_{\mathbf{k}, \mathbf{k}'} \left[\left({}^1A_{\mathbf{k}, \mathbf{k}'}^U + {}^1A_{\mathbf{k}, \mathbf{k}'}^{U^2} \right) \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}'} + \left({}^2A_{\mathbf{k}, \mathbf{k}'}^U + {}^2A_{\mathbf{k}, \mathbf{k}'}^{U^2} \right) \hat{b}_{\mathbf{k}} \hat{b}_{\mathbf{k}'}^\dagger \right. \\ & \left. + \left({}^3A_{\mathbf{k}, \mathbf{k}'}^U + {}^3A_{\mathbf{k}, \mathbf{k}'}^{U^2} \right) \hat{b}_{\mathbf{k}} \hat{b}_{\mathbf{k}'} + \left({}^4A_{\mathbf{k}, \mathbf{k}'}^U + {}^4A_{\mathbf{k}, \mathbf{k}'}^{U^2} \right) \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}'}^\dagger \right], \end{aligned} \quad (105)$$

where the corresponding coefficient matrices, which are linear in the random potential U , take the form

$${}^1A_{\mathbf{k}, \mathbf{k}'}^U = \frac{1}{\sqrt{V}} \{ [U(\mathbf{k} - \mathbf{k}') + 4g \psi_{00} \psi_{01}(\mathbf{k} - \mathbf{k}')] u_{\mathbf{k}} u_{\mathbf{k}'} + g \psi_{00} \psi_{01}(\mathbf{k} - \mathbf{k}') (u_{\mathbf{k}} v_{\mathbf{k}'} + u_{\mathbf{k}'} v_{\mathbf{k}}) \}, \quad (106)$$

$${}^2A_{\mathbf{k}, \mathbf{k}'}^U = \frac{1}{\sqrt{V}} \{ [U(-\mathbf{k} + \mathbf{k}') + 4g \psi_{00} \psi_{01}(-\mathbf{k} + \mathbf{k}')] v_{\mathbf{k}} v_{\mathbf{k}'} + g \psi_{00} \psi_{01}(-\mathbf{k} + \mathbf{k}') (u_{\mathbf{k}} v_{\mathbf{k}'} + u_{\mathbf{k}'} v_{\mathbf{k}}) \}, \quad (107)$$

$${}^3A_{\mathbf{k}, \mathbf{k}'}^U = \frac{1}{\sqrt{V}} \{ [U(-\mathbf{k} - \mathbf{k}') + 4g \psi_{00} \psi_{01}(-\mathbf{k} - \mathbf{k}')] u_{\mathbf{k}'} v_{\mathbf{k}} + g \psi_{00} \psi_{01}(-\mathbf{k} - \mathbf{k}') (u_{\mathbf{k}} u_{\mathbf{k}'} + v_{\mathbf{k}} v_{\mathbf{k}'}) \}, \quad (108)$$

$${}^4A_{\mathbf{k}, \mathbf{k}'}^U = \frac{1}{\sqrt{V}} \{ [U(\mathbf{k} + \mathbf{k}') + 4g \psi_{00} \psi_{01}(\mathbf{k} + \mathbf{k}')] u_{\mathbf{k}} v_{\mathbf{k}'} + g \psi_{00} \psi_{01}(\mathbf{k} + \mathbf{k}') (u_{\mathbf{k}} u_{\mathbf{k}'} + v_{\mathbf{k}} v_{\mathbf{k}'}) \}, \quad (109)$$

and those, which are quadratic in the disorder potential U , are defined via

$${}^1A_{\mathbf{k}, \mathbf{k}'}^{U^2} = \frac{1}{\sqrt{V}} \left[g \psi_{00} \psi_{02}(\mathbf{k} - \mathbf{k}') + \frac{g}{2\sqrt{V}} \sum_{\mathbf{k}''} \psi_{01}(\mathbf{k}'') \psi_{01}(\mathbf{k} - \mathbf{k}' - \mathbf{k}'') \right] (4u_{\mathbf{k}} u_{\mathbf{k}'} + u_{\mathbf{k}} v_{\mathbf{k}'} + u_{\mathbf{k}'} v_{\mathbf{k}}), \quad (110)$$

$$\begin{aligned} {}^2A_{\mathbf{k}, \mathbf{k}'}^{U^2} = & \frac{1}{\sqrt{V}} \left[g \psi_{00} \psi_{02}(-\mathbf{k} + \mathbf{k}') + \frac{g}{2\sqrt{V}} \sum_{\mathbf{k}''} \psi_{01}(\mathbf{k}'') \psi_{01}(-\mathbf{k} + \mathbf{k}' - \mathbf{k}'') \right] \\ & \times (4v_{\mathbf{k}} v_{\mathbf{k}'} + u_{\mathbf{k}} v_{\mathbf{k}'} + u_{\mathbf{k}'} v_{\mathbf{k}}), \end{aligned} \quad (111)$$

$$\begin{aligned} {}^3A_{\mathbf{k}, \mathbf{k}'}^{U^2} = & \frac{1}{\sqrt{V}} \left[g \psi_{00} \psi_{02}(-\mathbf{k} - \mathbf{k}') + \frac{g}{2\sqrt{V}} \sum_{\mathbf{k}''} \psi_{01}(\mathbf{k}'') \psi_{01}(-\mathbf{k} - \mathbf{k}' - \mathbf{k}'') \right] \\ & \times (4u_{\mathbf{k}'} v_{\mathbf{k}} + u_{\mathbf{k}} u_{\mathbf{k}'} + v_{\mathbf{k}} v_{\mathbf{k}'}), \end{aligned} \quad (112)$$

$${}^4A_{\mathbf{k}, \mathbf{k}'}^{U^2} = \frac{1}{\sqrt{V}} \left[g \psi_{00} \psi_{02}(\mathbf{k} + \mathbf{k}') + \frac{g}{2\sqrt{V}} \sum_{\mathbf{k}''} \psi_{01}(\mathbf{k}'') \psi_{01}(\mathbf{k} + \mathbf{k}' - \mathbf{k}'') \right] (4u_{\mathbf{k}} v_{\mathbf{k}'} + u_{\mathbf{k}} u_{\mathbf{k}'} + v_{\mathbf{k}} v_{\mathbf{k}'}). \quad (113)$$

Due to the symmetry of the transfer matrices U and V given in (98) and (99) we conclude the following symmetry relations of the coefficients $A_{\mathbf{k}, \mathbf{k}'}^j$

$${}^1A_{\mathbf{k}, \mathbf{k}'}^j = {}^1A_{-\mathbf{k}', -\mathbf{k}}^j \quad (114)$$

$${}^2A_{\mathbf{k}, \mathbf{k}'}^j = {}^2A_{-\mathbf{k}', -\mathbf{k}}^j \quad (115)$$

$${}^4A_{\mathbf{k}, \mathbf{k}'}^j = {}^3A_{-\mathbf{k}', -\mathbf{k}}^j \quad (116)$$

where $j = U, U^2$. These relations correspond to the fact that \hat{C} is a hermitian operator and the coefficients consist only of real constants and functions that are real in \mathbf{x} -space according to (3) and (26) and therefore fulfill in \mathbf{k} -space

$$f^*(\mathbf{k}) = f(-\mathbf{k}) \quad (117)$$

as can be read off (79) and (80) by using $f(\mathbf{x}) = f^\dagger(\mathbf{x})$. The relation (117) gives for the coefficients

$${}^iA_{\mathbf{k}, \mathbf{k}'}^{j,*} = {}^iA_{-\mathbf{k}, -\mathbf{k}'}^j, \quad (118)$$

for $i = 1, 2, 3, 4$ and $j = U, U^2$.

4 Calculation of Correlation Functions

To obtain the condensate density n_0 and the additional depletion terms in (44) we have next to calculate the correlation functions $\langle \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \rangle$ and $\langle \hat{\psi}(\mathbf{x}) \hat{\psi}(\mathbf{x}) \rangle$ in zeroth, first and second order in the disorder potential for a grand-canonical Hamiltonian as given in (105). Therefore we introduce the Dirac picture of quantum mechanics.

4.1 Dirac Picture

We can describe quantum mechanics basically in three different pictures, the Schrödinger picture, the Heisenberg picture and the Dirac picture [27]. The fundamental difference of these pictures is the interpretation of the propagation in time of the system. In a system which does not explicitly depend on time, we can define the state of the system as time dependent and the observables as time independent. This definition corresponds to the Schrödinger picture. The Heisenberg picture is defined in the same manner but with interchanged roles of the state and the observable referring to the time dependence. In the Dirac picture both the observables and the states propagate in time. We will define the Dirac picture out of the Schrödinger picture but directly with an imaginary time

$$t \rightarrow -i\tau, \quad (119)$$

because we are interested in quantum statistical calculations.

Considering a grand-canonical Hamiltonian

$$\hat{G} = \hat{G}_0 + \hat{V}, \quad (120)$$

where \hat{G}_0 corresponds to a free, which means a diagonalized system, and \hat{V} denotes the interacting parts of \hat{G} , we define for a state $|\psi(\tau)\rangle$ in the Schrödinger picture the transition to the Dirac picture by

$$|\psi_D(\tau)\rangle = \hat{U}_0(\tau_0, \tau) |\psi(\tau)\rangle, \quad (121)$$

where τ_0 denotes a fixed initial time and the time evolution operator \hat{U}_0 is defined via

$$\hat{U}_0(\tau, \tau') = \exp \left[-\frac{1}{\hbar} \hat{G}_0(\tau - \tau') \right]. \quad (122)$$

On the other hand, we describe the evolution of the state in the Dirac picture by

$$|\psi_D(\tau)\rangle = \hat{U}_D(\tau, \tau') |\psi_D(\tau')\rangle. \quad (123)$$

By considering a state in the Dirac picture $|\psi_D(\tau)\rangle$, changing to the Schrödinger picture with (121), expressing the state $|\psi(\tau)\rangle$ through $|\psi(\tau')\rangle$ by using the Schrödinger time evolution operator $\hat{U}(\tau, \tau')$ and changing back to the Dirac picture with (121) we obtain an explicit representation for $\hat{U}_D(\tau, \tau')$:

$$\hat{U}_D(\tau, \tau') = \hat{U}_0(\tau_0, \tau) \hat{U}(\tau, \tau') \hat{U}_0(\tau', \tau_0), \quad (124)$$

where the time evolution operator in the Schrödinger picture $\hat{U}(\tau, \tau')$ for a time independent grand-canonical Hamiltonian \hat{G} takes the form

$$\hat{U}(\tau, \tau') = \exp \left[-\frac{1}{\hbar} \hat{G}(\tau - \tau') \right]. \quad (125)$$

If we demand the equality of the expectation value of any operator in the Dirac and the Schrödinger picture, i. e.

$$\langle \psi_D(\tau) | \hat{A}_D(\tau) | \psi_D(\tau) \rangle = \langle \psi(\tau) | \hat{A} | \psi(\tau) \rangle, \quad (126)$$

we obtain by inserting (121) and the respective relation for the adjoint

$$\hat{A}_D(\tau) = \hat{U}_0(\tau_0, \tau) \hat{A} \hat{U}_0(\tau, \tau_0). \quad (127)$$

Calculating the time derivative $\frac{\partial}{\partial \tau} \psi_D(\tau)$ by changing to the Schrödinger picture with (121), using the imaginary-time dependent Schrödinger equation

$$-\hbar \frac{\partial}{\partial \tau} |\psi(\tau)\rangle = \hat{G} |\psi(\tau)\rangle \quad (128)$$

and returning into the Dirac picture yields

$$-\hbar \frac{\partial}{\partial \tau} |\psi_D(\tau)\rangle = \hat{V}_D(\tau) |\psi_D(\tau)\rangle. \quad (129)$$

By using (123) and realising that τ and τ' are independent variables we obtain

$$-\hbar \frac{\partial}{\partial \tau} \hat{U}_D(\tau, \tau') = \hat{V}_D(\tau) \hat{U}_D(\tau, \tau'), \quad (130)$$

which we can integrate to end up with the Dyson series

$$\hat{U}_D(\tau, \tau') = I - \frac{1}{\hbar} \int_{\tau'}^{\tau} d\tau'' \hat{V}_D(\tau'') + \frac{1}{\hbar^2} \int_{\tau'}^{\tau} d\tau'' \int_{\tau'}^{\tau''} d\tau''' \hat{V}_D(\tau'') \hat{V}_D(\tau''') + \dots, \quad (131)$$

where I denotes again the identity.

In order to rewrite (131) more conveniently we introduce the imaginary time-ordering operator

$$\hat{T}_\tau \left[\hat{A}(\tau_1) \hat{B}(\tau_2) \right] = \begin{cases} \hat{A}(\tau_1) \hat{B}(\tau_2) & \text{if } \tau_1 > \tau_2, \\ \hat{B}(\tau_2) \hat{A}(\tau_1) & \text{if } \tau_2 > \tau_1, \end{cases} \quad (132)$$

to end up with

$$\hat{U}_D(\tau, \tau') = \hat{T}_\tau \left[I - \frac{1}{\hbar} \int_{\tau'}^{\tau} d\tau'' \hat{V}_D(\tau'') + \frac{1}{2\hbar^2} \int_{\tau'}^{\tau} d\tau'' \int_{\tau'}^{\tau''} d\tau''' \hat{V}_D(\tau'') \hat{V}_D(\tau''') + \dots \right]. \quad (133)$$

4.2 Perturbation Theory for Quantum Averages and Wick Theorem

Using the time evolution operator in the Dirac picture (124) and in the Schrödinger picture (125) yields with (120) and (122)

$$\exp(-\beta \hat{G}) = \exp(-\beta \hat{G}_0) \hat{U}_D(\hbar\beta, 0). \quad (134)$$

Therefore we can write the quantum average of $\hat{A}_1 \dots \hat{A}_n$ with (6), (7) and (134) as

$$\langle \hat{A}_1 \dots \hat{A}_n \rangle = \frac{\text{Tr} \left[\hat{U}_D(\hbar\beta, 0) \hat{A}_1 \dots \hat{A}_n \exp(-\beta \hat{G}_0) \right]}{\text{Tr} \left[\exp(-\beta \hat{G}_0) \hat{U}_D(\hbar\beta, 0) \right]}. \quad (135)$$

Additionally we obtain from the relation of an operator in the Dirac and the Schrödinger picture (122) and (127)

$$\lim_{\tau \rightarrow \tau_0} \hat{A}_D(\tau) = \hat{A}. \quad (136)$$

Therefore we can rewrite the trace over n operators $\hat{A}_1 \dots \hat{A}_n$ as

$$\text{Tr} \left(\hat{A}_1 \dots \hat{A}_n \right) = \lim_{\tau_n \uparrow \tau_{n-1}} \dots \lim_{\tau_1 \uparrow 0} \text{Tr} \left\{ \hat{T}_\tau \left[\hat{A}_{1D}(\tau_1) \dots \hat{A}_{nD}(\tau_n) \right] \right\}, \quad (137)$$

where the limits $\lim_{\tau_n \uparrow \tau_{n-1}} \dots \lim_{\tau_1 \uparrow 0}$ have to be evaluated from the left to the right and $\lim_{\tau_n \uparrow \tau_{n-1}}$ denotes $\lim_{\substack{\tau_n \rightarrow \tau_{n-1} \\ \tau_n < \tau_{n-1}}}$.

Note that due to the introduced imaginary times τ_1, \dots, τ_n with $\tau_i \neq \tau_j$ for $i \neq j$, the commutation relations of the operators $\hat{A}_1, \dots, \hat{A}_n$ are replaced by their time order since due to the definition of the time ordering operator \hat{T}_τ in (132) we obtain

$$\hat{T}_\tau \left(\hat{A}_1(\tau_1) \hat{A}_2(\tau_2) \right) = \hat{T}_\tau \left(\hat{A}_2(\tau_2) \hat{A}_1(\tau_1) \right). \quad (138)$$

Therefore we have to be careful when introducing τ_1, \dots, τ_n to keep the natural ordering of the operators which means

$$\text{Tr} \left(\hat{A}_1 \hat{A}_2 \right) = \lim_{\tau_2 \uparrow \tau_1} \lim_{\tau_1 \uparrow 0} \text{Tr} \left\{ \hat{T}_\tau \left[\hat{A}_1(\tau_1) \hat{A}_2(\tau_2) \right] \right\}. \quad (139)$$

Using instead the limits $\lim_{\tau_1 \uparrow \tau_2} \lim_{\tau_2 \uparrow 0}$ would reverse the order of the operators and thus lead to a different expression.

With the consideration in (137) we can rewrite (135) as

$$\begin{aligned} \langle \hat{A}_1 \dots \hat{A}_n \rangle &= \lim_{\tau_n \uparrow \tau_{n-1}} \dots \lim_{\tau_1 \uparrow 0} \text{Tr} \left(\hat{T}_\tau \left\{ \left[I - \frac{1}{\hbar} \int_0^{\hbar\beta} d\tau \hat{V}_D(\tau) + \frac{1}{2\hbar^2} \int_0^{\hbar\beta} d\tau \int_0^{\hbar\beta} d\tau' \hat{V}_D(\tau) \hat{V}_D(\tau') + \dots \right] \right. \right. \\ &\quad \times \left. \left. \hat{A}_{1D}(\tau_1) \dots \hat{A}_{nD}(\tau_n) \right\} \exp(-\beta \hat{G}_0) \right) \left(\text{Tr} \left\{ \exp(-\beta \hat{G}_0) \hat{T}_\tau \left[I - \frac{1}{\hbar} \int_0^{\hbar\beta} d\tau \hat{V}_D(\tau) \right. \right. \right. \\ &\quad \left. \left. \left. + \frac{1}{2\hbar^2} \int_0^{\hbar\beta} d\tau \int_0^{\hbar\beta} d\tau' \hat{V}_D(\tau) \hat{V}_D(\tau') + \dots \right] \right\} \right)^{-1}, \end{aligned} \quad (140)$$

where we inserted the explicit form of the time evolution operator U_D in (133). Note that in (137) we also were allowed to use the general limit $\tau_1 \rightarrow 0$ but in (140) we have to set $\tau_1 < 0$ in order to keep the natural order of the operators because τ and τ' run from 0 to $\hbar\beta > 0$.

With the definition

$$\left\langle \hat{A}_{1D}(\tau_1) \dots \hat{A}_{nD} \right\rangle_0^\tau = \frac{\text{Tr} \left\{ \hat{T}_\tau \left[\hat{A}_{1D}(\tau_1) \dots \hat{A}_{nD}(\tau_n) \right] \exp(-\beta \hat{G}_0) \right\}}{\text{Tr}[\exp(-\beta \hat{G}_0)]}, \quad (141)$$

where τ denotes that the average is time-ordered by \hat{T}_τ and 0 denotes that the average is taken with respect to \hat{G}_0 , we can write the expansion of (140) in \hat{V}_D as

$$\begin{aligned} \left\langle \hat{A}_1 \dots \hat{A}_n \right\rangle &= \lim_{\tau_n \uparrow \tau_{n-1}} \dots \lim_{\tau_1 \uparrow 0} \left\{ \left\langle \hat{A}_{1D}(\tau_1) \dots \hat{A}_{nD}(\tau_n) \right\rangle_0^\tau + \int_0^{\hbar\beta} \frac{d\tau}{\hbar} \left[- \left\langle \hat{V}_D(\tau) \hat{A}_{1D}(\tau_1) \dots \hat{A}_{nD}(\tau_n) \right\rangle_0^\tau \right. \right. \\ &\quad \left. \left. + \left\langle \hat{V}_D(\tau) \right\rangle_0^\tau \left\langle \hat{A}_{1D}(\tau_1) \dots \hat{A}_{nD}(\tau_n) \right\rangle_0^\tau \right] \\ &\quad + \int_0^{\hbar\beta} \frac{d\tau}{\hbar} \int_0^{\hbar\beta} \frac{d\tau'}{\hbar} \left[\frac{1}{2} \left\langle \hat{V}_D(\tau) \hat{V}_D(\tau') \hat{A}_{1D}(\tau_1) \dots \hat{A}_{nD}(\tau_n) \right\rangle_0^\tau \right. \\ &\quad - \frac{1}{2} \left\langle \hat{V}_D(\tau) \hat{V}_D(\tau') \right\rangle_0^\tau \left\langle \hat{A}_{1D}(\tau_1) \dots \hat{A}_{nD}(\tau_n) \right\rangle_0^\tau \\ &\quad - \left\langle \hat{V}_D(\tau) \hat{A}_{1D}(\tau_1) \dots \hat{A}_{nD}(\tau_n) \right\rangle_0^\tau \left\langle \hat{V}_D(\tau') \right\rangle_0^\tau \\ &\quad \left. \left. + \left\langle \hat{A}_{1D}(\tau_1) \dots \hat{A}_{nD}(\tau_n) \right\rangle_0^\tau \left\langle \hat{V}_D(\tau) \right\rangle_0^\tau \left\langle \hat{V}_D(\tau') \right\rangle_0^\tau \right] \right\}. \quad (142) \end{aligned}$$

In order to calculate expressions like $\left\langle \hat{A}_1(\tau) \dots \hat{A}_n(\tau) \right\rangle_0^\tau$, where n is an even number, we will use a generalized form of Wick's theorem [27] which can be written recursively

$$\begin{aligned} \left\langle \hat{A}_{1D}(\tau_1) \dots \hat{A}_{nD}(\tau_n) \right\rangle_0^\tau &= \sum_{i=2}^n \left\langle \hat{A}_{1D}(\tau_1) \hat{A}_{iD}(\tau_i) \right\rangle_0^\tau \left\langle \hat{A}_{2D}(\tau_2) \dots \hat{A}_{(i-1)D}(\tau_{i-1}) \hat{A}_{(i+1)D}(\tau_{i+1}) \dots \right. \\ &\quad \left. \times \hat{A}_{nD}(\tau_n) \right\rangle_0^\tau. \quad (143) \end{aligned}$$

4.3 Disorder Expansion of Correlations

In order to determine the condensate density n_0 as a function of the particle density n by (44), which involves the relations given in (27)–(31), we have to calculate the correlations of the fluctuation operators $\left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_0$, $\left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_U$, $\left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_{U^2}$, $\left\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_0$ and $\left\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_U$.

Decomposing the grand-canonical Hamiltonian (22) into

$$\hat{C} = \hat{C}_{\text{GP}} + \hat{C}_{\text{Bog}}, \quad (144)$$

where \hat{C}_{GP} denotes the Gross-Pitaevskii terms of \hat{C} that do not depend on the fluctuation operators $\delta\hat{\psi}$ or $\delta\hat{\psi}^\dagger$ and \hat{C}_{Bog} is given by (105), the quantum average (7) simplifies to

$$\langle \cdot \rangle = \frac{\text{Tr} \left[\cdot \exp \left(-\beta \hat{C}_{\text{Bog}} \right) \right]}{\text{Tr} \left[\exp \left(-\beta \hat{C}_{\text{Bog}} \right) \right]}, \quad (145)$$

as the factor involving \hat{C}_{GP} can be taken out of the trace and cancels between denominator and numerator. Thus we can identify with (105) referring to the notation of Section 4.1

$$\hat{G} \longleftrightarrow \hat{C}_{\text{Bog}} \quad (146)$$

$$\hat{G}_0 \longleftrightarrow \sum_{\mathbf{k}} \left(\lambda_{\mathbf{k}} \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} - \frac{\varepsilon_{\mathbf{k}} - \lambda_{\mathbf{k}}}{2} \right) \quad (147)$$

$$\begin{aligned} \hat{V} \longleftrightarrow \sum_{\mathbf{k}, \mathbf{k}'} \left[\left({}^1 A_{\mathbf{k}, \mathbf{k}'}^U + {}^1 A_{\mathbf{k}, \mathbf{k}'}^{U^2} \right) \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}'} + \left({}^2 A_{\mathbf{k}, \mathbf{k}'}^U + {}^2 A_{\mathbf{k}, \mathbf{k}'}^{U^2} \right) \hat{b}_{\mathbf{k}} \hat{b}_{\mathbf{k}'}^\dagger + \left({}^3 A_{\mathbf{k}, \mathbf{k}'}^U + {}^3 A_{\mathbf{k}, \mathbf{k}'}^{U^2} \right) \hat{b}_{\mathbf{k}} \hat{b}_{\mathbf{k}'} \right. \\ \left. + \left({}^4 A_{\mathbf{k}, \mathbf{k}'}^U + {}^4 A_{\mathbf{k}, \mathbf{k}'}^{U^2} \right) \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}'}^\dagger \right]. \quad (148) \end{aligned}$$

Note that V is of first and second order in U and the non- \hat{b} or \hat{b}^\dagger -dependent term of \hat{G}_0 cancels between denominator and numerator in case of calculating averages.

Using the \mathbf{k} -representation of $\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x})$ and $\delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x})$ as defined in (79)–(82), inserting the Bogoliubov transformation in (103) and (104) and using the symmetry relations given in (98) and (99) we obtain explicitly

$$\begin{aligned} \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) &= \frac{1}{V} \sum_{\mathbf{k}_1\mathbf{k}_2} \exp[i\mathbf{x}(\mathbf{k}_1 + \mathbf{k}_2)] \left(u_{\mathbf{k}_1} u_{\mathbf{k}_2} \hat{b}_{-\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2} + v_{\mathbf{k}_1} v_{\mathbf{k}_2} \hat{b}_{\mathbf{k}_1} \hat{b}_{-\mathbf{k}_2}^\dagger + u_{\mathbf{k}_2} v_{\mathbf{k}_1} \hat{b}_{\mathbf{k}_1} \hat{b}_{\mathbf{k}_2} \right. \\ &\quad \left. + u_{\mathbf{k}_1} v_{\mathbf{k}_2} \hat{b}_{-\mathbf{k}_1}^\dagger \hat{b}_{-\mathbf{k}_2}^\dagger \right) \end{aligned} \quad (149)$$

$$\begin{aligned} \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) &= \frac{1}{V} \sum_{\mathbf{k}_1\mathbf{k}_2} \exp[i\mathbf{x}(\mathbf{k}_1 + \mathbf{k}_2)] \left(u_{\mathbf{k}_2} v_{\mathbf{k}_1} \hat{b}_{-\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2} + u_{\mathbf{k}_1} v_{\mathbf{k}_2} \hat{b}_{\mathbf{k}_1} \hat{b}_{-\mathbf{k}_2}^\dagger + u_{\mathbf{k}_1} u_{\mathbf{k}_2} \hat{b}_{\mathbf{k}_1} \hat{b}_{\mathbf{k}_2} \right. \\ &\quad \left. + v_{\mathbf{k}_1} v_{\mathbf{k}_2} \hat{b}_{-\mathbf{k}_1}^\dagger \hat{b}_{-\mathbf{k}_2}^\dagger \right) \end{aligned} \quad (150)$$

and therefore have to calculate the different orders in the random potential U of the correlations of all combinations of \hat{b} and \hat{b}^\dagger .

Due to Wick's theorem (143) we will use in the following several times

$$\left\langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \right\rangle_0^\tau = \exp\left(\frac{\tau_1 - \tau_2}{\hbar} \lambda_{\mathbf{k}_1}\right) \delta_{\mathbf{k}_1, \mathbf{k}_2} \times \begin{cases} \langle \hat{n}_{\mathbf{k}_1} \rangle_0 & \text{if } \tau_1 > \tau_2, \\ \langle \langle \hat{n}_{\mathbf{k}_1} \rangle_0 + 1 \rangle & \text{if } \tau_2 > \tau_1, \end{cases} \quad (151)$$

$$\left\langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\mathbf{k}_2\text{D}}^\dagger(\tau_2) \right\rangle_0^\tau = 0, \quad (152)$$

$$\left\langle \hat{b}_{\mathbf{k}_1\text{D}}(\tau_1) \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \right\rangle_0^\tau = 0, \quad (153)$$

where we evaluated all traces in the occupation number representation of the quasi-particle operators \hat{b} and \hat{b}^\dagger and $\langle \hat{n}_{\mathbf{k}} \rangle_0$ denotes the thermal occupation of the state \mathbf{k} in the absence of the random potential with

$$\langle \hat{n}_{\mathbf{k}} \rangle_0 = \frac{-\frac{\partial}{\partial \beta \lambda_{\mathbf{k}}} \sum_{\{n_{\mathbf{k}'}\}} \exp\left(-\beta \sum_{\mathbf{k}'} \lambda_{\mathbf{k}'} n_{\mathbf{k}'}\right)}{\sum_{\{n_{\mathbf{k}'}\}} \exp\left(-\beta \sum_{\mathbf{k}'} \lambda_{\mathbf{k}'} n_{\mathbf{k}'}\right)}. \quad (154)$$

Evaluating sums over all possible state configurations $\{n_{\mathbf{k}'}\}$ by canceling all not \mathbf{k} dependent terms between denominator and numerator as well as identifying the remaining sums as geometric sequences yields the Bose-Einstein distribution

$$\langle \hat{n}_{\mathbf{k}} \rangle_0 = \frac{1}{\exp(\beta \lambda_{\mathbf{k}}) - 1}. \quad (155)$$

4.3.1 Zeroth Order in Disorder Potentials

From the zeroth order term in (142) and the interim results (151)–(153) we immediately obtain

$$\left\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2} \right\rangle_0 = \delta_{\mathbf{k}_1, \mathbf{k}_2} \langle \hat{n}_{\mathbf{k}_1} \rangle_0, \quad (156)$$

$$\left\langle \hat{b}_{\mathbf{k}_1} \hat{b}_{\mathbf{k}_2}^\dagger \right\rangle_0 = \delta_{\mathbf{k}_1, \mathbf{k}_2} (\langle \hat{n}_{\mathbf{k}_1} \rangle_0 + 1), \quad (157)$$

$$\left\langle \hat{b}_{\mathbf{k}_1} \hat{b}_{\mathbf{k}_2} \right\rangle_0 = 0, \quad (158)$$

$$\left\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2}^\dagger \right\rangle_0 = 0. \quad (159)$$

Taking the quantum average of (149) and (150) and inserting (156)–(159) leads to

$$\left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_0 = \frac{1}{V} \sum_{\mathbf{k}} [u_{\mathbf{k}} u_{\mathbf{k}} \langle \hat{n}_{\mathbf{k}} \rangle_0 + v_{\mathbf{k}} v_{\mathbf{k}} (\langle \hat{n}_{\mathbf{k}} \rangle_0 + 1)], \quad (160)$$

$$\left\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_0 = \frac{1}{V} \sum_{\mathbf{k}} [u_{\mathbf{k}} v_{\mathbf{k}} \langle \hat{n}_{\mathbf{k}} \rangle_0 + u_{\mathbf{k}} v_{\mathbf{k}} (\langle \hat{n}_{\mathbf{k}} \rangle_0 + 1)]. \quad (161)$$

With (87) and the result for $\langle \hat{n}_{\mathbf{k}} \rangle_0$ in (155) the limit $T \rightarrow 0$ or $\beta \rightarrow \infty$ yields

$$\left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_{0, T=0} = \frac{1}{V} \sum_{\mathbf{k}} v_{\mathbf{k}} v_{\mathbf{k}}, \quad (162)$$

$$\left\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_{0, T=0} = \frac{1}{V} \sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}}. \quad (163)$$

4.3.2 First Order in Disorder Potentials

Due to (152) and (153), only averages of products of \hat{b} and \hat{b}^\dagger with an equal number of creation and annihilation operators do not vanish. If we would apply Wick's theorem (143) to a product with an unequal number of creation and annihilation operators, we will end up with a term proportional to the correlation of two creation or annihilation operators which vanishes due to (152) and (153). Therefore we have to take into account only those terms of V in (148) which give in total averages of products of as many creation as annihilation operators.

In the following of this Subsection we will use several times (152) and (153) without further mentioning. Firstly we calculate $\langle \hat{b}^\dagger \hat{b} \rangle_U$ with the first order term of (142) and (148) in the disorder potential U

$$\begin{aligned} \langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2} \rangle_U &= - \lim_{\tau_2 \uparrow \tau_1} \lim_{\tau_1 \uparrow 0} \left(\int_0^{\hbar\beta} \frac{d\tau}{\hbar} \lim_{\tau' \uparrow \tau} \sum_{\mathbf{k}, \mathbf{k}'} \left\{ \left\langle \left[{}^1 A_{\mathbf{k}, \mathbf{k}'}^U \hat{b}_{\mathbf{k}_D}^\dagger(\tau) \hat{b}_{\mathbf{k}'_D}(\tau') + {}^2 A_{\mathbf{k}, \mathbf{k}'}^U \hat{b}_{\mathbf{k}_D}(\tau) \hat{b}_{\mathbf{k}'_D}^\dagger(\tau') \right] \right. \right. \\ &\quad \times \hat{b}_{\mathbf{k}_1 D}^\dagger(\tau_1) \hat{b}_{\mathbf{k}_2 D}(\tau_2) \Big\rangle_0^\tau - \left\langle {}^1 A_{\mathbf{k}, \mathbf{k}'}^U \hat{b}_{\mathbf{k}_D}^\dagger(\tau) \hat{b}_{\mathbf{k}'_D}(\tau') + {}^2 A_{\mathbf{k}, \mathbf{k}'}^U \hat{b}_{\mathbf{k}_D}(\tau) \hat{b}_{\mathbf{k}'_D}^\dagger(\tau') \right\rangle_0^\tau \\ &\quad \left. \left. \times \left\langle \hat{b}_{\mathbf{k}_1 D}^\dagger(\tau_1) \hat{b}_{\mathbf{k}_2 D}(\tau_2) \right\rangle_0^\tau \right\} \right). \end{aligned} \quad (164)$$

Here we have introduced τ' in order to be able to evaluate the time-ordered average because a time ordering of two operators, which are taken at the same time τ , is not possible. The relation of τ and τ' , $\tau > \tau'$, follows in the same manner as argued before in (140). By applying Wick's theorem (143) to (164) we obtain

$$\begin{aligned} \langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2} \rangle_U &= - \lim_{\tau_2 \uparrow \tau_1} \lim_{\tau_1 \uparrow 0} \left\{ \int_0^{\hbar\beta} \frac{d\tau}{\hbar} \lim_{\tau' \uparrow \tau} \sum_{\mathbf{k}, \mathbf{k}'} \left[{}^1 A_{\mathbf{k}, \mathbf{k}'}^U \left\langle \hat{b}_{\mathbf{k}_D}^\dagger(\tau) \hat{b}_{\mathbf{k}_2 D}(\tau_2) \right\rangle_0^\tau \left\langle \hat{b}_{\mathbf{k}'_D}(\tau') \hat{b}_{\mathbf{k}_1 D}^\dagger(\tau_1) \right\rangle_0^\tau \right. \right. \\ &\quad \left. \left. + {}^2 A_{\mathbf{k}, \mathbf{k}'}^U \left\langle \hat{b}_{\mathbf{k}_D}(\tau) \hat{b}_{\mathbf{k}_1 D}^\dagger(\tau_1) \right\rangle_0^\tau \left\langle \hat{b}_{\mathbf{k}'_D}^\dagger(\tau') \hat{b}_{\mathbf{k}_2 D}(\tau_2) \right\rangle_0^\tau \right] \right\}. \end{aligned} \quad (165)$$

The limit $\tau' \uparrow \tau$ can now be evaluated. Note that due to carrying out Wick's theorem, all not connected terms referring to the imaginary time canceled out, which means only terms of the structure $\langle \hat{A}_1(\tau_1) \hat{A}_2(\tau) \rangle \langle \hat{A}_3(\tau) \hat{A}_4(\tau_2) \rangle$ are left.

Inserting (151) and performing the τ -integral and evaluating the limits of τ_2 and τ_1 and the sum over \mathbf{k} and \mathbf{k}' leads to

$$\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2} \rangle_U = - ({}^1 A_{\mathbf{k}_2, \mathbf{k}_1}^U + {}^2 A_{\mathbf{k}_1, \mathbf{k}_2}^U) \langle \hat{n}_{\mathbf{k}_2} \rangle_0 (\langle \hat{n}_{\mathbf{k}_1} \rangle_0 + 1) \times \begin{cases} \frac{\exp[\beta(\lambda_{\mathbf{k}_2} - \lambda_{\mathbf{k}_1})] - 1}{\lambda_{\mathbf{k}_2} - \lambda_{\mathbf{k}_1}} & \text{if } \lambda_{\mathbf{k}_1} \neq \lambda_{\mathbf{k}_2}, \\ \beta & \text{if } \lambda_{\mathbf{k}_1} = \lambda_{\mathbf{k}_2}. \end{cases} \quad (166)$$

In the same manner we calculate $\langle \hat{b}_{\mathbf{k}_1} \hat{b}_{\mathbf{k}_2}^\dagger \rangle_U$, $\langle \hat{b}_{\mathbf{k}_1} \hat{b}_{\mathbf{k}_2} \rangle_U$ and $\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2}^\dagger \rangle_U$ and obtain

$$\langle \hat{b}_{\mathbf{k}_1} \hat{b}_{\mathbf{k}_2}^\dagger \rangle_U = - ({}^1 A_{\mathbf{k}_1, \mathbf{k}_2}^U + {}^2 A_{\mathbf{k}_2, \mathbf{k}_1}^U) \langle \hat{n}_{\mathbf{k}_1} \rangle_0 (\langle \hat{n}_{\mathbf{k}_2} \rangle_0 + 1) \times \begin{cases} \frac{\exp[\beta(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2}} & \text{if } \lambda_{\mathbf{k}_1} \neq \lambda_{\mathbf{k}_2}, \\ \beta & \text{if } \lambda_{\mathbf{k}_1} = \lambda_{\mathbf{k}_2}, \end{cases} \quad (167)$$

$$\langle \hat{b}_{\mathbf{k}_1} \hat{b}_{\mathbf{k}_2} \rangle_U = - ({}^4 A_{\mathbf{k}_1, \mathbf{k}_2}^U + {}^4 A_{\mathbf{k}_2, \mathbf{k}_1}^U) \langle \hat{n}_{\mathbf{k}_1} \rangle_0 \langle \hat{n}_{\mathbf{k}_2} \rangle_0 \frac{\exp[\beta(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2}}, \quad (168)$$

$$\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2}^\dagger \rangle_U = - ({}^3 A_{\mathbf{k}_1, \mathbf{k}_2}^U + {}^3 A_{\mathbf{k}_2, \mathbf{k}_1}^U) (\langle \hat{n}_{\mathbf{k}_1} \rangle_0 + 1) (\langle \hat{n}_{\mathbf{k}_2} \rangle_0 + 1) \frac{\exp[\beta(-\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{-\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2}}. \quad (169)$$

Note that inserting the explicit expression for $\langle \hat{n}_{\mathbf{k}} \rangle_0$ in (155) leads to

$$(\langle \hat{n}_{\mathbf{k}_1} \rangle_0 + 1) \langle \hat{n}_{\mathbf{k}_2} \rangle_0 \frac{\exp[\beta(-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2}} = \langle \hat{n}_{\mathbf{k}_1} \rangle_0 (\langle \hat{n}_{\mathbf{k}_2} \rangle_0 + 1) \frac{\exp[\beta(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2}}, \quad (170)$$

$$\langle \hat{n}_{\mathbf{k}_1} \rangle_0 \langle \hat{n}_{\mathbf{k}_2} \rangle_0 \frac{\exp[\beta(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2}} = (\langle \hat{n}_{\mathbf{k}_1} \rangle_0 + 1) (\langle \hat{n}_{\mathbf{k}_2} \rangle_0 + 1) \frac{\exp[\beta(-\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{-\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2}}. \quad (171)$$

The results for $\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2} \rangle_U$ and $\langle \hat{b}_{\mathbf{k}_1} \hat{b}_{\mathbf{k}_2}^\dagger \rangle_U$ correspond to each other if we interchange \mathbf{k}_1 and \mathbf{k}_2 in one of these two expressions, for all \mathbf{k}_1 and \mathbf{k}_2 . This can be seen as a consequence of the fact that the commutation relation $[\hat{b}_{\mathbf{k}_1}, \hat{b}_{\mathbf{k}_2}^\dagger] = \delta_{\mathbf{k}_1, \mathbf{k}_2}$ is already included in the zeroth order result (156) and (157)

and therefore does not affect on the first order results. By using (114), (115) and (170) we obtain in addition the equality of $\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2} \rangle_U$ and $\langle \hat{b}_{-\mathbf{k}_2}^\dagger \hat{b}_{-\mathbf{k}_1} \rangle_U$ as $\lambda_{-\mathbf{k}} = \lambda_{\mathbf{k}}$. Therefore we obtain in total the symmetry relations

$$\langle \hat{b}_{\mathbf{k}_2} \hat{b}_{\mathbf{k}_1}^\dagger \rangle_U = \langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2} \rangle_U = \langle \hat{b}_{-\mathbf{k}_2}^\dagger \hat{b}_{-\mathbf{k}_1} \rangle_U \quad (172)$$

even for the case $\mathbf{k}_1 = \mathbf{k}_2$.

The correlations $\langle \hat{b}_{\mathbf{k}_1} \hat{b}_{\mathbf{k}_2} \rangle_U$ and $\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2}^\dagger \rangle_U$ are the hermitian conjugate of each other which can explicitly be seen by using (116), (118) and (171) in (168). Moreover we obtain by using (116) and (171) in (169) the equality of $\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2}^\dagger \rangle_U$ and $\langle \hat{b}_{-\mathbf{k}_1} \hat{b}_{-\mathbf{k}_2} \rangle_U$ which gives in total the symmetry relation

$$\langle \hat{b}_{\mathbf{k}_1} \hat{b}_{\mathbf{k}_2} \rangle_U^\dagger = \langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2}^\dagger \rangle_U = \langle \hat{b}_{-\mathbf{k}_1} \hat{b}_{-\mathbf{k}_2} \rangle_U. \quad (173)$$

We can insert the results (166)–(169) in the averages of (149) and (150) and use, in order to avoid the distinction of cases in (166) and (167), the limit

$$\lim_{\alpha \rightarrow 0} \frac{\exp(\beta\alpha) - 1}{\alpha} = \beta, \quad (174)$$

which can be easily seen by using the Taylor expansion of the exponential, and the relations (172) and (173) to end up with

$$\begin{aligned} \langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \rangle_U &= -\frac{1}{V} \sum_{\mathbf{k}_1 \mathbf{k}_2} \exp[i\mathbf{x}(\mathbf{k}_1 + \mathbf{k}_2)] \left\{ (u_{\mathbf{k}_1} v_{\mathbf{k}_2} + u_{\mathbf{k}_2} v_{\mathbf{k}_1}) ({}^4 A_{\mathbf{k}_1, \mathbf{k}_2}^U + {}^4 A_{\mathbf{k}_2, \mathbf{k}_1}^U) \langle \hat{n}_{\mathbf{k}_1} \rangle_0 \right. \\ &\quad \times \langle \hat{n}_{\mathbf{k}_2} \rangle_0 \frac{\exp[\beta(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2}} + (u_{\mathbf{k}_1} u_{\mathbf{k}_2} + v_{\mathbf{k}_1} v_{\mathbf{k}_2}) ({}^1 A_{\mathbf{k}_1, -\mathbf{k}_2}^U + {}^2 A_{-\mathbf{k}_2, \mathbf{k}_1}^U) \\ &\quad \left. \times \langle \hat{n}_{\mathbf{k}_1} \rangle_0 (\langle \hat{n}_{\mathbf{k}_2} \rangle_0 + 1) \frac{\exp[\beta(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2}} \right\}, \quad (175) \end{aligned}$$

$$\begin{aligned} \langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}^\dagger(\mathbf{x}) \rangle_U &= -\frac{1}{V} \sum_{\mathbf{k}_1 \mathbf{k}_2} \exp[i\mathbf{x}(\mathbf{k}_1 + \mathbf{k}_2)] \left\{ (v_{\mathbf{k}_1} v_{\mathbf{k}_2} + u_{\mathbf{k}_1} u_{\mathbf{k}_2}) ({}^4 A_{\mathbf{k}_1, \mathbf{k}_2}^U + {}^4 A_{\mathbf{k}_2, \mathbf{k}_1}^U) \langle \hat{n}_{\mathbf{k}_1} \rangle_0 \right. \\ &\quad \times \langle \hat{n}_{\mathbf{k}_2} \rangle_0 \frac{\exp[\beta(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2}} + (u_{\mathbf{k}_1} v_{\mathbf{k}_2} + u_{\mathbf{k}_2} v_{\mathbf{k}_1}) ({}^1 A_{\mathbf{k}_1, -\mathbf{k}_2}^U + {}^2 A_{-\mathbf{k}_2, \mathbf{k}_1}^U) \\ &\quad \left. \times \langle \hat{n}_{\mathbf{k}_1} \rangle_0 (\langle \hat{n}_{\mathbf{k}_2} \rangle_0 + 1) \frac{\exp[\beta(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2}} \right\}. \quad (176) \end{aligned}$$

Taking into account the limit

$$\lim_{\beta \rightarrow \infty} \langle \hat{n}_{\mathbf{k}} \rangle_0 \exp(\beta\lambda_{\mathbf{k}}) = \lim_{\beta \rightarrow \infty} \frac{1}{1 - \exp(-\beta\lambda_{\mathbf{k}})} = 1, \quad (177)$$

where we used (87) and (155), leads to the zero-temperature results

$$\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \rangle_{U, T=0} = -\frac{1}{V} \sum_{\mathbf{k}_1 \mathbf{k}_2} \exp[i\mathbf{x}(\mathbf{k}_1 + \mathbf{k}_2)] (u_{\mathbf{k}_1} v_{\mathbf{k}_2} + u_{\mathbf{k}_2} v_{\mathbf{k}_1}) \frac{{}^4 A_{\mathbf{k}_1, \mathbf{k}_2}^U + {}^4 A_{\mathbf{k}_2, \mathbf{k}_1}^U}{\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2}}, \quad (178)$$

$$\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}^\dagger(\mathbf{x}) \rangle_{U, T=0} = -\frac{1}{V} \sum_{\mathbf{k}_1 \mathbf{k}_2} \exp[i\mathbf{x}(\mathbf{k}_1 + \mathbf{k}_2)] (u_{\mathbf{k}_1} u_{\mathbf{k}_2} + v_{\mathbf{k}_1} v_{\mathbf{k}_2}) \frac{{}^4 A_{\mathbf{k}_1, \mathbf{k}_2}^U + {}^4 A_{\mathbf{k}_2, \mathbf{k}_1}^U}{\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2}}. \quad (179)$$

4.3.3 Second Order in Disorder Potentials

The second order terms of the correlations are composed of two different contributions. The first contribution are the second order terms of (142) in V with the terms of V in (148) which are linear in the random potential U . The second contribution results from the first order terms of (142) in V with the terms of V in (148) which are quadratic in the disorder potential U . The latter contribution has the same structure as the results given in (175) and (176) except of the replacement

$${}^i A_{\mathbf{k}, \mathbf{k}'}^U \rightarrow {}^i A_{\mathbf{k}, \mathbf{k}'}^{U^2}. \quad (180)$$

With the same argumentation as in Subsection 4.3.2, all averages of products of creation and annihilation operators with an unequal number of creation and annihilation operators will vanish. Additionally we will again use in the following (152) and (153) several times without further mentioning in this Subsection.

It is left to calculate the first contribution terms $\langle \cdot \rangle_{U^2}^1$ with (142) and (148). We will begin with computing $\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2} \rangle_{U^2}^1$ where we directly apply Wick's theorem (143) which again cancels out all not connected terms as in (165)

$$\begin{aligned}
\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2} \rangle_{U^2}^1 &= \lim_{\tau_2 \uparrow \tau_1} \lim_{\tau_1 \uparrow 0} \left(\int_0^{\hbar\beta} \frac{d\tau}{\hbar} \int_0^{\hbar\beta} \frac{d\tilde{\tau}}{\hbar} \lim_{\tau' \uparrow \tau} \lim_{\tilde{\tau}' \uparrow \tilde{\tau}} \sum_{\mathbf{k}, \mathbf{k}', \tilde{\mathbf{k}}, \tilde{\mathbf{k}'}} \frac{1}{2} \left\{ {}^1 A_{\mathbf{k}, \mathbf{k}'}^U {}^1 A_{\tilde{\mathbf{k}}, \tilde{\mathbf{k}'}}^U \left[\langle \hat{b}_{\mathbf{kD}}^\dagger(\tau) \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \right. \right. \\
&\times \langle \hat{b}_{\mathbf{kD}}^\dagger(\tilde{\tau}) \hat{b}_{\mathbf{k}'\text{D}}(\tau') \rangle_0^\tau \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')} \rangle_0^\tau + \langle \hat{b}_{\mathbf{kD}}^\dagger(\tau) \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')} \rangle_0^\tau \langle \hat{b}_{\mathbf{kD}}^\dagger(\tilde{\tau}) \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \\
&\times \left. \left. \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\mathbf{k}'\text{D}}(\tau') \rangle_0^\tau \right] \right. \\
&+ {}^1 A_{\mathbf{k}, \mathbf{k}'}^U {}^2 A_{\tilde{\mathbf{k}}, \tilde{\mathbf{k}'}}^U \left[\langle \hat{b}_{\mathbf{kD}}^\dagger(\tau) \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \langle \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')}^\dagger \hat{b}_{\mathbf{k}'\text{D}}(\tau') \rangle_0^\tau \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\tilde{\mathbf{kD}}(\tilde{\tau})} \rangle_0^\tau \right. \\
&+ \langle \hat{b}_{\mathbf{kD}}^\dagger(\tau) \hat{b}_{\tilde{\mathbf{kD}}(\tilde{\tau})} \rangle_0^\tau \langle \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')}^\dagger \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\mathbf{k}'\text{D}}(\tau') \rangle_0^\tau \left. \right] \\
&+ {}^2 A_{\mathbf{k}, \mathbf{k}'}^U {}^1 A_{\tilde{\mathbf{k}}, \tilde{\mathbf{k}'}}^U \left[\langle \hat{b}_{\mathbf{kD}}^\dagger(\tilde{\tau}) \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \langle \hat{b}_{\mathbf{k}'\text{D}}^\dagger(\tau') \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')} \rangle_0^\tau \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\mathbf{kD}}(\tau) \rangle_0^\tau \right. \\
&+ \langle \hat{b}_{\mathbf{kD}}^\dagger(\tilde{\tau}) \hat{b}_{\mathbf{kD}}(\tau) \rangle_0^\tau \langle \hat{b}_{\mathbf{k}'\text{D}}^\dagger(\tau') \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')} \rangle_0^\tau \left. \right] \\
&+ {}^2 A_{\mathbf{k}, \mathbf{k}'}^U {}^2 A_{\tilde{\mathbf{k}}, \tilde{\mathbf{k}'}}^U \left[\langle \hat{b}_{\mathbf{k}'\text{D}}^\dagger(\tau') \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \langle \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')}^\dagger \hat{b}_{\mathbf{kD}}(\tau) \rangle_0^\tau \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\mathbf{kD}}(\tau) \rangle_0^\tau \right. \\
&+ \langle \hat{b}_{\mathbf{k}'\text{D}}^\dagger(\tau') \hat{b}_{\tilde{\mathbf{kD}}(\tilde{\tau})} \rangle_0^\tau \langle \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')}^\dagger \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\mathbf{kD}}(\tau) \rangle_0^\tau \left. \right] \\
&+ {}^3 A_{\mathbf{k}, \mathbf{k}'}^U {}^4 A_{\tilde{\mathbf{k}}, \tilde{\mathbf{k}'}}^U \left[\langle \hat{b}_{\mathbf{kD}}^\dagger(\tilde{\tau}) \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \langle \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')}^\dagger \hat{b}_{\mathbf{kD}}(\tau) \rangle_0^\tau \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\mathbf{k}'\text{D}}(\tau') \rangle_0^\tau \right. \\
&+ \langle \hat{b}_{\mathbf{kD}}^\dagger(\tilde{\tau}) \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \langle \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')}^\dagger \hat{b}_{\mathbf{k}'\text{D}}(\tau') \rangle_0^\tau \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\mathbf{kD}}(\tau) \rangle_0^\tau + \langle \hat{b}_{\mathbf{kD}}^\dagger(\tilde{\tau}) \hat{b}_{\mathbf{kD}}(\tau) \rangle_0^\tau \\
&\times \langle \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')}^\dagger \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\mathbf{k}'\text{D}}(\tau') \rangle_0^\tau + \langle \hat{b}_{\mathbf{kD}}^\dagger(\tilde{\tau}) \hat{b}_{\mathbf{k}'\text{D}}(\tau') \rangle_0^\tau \\
&\times \left. \left. \langle \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')}^\dagger \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\mathbf{kD}}(\tau) \rangle_0^\tau \right] \right. \\
&+ {}^4 A_{\mathbf{k}, \mathbf{k}'}^U {}^3 A_{\tilde{\mathbf{k}}, \tilde{\mathbf{k}'}}^U \left[\langle \hat{b}_{\mathbf{kD}}^\dagger(\tau) \hat{b}_{\tilde{\mathbf{kD}}(\tilde{\tau})} \rangle_0^\tau \langle \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')}^\dagger \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')} \rangle_0^\tau \right. \\
&+ \langle \hat{b}_{\mathbf{kD}}^\dagger(\tau) \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')} \rangle_0^\tau \langle \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')}^\dagger \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\tilde{\mathbf{kD}}(\tilde{\tau})} \rangle_0^\tau + \langle \hat{b}_{\mathbf{kD}}^\dagger(\tau) \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \\
&\times \langle \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')}^\dagger \hat{b}_{\tilde{\mathbf{kD}}(\tilde{\tau})} \rangle_0^\tau \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')} \rangle_0^\tau + \langle \hat{b}_{\mathbf{kD}}^\dagger(\tau) \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \\
&\times \left. \left. \langle \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')}^\dagger \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')} \rangle_0^\tau \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\tilde{\mathbf{kD}}(\tilde{\tau})} \rangle_0^\tau \right] \right\}. \tag{181}
\end{aligned}$$

The limits $\tau' \uparrow \tau$ and $\tilde{\tau}' \uparrow \tilde{\tau}$ can now easily be evaluated. By relabeling the summation indices \mathbf{k} , \mathbf{k}' , $\tilde{\mathbf{k}}$ and $\tilde{\mathbf{k}'}$ and the integration variables τ and $\tilde{\tau}$ for every summand in a convenient way we obtain

$$\begin{aligned}
\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2} \rangle_{U^2}^1 &= \lim_{\tau_2 \uparrow \tau_1} \lim_{\tau_1 \uparrow 0} \left\{ \int_0^{\hbar\beta} \frac{d\tau}{\hbar} \int_0^{\hbar\beta} \frac{d\tilde{\tau}}{\hbar} \sum_{\mathbf{k}, \mathbf{k}', \tilde{\mathbf{k}}, \tilde{\mathbf{k}'}} \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\mathbf{k}'\text{D}}(\tau) \rangle_0^\tau \langle \hat{b}_{\tilde{\mathbf{k}'\text{D}}(\tilde{\tau}')}^\dagger \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \right. \\
&\times \left[\langle \hat{b}_{\mathbf{kD}}^\dagger(\tau) \hat{b}_{\tilde{\mathbf{kD}}(\tilde{\tau})} \rangle_0^\tau ({}^1 A_{\mathbf{k}, \mathbf{k}'}^U + {}^2 A_{\mathbf{k}', \mathbf{k}}^U) ({}^1 A_{\tilde{\mathbf{k}}, \tilde{\mathbf{k}'}}^U + {}^2 A_{\tilde{\mathbf{k}'}, \tilde{\mathbf{k}}}^U) + \langle \hat{b}_{\mathbf{kD}}^\dagger(\tilde{\tau}) \hat{b}_{\mathbf{kD}}(\tau) \rangle_0^\tau \right. \\
&\times \left. \left. ({}^3 A_{\mathbf{k}, \mathbf{k}'}^U + {}^3 A_{\mathbf{k}', \mathbf{k}}^U) ({}^4 A_{\tilde{\mathbf{k}}, \tilde{\mathbf{k}'}}^U + {}^4 A_{\tilde{\mathbf{k}'}, \tilde{\mathbf{k}}}^U) \right] \right\}. \tag{182}
\end{aligned}$$

In order to insert our results of (151) we need one of the conditions $\tau < \tilde{\tau}$ or $\tau > \tilde{\tau}$. Therefore we split the integral over $\tilde{\tau}$ into two parts

$$\int_0^{\hbar\beta} d\tilde{\tau} = \int_0^\tau d\tilde{\tau} + \int_\tau^{\hbar\beta} d\tilde{\tau}. \tag{183}$$

Realising that

$$\int_0^{\hbar\beta} d\tau \int_\tau^{\hbar\beta} d\tilde{\tau} = \int_0^{\hbar\beta} d\tilde{\tau} \int_0^{\tilde{\tau}} d\tau, \tag{184}$$

and relabeling the integration variable τ and $\tilde{\tau}$ in half of the terms leads to

$$\begin{aligned}
\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2} \rangle_{U^2}^1 &= \lim_{\tau_2 \uparrow \tau_1} \lim_{\tau_1 \uparrow 0} \left(\int_0^{\hbar\beta} \frac{d\tau}{\hbar} \int_0^\tau \frac{d\tilde{\tau}}{\hbar} \sum_{\mathbf{k}, \mathbf{k}', \tilde{\mathbf{k}}, \tilde{\mathbf{k}'}} \left\{ \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\mathbf{k}'\text{D}}(\tau) \rangle_0^\tau \langle \hat{b}_{\mathbf{k}'\text{D}}^\dagger(\tilde{\tau}) \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \right. \right. \\
&\quad \times \left[\langle \hat{b}_{\mathbf{k}\text{D}}^\dagger(\tau) \hat{b}_{\tilde{\mathbf{k}}\text{D}}(\tilde{\tau}) \rangle_0^\tau \left({}^1A_{\mathbf{k}, \mathbf{k}'}^U + {}^2A_{\mathbf{k}', \mathbf{k}}^U \right) \left({}^1A_{\tilde{\mathbf{k}}, \tilde{\mathbf{k}'}}^U + {}^2A_{\tilde{\mathbf{k}'}, \tilde{\mathbf{k}}}^U \right) + \langle \hat{b}_{\mathbf{k}\text{D}}^\dagger(\tilde{\tau}) \hat{b}_{\mathbf{k}\text{D}}(\tau) \rangle_0^\tau \right. \\
&\quad \times \left({}^3A_{\mathbf{k}, \mathbf{k}'}^U + {}^3A_{\mathbf{k}', \mathbf{k}}^U \right) \left({}^4A_{\tilde{\mathbf{k}}, \tilde{\mathbf{k}}}^U + {}^4A_{\tilde{\mathbf{k}'}, \tilde{\mathbf{k}'}}^U \right) \left. \right] + \langle \hat{b}_{\mathbf{k}_1\text{D}}^\dagger(\tau_1) \hat{b}_{\mathbf{k}'\text{D}}(\tilde{\tau}) \rangle_0^\tau \langle \hat{b}_{\mathbf{k}'\text{D}}^\dagger(\tau) \hat{b}_{\mathbf{k}_2\text{D}}(\tau_2) \rangle_0^\tau \\
&\quad \times \left[\langle \hat{b}_{\mathbf{k}\text{D}}^\dagger(\tilde{\tau}) \hat{b}_{\tilde{\mathbf{k}}\text{D}}(\tau) \rangle_0^\tau \left({}^1A_{\mathbf{k}, \mathbf{k}'}^U + {}^2A_{\mathbf{k}', \mathbf{k}}^U \right) \left({}^1A_{\tilde{\mathbf{k}}, \tilde{\mathbf{k}'}}^U + {}^2A_{\tilde{\mathbf{k}'}, \tilde{\mathbf{k}}}^U \right) + \langle \hat{b}_{\mathbf{k}\text{D}}^\dagger(\tau) \hat{b}_{\mathbf{k}\text{D}}(\tilde{\tau}) \rangle_0^\tau \right. \\
&\quad \left. \left. \times \left({}^3A_{\mathbf{k}, \mathbf{k}'}^U + {}^3A_{\mathbf{k}', \mathbf{k}}^U \right) \left({}^4A_{\tilde{\mathbf{k}}, \tilde{\mathbf{k}}}^U + {}^4A_{\tilde{\mathbf{k}'}, \tilde{\mathbf{k}'}}^U \right) \right] \right\}. \tag{185}
\end{aligned}$$

As in Section 4.3.2 we now insert (151), perform the limits $\tau_2 \uparrow \tau_1$ and $\tau_1 \uparrow 0$ and evaluate the sum for three of the four indices with the corresponding Kronecker delta from the correlations. The remaining integrals are of the form

$$I_{\alpha\tilde{\alpha}} = \int_0^\beta d\tau \int_0^\tau d\tilde{\tau} \exp(\tau\alpha) \exp(\tilde{\tau}\tilde{\alpha}), \tag{186}$$

where we have to evaluate $I_{\alpha\tilde{\alpha}}$ for five different case:

$$\text{case 1: } \alpha \neq 0, \tilde{\alpha} \neq 0, \alpha + \tilde{\alpha} \neq 0, \tag{187}$$

$$\text{case 2: } \alpha = 0, \tilde{\alpha} \neq 0, \alpha + \tilde{\alpha} \neq 0, \tag{188}$$

$$\text{case 3: } \alpha \neq 0, \tilde{\alpha} = 0, \alpha + \tilde{\alpha} \neq 0, \tag{189}$$

$$\text{case 4: } \alpha \neq 0, \tilde{\alpha} \neq 0, \alpha + \tilde{\alpha} = 0, \tag{190}$$

$$\text{case 5: } \alpha = 0, \tilde{\alpha} = 0, \alpha + \tilde{\alpha} = 0. \tag{191}$$

We obtain

$$I_{\alpha\tilde{\alpha}}^1 = \frac{\exp[\beta(\alpha + \tilde{\alpha})] - 1}{\tilde{\alpha}(\alpha + \tilde{\alpha})} - \frac{\exp(\beta\alpha) - 1}{\alpha\tilde{\alpha}}, \tag{192}$$

$$I_{0\tilde{\alpha}}^2 = \frac{\exp(\beta\tilde{\alpha}) - 1}{\tilde{\alpha}^2} - \frac{\beta}{\tilde{\alpha}}, \tag{193}$$

$$I_{\alpha 0}^3 = -\frac{\exp(\beta\alpha) - 1}{\alpha^2} + \frac{\beta \exp(\beta\alpha)}{\alpha}, \tag{194}$$

$$I_{\alpha-\alpha}^4 = \frac{\exp(\beta\alpha) - 1}{\alpha^2} - \frac{\beta}{\alpha}, \tag{195}$$

$$I_{00}^5 = \frac{1}{2}\beta^2. \tag{196}$$

It can be easily shown by using the Taylor expansion of the exponential that

$$\lim_{\alpha \rightarrow 0} I_{\alpha\tilde{\alpha}}^1 = I_{0\tilde{\alpha}}^2, \tag{197}$$

$$\lim_{\tilde{\alpha} \rightarrow -\alpha} I_{\alpha\tilde{\alpha}}^1 = I_{\alpha-\alpha}^4. \tag{198}$$

In order to evaluate the limit $\tilde{\alpha} \rightarrow 0$ and the combined limit of $\alpha \rightarrow 0$ and $\tilde{\alpha} \rightarrow 0$ we use the Taylor expansion of the exponential and the binomial theorem to obtain

$$I_{\alpha\tilde{\alpha}}^1 = \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} \frac{\beta^n}{n!} \binom{n-1}{k} \tilde{\alpha}^{k-1} \alpha^{n-1-k}. \tag{199}$$

From (199) we can read off

$$\lim_{\substack{\alpha \rightarrow 0 \\ \tilde{\alpha} \rightarrow 0}} I_{\alpha\tilde{\alpha}}^1 = I_{00}^5, \tag{200}$$

and by completing the respective sums we obtain

$$\lim_{\tilde{\alpha} \rightarrow 0} I_{\alpha\tilde{\alpha}}^1 = I_{\alpha 0}^3. \tag{201}$$

With (197), (198), (200) and (201) we conclude that the expression for case 1 in (192) can be continuously continued in order to include the cases 2 to 5 in (193)–(196). Therefore we can write (185) as

$$\begin{aligned}
\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2} \rangle_{U^2}^1 &= (\langle \hat{n}_{\mathbf{k}_1} \rangle_0 + 1) \langle \hat{n}_{\mathbf{k}_2} \rangle \sum_{\mathbf{k}} \left[({}^1A_{\mathbf{k},\mathbf{k}_1}^U + {}^2A_{\mathbf{k}_1,\mathbf{k}}^U) ({}^1A_{\mathbf{k}_2,\mathbf{k}}^U + {}^2A_{\mathbf{k},\mathbf{k}_2}^U) \left(\langle \hat{n}_{\mathbf{k}} \rangle_0 \right. \right. \\
&\times \left. \left\{ \frac{\exp[\beta(-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})(-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})] - 1}{(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})} \right\} + (\langle \hat{n}_{\mathbf{k}} \rangle_0 + 1) \right. \\
&\times \left. \left. \left\{ \frac{\exp[\beta(-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})(-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})} \right\} \right) + ({}^3A_{\mathbf{k},\mathbf{k}_1}^U + {}^3A_{\mathbf{k}_1,\mathbf{k}}^U) \\
&\times ({}^4A_{\mathbf{k},\mathbf{k}_2}^U + {}^4A_{\mathbf{k}_2,\mathbf{k}}^U) \left(\langle \hat{n}_{\mathbf{k}} \rangle_0 \left\{ \frac{\exp[\beta(-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})(-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})} \right\} \right. \\
&\left. \left. + (\langle \hat{n}_{\mathbf{k}} \rangle_0 + 1) \left\{ \frac{\exp[\beta(-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})(-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})] - 1}{(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})} \right\} \right) \right]. \quad (202)
\end{aligned}$$

As argued above we can express the second contribution to $\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2} \rangle_{U^2}$ through the result for $\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2} \rangle_U$ in (166) with the replacement (180). Therefore we obtain in total

$$\begin{aligned}
\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2} \rangle_{U^2} &= (\langle \hat{n}_{\mathbf{k}_1} \rangle_0 + 1) \langle \hat{n}_{\mathbf{k}_2} \rangle \left\{ \sum_{\mathbf{k}} \left[({}^1A_{\mathbf{k},\mathbf{k}_1}^U + {}^2A_{\mathbf{k}_1,\mathbf{k}}^U) ({}^1A_{\mathbf{k}_2,\mathbf{k}}^U + {}^2A_{\mathbf{k},\mathbf{k}_2}^U) \left(\langle \hat{n}_{\mathbf{k}} \rangle_0 \right. \right. \right. \\
&\times \left. \left\{ \frac{\exp[\beta(-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})(-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})] - 1}{(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})} \right\} + (\langle \hat{n}_{\mathbf{k}} \rangle_0 + 1) \right. \\
&\times \left. \left. \left\{ \frac{\exp[\beta(-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})(-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})} \right\} \right) + ({}^3A_{\mathbf{k},\mathbf{k}_1}^U + {}^3A_{\mathbf{k}_1,\mathbf{k}}^U) \\
&\times ({}^4A_{\mathbf{k},\mathbf{k}_2}^U + {}^4A_{\mathbf{k}_2,\mathbf{k}}^U) \left(\langle \hat{n}_{\mathbf{k}} \rangle_0 \left\{ \frac{\exp[\beta(-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})(-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})} \right\} \right. \\
&\left. \left. + (\langle \hat{n}_{\mathbf{k}} \rangle_0 + 1) \left\{ \frac{\exp[\beta(-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})(-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})] - 1}{(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})} \right\} \right) \right] \\
&- \left({}^1A_{\mathbf{k}_2,\mathbf{k}_1}^{U^2} + {}^2A_{\mathbf{k}_1,\mathbf{k}_2}^{U^2} \right) \frac{\exp[\beta(-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{-\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2}} \Big\}. \quad (203)
\end{aligned}$$

Performing the same calculation for $\langle \hat{b}_{\mathbf{k}_1} \hat{b}_{\mathbf{k}_2}^\dagger \rangle_{U^2}$, $\langle \hat{b}_{\mathbf{k}_1} \hat{b}_{\mathbf{k}_2} \rangle_{U^2}$ and $\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2}^\dagger \rangle_{U^2}$ yields

$$\begin{aligned}
\langle \hat{b}_{\mathbf{k}_1} \hat{b}_{\mathbf{k}_2}^\dagger \rangle_{U^2} &= \langle \hat{n}_{\mathbf{k}_1} \rangle_0 (\langle \hat{n}_{\mathbf{k}_2} \rangle_0 + 1) \left\{ \sum_{\mathbf{k}} \left[({}^1A_{\mathbf{k}_1,\mathbf{k}}^U + {}^2A_{\mathbf{k},\mathbf{k}_1}^U) ({}^1A_{\mathbf{k},\mathbf{k}_2}^U + {}^2A_{\mathbf{k}_2,\mathbf{k}}^U) \left(\langle \hat{n}_{\mathbf{k}} \rangle_0 \right. \right. \right. \\
&\times \left. \left\{ \frac{\exp[\beta(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})} \right\} + (\langle \hat{n}_{\mathbf{k}} \rangle_0 + 1) \right. \\
&\times \left. \left. \left\{ \frac{\exp[\beta(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})] - 1}{(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})} \right\} \right) + ({}^4A_{\mathbf{k},\mathbf{k}_1}^U + {}^4A_{\mathbf{k}_1,\mathbf{k}}^U) \\
&\times ({}^3A_{\mathbf{k},\mathbf{k}_2}^U + {}^3A_{\mathbf{k}_2,\mathbf{k}}^U) \left(\langle \hat{n}_{\mathbf{k}} \rangle_0 \left\{ \frac{\exp[\beta(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})] - 1}{(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})} \right\} \right. \\
&\left. \left. + (\langle \hat{n}_{\mathbf{k}} \rangle_0 + 1) \left\{ \frac{\exp[\beta(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})} \right\} \right) \right] \\
&- \left({}^1A_{\mathbf{k}_1,\mathbf{k}_2}^{U^2} + {}^2A_{\mathbf{k}_2,\mathbf{k}_1}^{U^2} \right) \frac{\exp[\beta(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2}} \Big\}, \quad (204)
\end{aligned}$$

$$\begin{aligned}
\langle \hat{b}_{\mathbf{k}_1} \hat{b}_{\mathbf{k}_2} \rangle_{U^2} &= \langle \hat{n}_{\mathbf{k}_1} \rangle_0 \langle \hat{n}_{\mathbf{k}_2} \rangle_0 \left\{ \sum_{\mathbf{k}} \left[({}^1A_{\mathbf{k}_1, \mathbf{k}}^U + {}^2A_{\mathbf{k}, \mathbf{k}_1}^U) ({}^4A_{\mathbf{k}, \mathbf{k}_2}^U + {}^4A_{\mathbf{k}_2, \mathbf{k}}^U) \left(\langle \hat{n}_{\mathbf{k}} \rangle_0 \right. \right. \right. \\
&\times \left. \left. \left\{ \frac{\exp[\beta(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})} \right\} + (\langle \hat{n}_{\mathbf{k}} \rangle_0 + 1) \right. \right. \\
&\times \left. \left. \left\{ \frac{\exp[\beta(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})] - 1}{(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})} \right\} + ({}^4A_{\mathbf{k}, \mathbf{k}_1}^U + {}^4A_{\mathbf{k}_1, \mathbf{k}}^U) \right. \\
&\times ({}^1A_{\mathbf{k}_2, \mathbf{k}}^U + {}^2A_{\mathbf{k}, \mathbf{k}_2}^U) \left(\langle \hat{n}_{\mathbf{k}} \rangle_0 \left\{ \frac{\exp[\beta(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})] - 1}{(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})} \right\} \right. \\
&+ \left. \left. (\langle \hat{n}_{\mathbf{k}} \rangle_0 + 1) \left\{ \frac{\exp[\beta(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})} \right\} \right) \right] \\
&- \left. \left. \left({}^4A_{\mathbf{k}_1, \mathbf{k}_2}^{U^2} + {}^4A_{\mathbf{k}_2, \mathbf{k}_1}^{U^2} \right) \frac{\exp[\beta(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})} \right\}, \tag{205}
\end{aligned}$$

$$\begin{aligned}
\langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2}^\dagger \rangle_{U^2} &= (\langle \hat{n}_{\mathbf{k}_1} \rangle_0 + 1)(\langle \hat{n}_{\mathbf{k}_2} \rangle_0 + 1) \left\{ \sum_{\mathbf{k}} \left[({}^1A_{\mathbf{k}, \mathbf{k}_1}^U + {}^2A_{\mathbf{k}_1, \mathbf{k}}^U) ({}^3A_{\mathbf{k}, \mathbf{k}_2}^U + {}^3A_{\mathbf{k}_2, \mathbf{k}}^U) \left(\langle \hat{n}_{\mathbf{k}} \rangle_0 \right. \right. \right. \\
&\times \left. \left. \left\{ \frac{\exp[\beta(-\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})(-\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})] - 1}{(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})} \right\} + (\langle \hat{n}_{\mathbf{k}} \rangle_0 + 1) \right. \\
&\times \left. \left. \left\{ \frac{\exp[\beta(-\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})(-\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})} \right\} + ({}^3A_{\mathbf{k}, \mathbf{k}_1}^U + {}^3A_{\mathbf{k}_1, \mathbf{k}}^U) \right. \\
&\times ({}^1A_{\mathbf{k}, \mathbf{k}_2}^U + {}^2A_{\mathbf{k}_2, \mathbf{k}}^U) \left(\langle \hat{n}_{\mathbf{k}} \rangle_0 \left\{ \frac{\exp[\beta(-\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})(-\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})} \right\} \right. \\
&+ \left. \left. (\langle \hat{n}_{\mathbf{k}} \rangle_0 + 1) \left\{ \frac{\exp[\beta(-\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})(-\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})] - 1}{(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})} \right\} \right) \right] \\
&- \left. \left. \left({}^3A_{\mathbf{k}_1, \mathbf{k}_2}^{U^2} + {}^3A_{\mathbf{k}_2, \mathbf{k}_1}^{U^2} \right) \frac{\exp[\beta(-\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{-\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2}} \right\}. \tag{206}
\end{aligned}$$

With the relation $\lambda_{-\mathbf{k}} = \lambda_{\mathbf{k}}$ due to (96), the symmetry properties of the coefficients in (114)–(116) and (118) and the explicit form of $\langle \hat{n}_{\mathbf{k}} \rangle_0$ in (155) we obtain as in Subsection 4.3.2 the relations

$$\langle \hat{b}_{\mathbf{k}_2} \hat{b}_{\mathbf{k}_1}^\dagger \rangle_{U^2} = \langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2} \rangle_{U^2} = \langle \hat{b}_{-\mathbf{k}_2}^\dagger \hat{b}_{-\mathbf{k}_1} \rangle_{U^2}, \tag{207}$$

$$\langle \hat{b}_{\mathbf{k}_1} \hat{b}_{\mathbf{k}_2} \rangle_{U^2}^\dagger = \langle \hat{b}_{\mathbf{k}_1}^\dagger \hat{b}_{\mathbf{k}_2}^\dagger \rangle_{U^2} = \langle \hat{b}_{-\mathbf{k}_2}^\dagger \hat{b}_{-\mathbf{k}_1}^\dagger \rangle_{U^2}. \tag{208}$$

Inserting the results given in (203)–(206) into the average of (149), using (207) and (208) and relabeling

the summation indices in a convenient way leads finally to

$$\begin{aligned}
\left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_{U^2} &= \frac{1}{V} \sum_{\mathbf{k}_1, \mathbf{k}_2} \exp[i\mathbf{x}(\mathbf{k}_1 + \mathbf{k}_2)] \left(2(u_{\mathbf{k}_1}v_{\mathbf{k}_2} + u_{\mathbf{k}_2}v_{\mathbf{k}_1}) \langle \hat{n}_{\mathbf{k}_1} \rangle_0 \langle \hat{n}_{\mathbf{k}_2} \rangle_0 \right. \\
&\times \left\{ \sum_{\mathbf{k}} \left[\left({}^1A_{\mathbf{k}_1, \mathbf{k}}^U + {}^2A_{\mathbf{k}, \mathbf{k}_1}^U \right) \left({}^4A_{\mathbf{k}, \mathbf{k}_2}^U + {}^4A_{\mathbf{k}_2, \mathbf{k}}^U \right) \left(\langle \hat{n}_{\mathbf{k}} \rangle_0 \left\{ \frac{\exp[\beta(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})} \right. \right. \right. \\
&- \left. \left. \frac{\exp[\beta(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})} \right\} + (\langle \hat{n}_{\mathbf{k}} \rangle_0 + 1) \left\{ \frac{\exp[\beta(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})} \right. \right. \\
&- \left. \left. \frac{\exp[\beta(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})] - 1}{(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})} \right\} \right] - \left({}^4A_{\mathbf{k}_1, \mathbf{k}_2}^{U^2} + {}^4A_{\mathbf{k}_2, \mathbf{k}_1}^{U^2} \right) \frac{\exp[\beta(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})] - 1}{2(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})} \right\} \\
&+ (u_{\mathbf{k}_1}u_{\mathbf{k}_2} + v_{\mathbf{k}_1}v_{\mathbf{k}_2}) \langle \hat{n}_{\mathbf{k}_1} \rangle_0 (\langle \hat{n}_{\mathbf{k}_2} \rangle_0 + 1) \left\{ \sum_{\mathbf{k}} \left[\left({}^1A_{\mathbf{k}_1, \mathbf{k}}^U + {}^2A_{\mathbf{k}, \mathbf{k}_1}^U \right) \left({}^1A_{\mathbf{k}, -\mathbf{k}_2}^U \right. \right. \right. \\
&+ \left. \left. {}^2A_{-\mathbf{k}_2, \mathbf{k}}^U \right) \left(\langle \hat{n}_{\mathbf{k}} \rangle_0 \left\{ \frac{\exp[\beta(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})} \right\} \right. \right. \\
&+ \left. \left. (\langle \hat{n}_{\mathbf{k}} \rangle_0 + 1) \left\{ \frac{\exp[\beta(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})} - \frac{\exp[\beta(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})] - 1}{(-\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})} \right\} \right) + \left({}^4A_{\mathbf{k}, \mathbf{k}_1}^U \right. \right. \\
&+ \left. \left. {}^4A_{\mathbf{k}_1, \mathbf{k}}^U \right) \left({}^3A_{\mathbf{k}, -\mathbf{k}_2}^U + {}^3A_{-\mathbf{k}_2, \mathbf{k}}^U \right) \left(\langle \hat{n}_{\mathbf{k}} \rangle_0 \left\{ \frac{\exp[\beta(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})} \right. \right. \right. \\
&- \left. \left. \frac{\exp[\beta(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})] - 1}{(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})} \right\} + (\langle \hat{n}_{\mathbf{k}} \rangle_0 + 1) \left\{ \frac{\exp[\beta(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})} \right. \right. \\
&- \left. \left. \frac{\exp[\beta(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})] - 1}{(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(-\lambda_{\mathbf{k}} - \lambda_{\mathbf{k}_2})} \right\} \right] - \left({}^1A_{\mathbf{k}_1, -\mathbf{k}_2}^{U^2} + {}^2A_{-\mathbf{k}_2, \mathbf{k}_1}^{U^2} \right) \\
&\times \left. \frac{\exp[\beta(\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2})] - 1}{\lambda_{\mathbf{k}_1} - \lambda_{\mathbf{k}_2}} \right\}. \tag{209}
\end{aligned}$$

Due to the fact that (44) does not depend on $\psi_{12}(\mathbf{x})$, the explicit form of $\left\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_{U^2}$ is not needed, as it only contributes to $\psi_{12}(\mathbf{x})$ in (32).

Thus, we obtain with the limit in (177) for the case $T \rightarrow 0$

$$\begin{aligned}
\left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_{U^2, T=0} &= \frac{1}{V} \sum_{\mathbf{k}_1, \mathbf{k}_2} \exp[i\mathbf{x}(\mathbf{k}_1 + \mathbf{k}_2)] \left\{ 2(u_{\mathbf{k}_1}v_{\mathbf{k}_2} + u_{\mathbf{k}_2}v_{\mathbf{k}_1}) \right. \\
&\times \left[\sum_{\mathbf{k}} \frac{\left({}^1A_{\mathbf{k}_1, \mathbf{k}}^U + {}^2A_{\mathbf{k}, \mathbf{k}_1}^U \right) \left({}^4A_{\mathbf{k}, \mathbf{k}_2}^U + {}^4A_{\mathbf{k}_2, \mathbf{k}}^U \right)}{(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})} - \frac{\left({}^4A_{\mathbf{k}_1, \mathbf{k}_2}^{U^2} + {}^4A_{\mathbf{k}_2, \mathbf{k}_1}^{U^2} \right)}{2(\lambda_{\mathbf{k}_1} + \lambda_{\mathbf{k}_2})} \right] \\
&+ (u_{\mathbf{k}_1}u_{\mathbf{k}_2} + v_{\mathbf{k}_1}v_{\mathbf{k}_2}) \sum_{\mathbf{k}} \frac{\left({}^4A_{\mathbf{k}, \mathbf{k}_1}^U + {}^4A_{\mathbf{k}_1, \mathbf{k}}^U \right) \left({}^3A_{\mathbf{k}, -\mathbf{k}_2}^U + {}^3A_{-\mathbf{k}_2, \mathbf{k}}^U \right)}{(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_1})(\lambda_{\mathbf{k}} + \lambda_{\mathbf{k}_2})} \left. \right\}, \tag{210}
\end{aligned}$$

5 Calculation of Particle Density

With the results for the expansion in the disorder potential U of the correlations $\left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle$ and $\left\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle$ up to second order we are able to calculate the condensate density n_0 in terms of the particle density n by (44) where we restrict ourself to the case $T = 0$ and can therefore use (162), (163), (178), (179) and (210). In addition we have to apply the thermodynamic limit

$$N, V \rightarrow \infty, \tag{211}$$

where we keep the particle density n constant. The limit (211) implies

$$\sum_{\mathbf{k}} \rightarrow \frac{V}{(2\pi)^3} \int d^3k. \tag{212}$$

5.1 Bogoliubov Depletion

As described in Subsection 2.4, the term $\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \rangle_0$ represents the Bogoliubov depletion which occurs because of the interaction of the particles in zeroth order referring to the random potential U . With (90), (96), (101) and (162) we obtain in the thermodynamic limit (211) and (212)

$$\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \rangle_{0,T=0} = \int \frac{d^3k}{(2\pi)^3} \frac{\frac{\hbar^2\mathbf{k}^2}{2m} + \mu - \sqrt{\frac{\hbar^2\mathbf{k}^2}{2m}} \sqrt{\frac{\hbar^2\mathbf{k}^2}{2m} + 2\mu}}{2\sqrt{\frac{\hbar^2\mathbf{k}^2}{2m}} \sqrt{\frac{\hbar^2\mathbf{k}^2}{2m} + 2\mu}}, \quad (213)$$

where we inserted the solution

$$\psi_{00} = \sqrt{\frac{\mu}{g}} \quad (214)$$

of (27). Note that we can drop the second solution $\psi_{00} = 0$ because the whole system is described by ψ_{00} in zeroth order of the fluctuation operators $\delta\hat{\psi}^\dagger$ and $\delta\hat{\psi}$ and the limit of vanishing disorder potential and therefore the trivial solution of ψ_{00} would immediately lead in this case with (42) and (43) to the unphysical result $n = n_0 = 0$. Using spherical coordinates, performing the angular integrals and substituting

$$\epsilon = \frac{\hbar^2 k^2}{2m} \quad (215)$$

leads to

$$\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \rangle_{0,T=0} = \frac{1}{\pi^2} \left(\frac{m}{2\hbar^2} \right)^{3/2} \lim_{L \rightarrow \infty} \int_0^L d\epsilon \left(\sqrt{\epsilon + 2\mu} - \frac{\mu}{\sqrt{\epsilon + 2\mu}} - \sqrt{\epsilon} \right), \quad (216)$$

where we introduced the ultraviolet cutoff L in order to regularize the divergent radial integrals. Calculating the integrals and inserting the expansion $(1+x)^{n/2} = 1 + \frac{n}{2}x + \mathcal{O}(x^2)$ yields

$$\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \rangle_{0,T=0} = \frac{1}{\pi^2} \left(\frac{m}{2\hbar^2} \right)^{3/2} \lim_{L \rightarrow \infty} \left[\frac{2^{3/2}}{3} \sqrt{\mu}^3 + \frac{2}{3} \sqrt{L}^3 \mathcal{O}\left(\frac{1}{L^2}\right) - 2\mu\sqrt{L} \mathcal{O}\left(\frac{1}{L}\right) \right], \quad (217)$$

where we can evaluate the limit easily because the different integrands in (216) regularize each other and obtain

$$\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \rangle_{0,T=0} = \frac{1}{3\pi^2} \left(\frac{m\mu}{\hbar^2} \right)^{3/2}. \quad (218)$$

The term $\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \rangle_{0,T=0}$ itself is of first order in $\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x})$ and therefore we have to insert for the chemical potential μ the expansion given in (45) without μ_{10} and, as argued in Section 2.4, the vanishing μ_{01} contribution. With (5) and (46) we obtain in zeroth order of the disorder potential

$$\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \rangle_{0,T=0} = \frac{8}{3\sqrt{\pi}} (an)^{3/2} + \mathcal{O}(R), \quad (219)$$

where the term $\mathcal{O}(R)$ corresponds to the contribution due to μ_{02} which is of the order of U^2 but, as we perform the disorder average in (42), μ_{02} is proportional to the second cumulant R as it was defined in (37). This result correspond to the Bogoliubov depletion in (1) with the zeroth order identification $n = n_0$ of (44) in (1).

5.2 Gross-Pitaevskii Disorder Depletion and Chemical Potential

In order to calculate the depletion term on the level of the Gross-Pitaevskii theory in (44) and the expansion of μ in (45) with (42) we need to determine $\psi_{01}(\mathbf{x})$, $\psi_{02}(\mathbf{x})$ and ψ_{10} out of (28)–(30). Applying a Fourier transformation as defined in (79) to (28) yields with (214)

$$\psi_{01}(\mathbf{k}) = -\frac{\sqrt{\frac{\mu}{g}} U(\mathbf{k})}{\frac{\hbar^2\mathbf{k}^2}{2m} + 2\mu}. \quad (220)$$

Therefore we obtain in the thermodynamic limit

$$\overline{\psi_{01}(\mathbf{x})\psi_{01}(\mathbf{x})} = \frac{\mu V}{g} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \exp[i\mathbf{x}(\mathbf{k} + \mathbf{k}')] \frac{\overline{U(\mathbf{k})U(\mathbf{k}')}}{\left(\frac{\hbar^2\mathbf{k}^2}{2m} + 2\mu\right) \left(\frac{\hbar^2\mathbf{k}'^2}{2m} + 2\mu\right)}. \quad (221)$$

We calculate the disorder average $\overline{U(\mathbf{k})U(\mathbf{k}')}$ by inserting the inverse transformation of (79)

$$f(\mathbf{k}) = \frac{1}{\sqrt{V}} \int d^3x f(\mathbf{x}) \exp(-i\mathbf{x}\mathbf{k}) \quad (222)$$

for $U(\mathbf{k})$ and $U(\mathbf{k}')$, using the definition of the second cumulant in (37) and applying the Fourier transformation (79) to $R(\mathbf{x} - \mathbf{x}')$ and obtain by evaluating the integrals in the thermodynamic limit

$$\overline{U(\mathbf{k})U(\mathbf{k}')} = \frac{(2\pi)^3}{\sqrt{V}} R(\mathbf{k})\delta(\mathbf{k} + \mathbf{k}'). \quad (223)$$

Inserting this result into (221) yields

$$\overline{\psi_{01}(\mathbf{x})\psi_{01}(\mathbf{x})} = \frac{\mu\sqrt{V}}{g} \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})}{\left(\frac{\hbar^2\mathbf{k}^2}{2m} + 2\mu\right)^2}. \quad (224)$$

By Fourier transforming (29) we obtain for ψ_{02} in the thermodynamic limit (211) and (212)

$$\psi_{02}(\mathbf{k}) = -\sqrt{V} \int \frac{d^3k'}{(2\pi)^3} \frac{\psi_{01}(\mathbf{k}')U(\mathbf{k} - \mathbf{k}') + 3\sqrt{\mu g}\psi_{01}(\mathbf{k}')\psi_{01}(\mathbf{k} - \mathbf{k}')}{\frac{\hbar^2\mathbf{k}^2}{2m} + 2\mu}. \quad (225)$$

Inserting the result for ψ_{00} in (214), for $\psi_{01}(\mathbf{k})$ in (220) and the second cumulant of U in \mathbf{k} -space (223) leads to

$$\psi_{00}\overline{\psi_{02}(\mathbf{x})} = \frac{\sqrt{V}}{2g} \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})}{\frac{\hbar^2\mathbf{k}^2}{2m} + 2\mu} \left(1 - \frac{3\mu}{\frac{\hbar^2\mathbf{k}^2}{2m} + 2\mu}\right). \quad (226)$$

In order to calculate ψ_{10} by (30) we have to evaluate $\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \rangle_{0,T=0}$. By using (100), (101) and (163) we obtain in the thermodynamic limit

$$\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \rangle_{0,T=0} = - \int \frac{d^3k}{(2\pi)^3} \frac{\mu}{2\sqrt{\frac{\hbar^2\mathbf{k}^2}{2m}} \sqrt{\frac{\hbar^2\mathbf{k}^2}{2m} + 2\mu}}. \quad (227)$$

This integral can not be regularized by introducing an ultraviolet cutoff as we did in (216). Therefore we use the dimensional regularization as described e.g. in Ref. [28], i.e. we calculate the integral in d -dimensions and continue the solution analytically to $d = 3$. In order to do so we will use the identity

$$\frac{1}{a^y} = \frac{1}{\Gamma(y)} \int_0^\infty d\tau \exp(-a\tau)\tau^{y-1}, \quad (228)$$

which we will refer to as the Schwinger trick and where $\Gamma(y)$ denotes the gamma function of y . The analytical continuation of the results can be done by continuing the gamma function

$$\Gamma(y) = \int_0^\infty dt \exp(-t)t^{y-1}, \quad (229)$$

which is defined via (229) only for $y > 0$, with its functional equation

$$\Gamma(y+1) = y\Gamma(y), \quad (230)$$

to negative values of y . The integral in (227) in d -dimensions reads

$$\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \rangle_{0,T=0} = - S_d \sqrt{\frac{2m^d}{\hbar^2}} \frac{\mu}{4(2\pi)^d} \int_0^\infty d\epsilon \frac{\sqrt{\epsilon}^{d-3}}{\sqrt{\epsilon + 2\mu}} \Bigg|_{d=3}, \quad (231)$$

where S_d denotes the surface of the d -dimensional unit sphere, which is the result of the d -dimensional angular integral with the special value

$$S_3 = 4\pi. \quad (232)$$

Inserting the Schwinger trick (228) with $a = \epsilon + 2\mu$ and $y = \frac{1}{2}$ and performing the ϵ and τ integral leads to the result

$$\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \rangle_{0,T=0} = - S_d \sqrt{\frac{2m^d}{\hbar^2}} \frac{\mu}{4(2\pi)^d} \frac{\Gamma\left(\frac{d-1}{2}\right) \Gamma\left(1 - \frac{d}{2}\right)}{\Gamma\left(\frac{1}{2}\right) (2\mu)^{1-d/2}} \Bigg|_{d=3}. \quad (233)$$

Inserting $d = 3$ and (232) and using

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad (234)$$

$$\Gamma(1) = 1, \quad (235)$$

where the first equation can be easily shown by substituting $t = x^2$ in (229) and performing the Gaussian integral and the second equation follows immediately from (229), and (230) yields

$$\left\langle \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_{0,T=0} = \frac{1}{\pi^2} \left(\frac{m\mu}{\hbar^2} \right)^{3/2}. \quad (236)$$

Therefore we obtain for ψ_{10} with (30), (214), (218) and (236)

$$\psi_{10} = -\sqrt{\frac{g}{\mu}} \frac{5}{6\pi^2} \left(\frac{m\mu}{\hbar^2} \right)^{3/2}. \quad (237)$$

In order to evaluate the remaining integrals in (224) and (226) we have to assume a model for the disorder correlation $R(\mathbf{x} - \mathbf{x}')$. The most simple model we could choose is

$$R(\mathbf{x} - \mathbf{x}') = R\delta(\mathbf{x} - \mathbf{x}'), \quad (238)$$

which correspond to a complete randomly arranged potential with no correlation at two different places. Applying the Fourier transformation (79) on (238) leads to

$$R(\mathbf{k}) = \frac{1}{\sqrt{V}} R. \quad (239)$$

With the model given in (239), (224) takes the form

$$\overline{\psi_{01}(\mathbf{x})\psi_{01}(\mathbf{x})} = \frac{2}{\pi^2} \left(\frac{m}{2\hbar^2} \right)^{3/2} \frac{\mu}{g} R \int_0^\infty d\epsilon \frac{\sqrt{\epsilon}}{(\epsilon + 2\mu)^2}, \quad (240)$$

where we already have evaluated the angular integral and substituted $\epsilon = \frac{\hbar^2 k^2}{2m}$. Using the Schwinger trick (228) with $a = \epsilon + 2\mu$ and $y = 2$ and evaluating the integrals yields with (230), (234) and (235)

$$\overline{\psi_{01}(\mathbf{x})\psi_{01}(\mathbf{x})} = \sqrt{\frac{\pi}{2}} \left(\frac{m}{2\pi\hbar^2} \right)^{3/2} \frac{R}{g} \sqrt{\mu}. \quad (241)$$

Inserting (5) and (46) yields for the zeroth order in the fluctuation operators $\delta\hat{\psi}^\dagger$ and $\delta\hat{\psi}$

$$\overline{\psi_{01}(\mathbf{x})\psi_{01}(\mathbf{x})}_{\delta\hat{\psi}^\dagger=0, \delta\hat{\psi}=0} = \frac{m^2}{8\pi^{3/2}\hbar^4} \sqrt{\frac{n}{a}} R, \quad (242)$$

which is the same result as (2), where we can use again the zeroth order identification $n = n_0$ of (44) in (2), as the results of Ref. [8] does not include terms of the order of a Bogoliubov disorder depletion [20, 21] which would be the result of a consideration of higher than zeroth order terms of (44) in (1) or (2).

Using the model in (239) in (226) leads to a divergent integral which we again regularise dimensionally. Inserting (239) into (226), using the substitution (215), the Schwinger trick twice (228) with $a_1 = \epsilon + 2\mu$, $y_1 = 1$, $a_2 = \epsilon + 2\mu$ and $y_2 = 2$ and the definition of the gamma function (229) yields in d -dimensions

$$\psi_{00} \overline{\psi_{02}(\mathbf{x})} = \frac{2^{d/2-1} S_d}{\pi^{d/2}} \left(\frac{m}{2\pi\hbar^2} \right)^{d/2} \frac{R}{g} \mu^{d/2-1} \left[\frac{\Gamma(\frac{d}{2}) \Gamma(1 - \frac{d}{2})}{\Gamma(1)} - \frac{3}{2} \frac{\Gamma(\frac{d}{2}) \Gamma(2 - \frac{d}{2})}{\Gamma(2)} \right] \Bigg|_{d=3}. \quad (243)$$

Using (232) and evaluating the gamma functions for $d = 3$ with (230), (234) and (235) leads to

$$\psi_{00} \overline{\psi_{02}(\mathbf{x})} = -\frac{7}{2} \sqrt{\frac{\pi}{2}} \left(\frac{m}{2\pi\hbar^2} \right)^{3/2} \frac{R}{g} \sqrt{\mu}. \quad (244)$$

Inserting the expansion of the chemical potential in (45) and our previous results (214), (218) and (237) into (42) yields in zeroth order of the disorder potential

$$n = \frac{\mu_{00} + \mu_{10}}{g} - \frac{8}{3} \sqrt{\frac{2}{\pi}} \left(\frac{m}{2\pi\hbar^2} \right)^{3/2} \sqrt{\mu_{00}^3} + \dots \quad (245)$$

Solving (245) for μ_{10} and inserting (46) leads to

$$\mu_{10} = \frac{8}{3} \sqrt{\frac{2}{\pi}} \left(\frac{m}{2\pi\hbar^2} \right)^{3/2} \sqrt{g^5} \sqrt{n^3}. \quad (246)$$

By inserting (241) and (244) into (42) we obtain in zeroth order of the fluctuation operators

$$n = \frac{\mu_{00} + \mu_{02}}{g} - 6 \sqrt{\frac{\pi}{2}} \frac{R}{g} \left(\frac{m}{2\pi\hbar^2} \right)^{3/2} \sqrt{\mu_{00}} + \dots, \quad (247)$$

where we already used that μ_{01} is vanishing as argued in Subsection 2.4. Solving (247) for μ_{02} and inserting (46) yields

$$\mu_{02} = 6 \sqrt{\frac{\pi}{2}} R \left(\frac{m}{2\pi\hbar^2} \right)^{3/2} \sqrt{g} \sqrt{n}. \quad (248)$$

Note that due to the fact that we inserted (43) into (42) no chemical potential dependent term that is as well of zeroth order in the fluctuation operators as of zeroth in the disorder potential occurs in (44) and we therefore do not have to calculate μ_{12} , as all μ -dependent terms are itself of higher as zeroth order in the fluctuation operators or the random potential.

Inserting the expansion of the chemical potential μ in (45) up to first order in the fluctuation operators and zeroth order in the disorder potential into (241), expanding the result up to first order in $\frac{\mu_{10}}{\mu_{00}}$ and using the definition of the interaction strength g in (5) and the results for μ_{00} (46) and (246) leads to

$$\overline{\psi_{01}(\mathbf{x})\psi_{01}(\mathbf{x})} = \frac{m^2}{8\pi^{3/2}\hbar^4} \sqrt{\frac{n}{a}} R + \frac{2}{3} \frac{m^2}{\pi^2\hbar^4} anR. \quad (249)$$

Note that we only have to insert terms of the expansion of μ that are of zeroth order in the disorder potential U because $\overline{\psi_{01}(\mathbf{x})\psi_{01}(\mathbf{x})}$ itself is of second order in U .

Inserting the expansion of μ (45) up to second order in the random potential and zeroth order in fluctuation operators into (218), expanding the result up to first in $\frac{\mu_{02}}{\mu_{00}}$ and using (5), (46), $\mu_{01} = 0$ and (248) yields

$$\left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_{0,T=0} = \frac{8}{3\sqrt{\pi}} (an)^{3/2} + 3 \frac{m^2}{\pi^2\hbar^4} anR. \quad (250)$$

5.3 Bogoliubov Disorder Depletion

The Bogoliubov disorder depletion term in (44) is described by $\overline{\psi_{01}(\mathbf{x})\psi_{11}(\mathbf{x})}$ and $\left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_{U^2}$.

Inserting (178), (179), (214), (218), (236), (237) and the Fourier transformation (79) of $\psi_{11}(\mathbf{x})$ and $\psi_{01}(\mathbf{x})$ into (31) yields

$$\begin{aligned} \psi_{11}(\mathbf{k}) &= \frac{1}{\frac{\hbar^2\mathbf{k}^2}{2m} + 2\mu} \frac{1}{3\pi^2} \left(\frac{m\mu}{\hbar^2} \right)^{3/2} \left[10g\psi_{01}(\mathbf{k}) + \frac{5}{2} \sqrt{\frac{g}{\mu}} U(\mathbf{k}) \right] - \frac{\sqrt{g\mu}}{\frac{\hbar^2\mathbf{k}^2}{2m} + 2\mu} \frac{1}{\sqrt{V}} \\ &\times \sum_{\mathbf{k}'} \frac{4A_{\mathbf{k}-\mathbf{k}',\mathbf{k}'}^U + 4A_{\mathbf{k}',\mathbf{k}-\mathbf{k}'}^U}{\lambda_{\mathbf{k}-\mathbf{k}'} + \lambda_{\mathbf{k}'}} (u_{\mathbf{k}-\mathbf{k}'}u_{\mathbf{k}'} + v_{\mathbf{k}-\mathbf{k}'}v_{\mathbf{k}'} + 2u_{\mathbf{k}'}v_{\mathbf{k}-\mathbf{k}'} + 2u_{\mathbf{k}-\mathbf{k}'}v_{\mathbf{k}'}). \end{aligned} \quad (251)$$

Therefore we obtain by applying a Fourier transformation (79) and using (109), (214), (220) and (223) in the thermodynamic limit (211) and (212)

$$\begin{aligned} \overline{\psi_{01}(\mathbf{x})\psi_{11}(\mathbf{x})} &= -\sqrt{V} \int \frac{d^3k}{(2\pi)^3} \frac{5}{6\pi^2} \left(\frac{m\mu}{\hbar^2} \right)^{3/2} R(\mathbf{k}) \frac{\frac{\hbar^2\mathbf{k}^2}{2m} - 2\mu}{\left(\frac{\hbar^2\mathbf{k}^2}{2m} + 2\mu \right)^3} - \sqrt{V} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \mu R(\mathbf{k}) \\ &\times \frac{1}{\left(\frac{\hbar^2\mathbf{k}^2}{2m} + 2\mu \right)^2} \frac{u_{\mathbf{k}-\mathbf{k}'}u_{\mathbf{k}'} + v_{\mathbf{k}-\mathbf{k}'}v_{\mathbf{k}'} + 2u_{\mathbf{k}'}v_{\mathbf{k}-\mathbf{k}'} + 2u_{\mathbf{k}-\mathbf{k}'}v_{\mathbf{k}'}}{\lambda_{\mathbf{k}-\mathbf{k}'} + \lambda_{\mathbf{k}'}} \left[(u_{\mathbf{k}'}v_{\mathbf{k}-\mathbf{k}'} \right. \\ &\left. + u_{\mathbf{k}-\mathbf{k}'}v_{\mathbf{k}'}) - \frac{2\mu}{\frac{\hbar^2\mathbf{k}^2}{2m} + 2\mu} (u_{\mathbf{k}-\mathbf{k}'}u_{\mathbf{k}'} + v_{\mathbf{k}-\mathbf{k}'}v_{\mathbf{k}'} + 2u_{\mathbf{k}'}v_{\mathbf{k}-\mathbf{k}'} + 2u_{\mathbf{k}-\mathbf{k}'}v_{\mathbf{k}'}) \right]. \end{aligned} \quad (252)$$

Inserting the definition of the transformation coefficients $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ in (100) and (101) and using the

explicit form of $\epsilon_{\mathbf{k}}$ in (90) and of $\lambda_{\mathbf{k}}$ in (96) leads to

$$\begin{aligned}
\overline{\psi_{01}(\mathbf{x})\psi_{11}(\mathbf{x})} &= -\sqrt{V} \int \frac{d^3k}{(2\pi)^3} \frac{5}{6\pi^2} \left(\frac{m\mu}{\hbar^2}\right)^{3/2} R(\mathbf{k}) \frac{\frac{\hbar^2\mathbf{k}^2}{2m} - 2\mu}{\left(\frac{\hbar^2\mathbf{k}^2}{2m} + 2\mu\right)^3} - \sqrt{V} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \mu R(\mathbf{k}) \\
&\times \frac{1}{\left(\frac{\hbar^2\mathbf{k}^2}{2m} + 2\mu\right)^2} \frac{1}{\sqrt{\frac{\hbar^2\mathbf{k}'^2}{2m}} \sqrt{\frac{\hbar^2\mathbf{k}'^2}{2m} + 2\mu} \sqrt{\frac{\hbar^2(\mathbf{k}-\mathbf{k}')^2}{2m}} \sqrt{\frac{\hbar^2(\mathbf{k}-\mathbf{k}')^2}{2m} + 2\mu}} \\
&\times \frac{1}{\sqrt{\frac{\hbar^2\mathbf{k}'^2}{2m}} \sqrt{\frac{\hbar^2\mathbf{k}'^2}{2m} + 2\mu} + \sqrt{\frac{\hbar^2(\mathbf{k}-\mathbf{k}')^2}{2m}} \sqrt{\frac{\hbar^2(\mathbf{k}-\mathbf{k}')^2}{2m} + 2\mu}} \left(\left\{ \mu^2 + \frac{\hbar^2\mathbf{k}'^2}{2m} \frac{\hbar^2(\mathbf{k}-\mathbf{k}')^2}{2m} \right. \right. \\
&+ \frac{1}{2}\mu \left[\frac{\hbar^2\mathbf{k}'^2}{2m} + \frac{\hbar^2(\mathbf{k}-\mathbf{k}')^2}{2m} \right] - \sqrt{\frac{\hbar^2\mathbf{k}'^2}{2m}} \sqrt{\frac{\hbar^2\mathbf{k}'^2}{2m} + 2\mu} \sqrt{\frac{\hbar^2(\mathbf{k}-\mathbf{k}')^2}{2m}} \\
&\times \left. \left. \sqrt{\frac{\hbar^2(\mathbf{k}-\mathbf{k}')^2}{2m} + 2\mu} \right\} - \frac{2\mu}{\frac{\hbar^2\mathbf{k}^2}{2m} + 2\mu} \left\{ \mu^2 + \frac{5}{2} \frac{\hbar^2\mathbf{k}'^2}{2m} \frac{\hbar^2(\mathbf{k}-\mathbf{k}')^2}{2m} + \frac{1}{2}\mu \left[\frac{\hbar^2\mathbf{k}'^2}{2m} \right. \right. \right. \\
&+ \left. \left. \left. \frac{\hbar^2(\mathbf{k}-\mathbf{k}')^2}{2m} \right] - \frac{3}{2} \sqrt{\frac{\hbar^2\mathbf{k}'^2}{2m}} \sqrt{\frac{\hbar^2\mathbf{k}'^2}{2m} + 2\mu} \sqrt{\frac{\hbar^2(\mathbf{k}-\mathbf{k}')^2}{2m}} \sqrt{\frac{\hbar^2(\mathbf{k}-\mathbf{k}')^2}{2m} + 2\mu} \right\} \right). \quad (253)
\end{aligned}$$

In order to rewrite this expression we scale the length by a characteristic length Λ . As (253) is a term of first order in the fluctuations and of second order in the disorder potential, we replace the chemical potential μ by μ_{00} . With (5) and (46) we obtain

$$\frac{\hbar^2\mathbf{k}^2}{2m} + 2\mu = \frac{\hbar^2}{2m} 16\pi a n \left(\frac{\mathbf{k}^2}{16\pi a n} + 1 \right), \quad (254)$$

which suggests the choice

$$\Lambda = \frac{1}{4\sqrt{\pi}\sqrt{an}}. \quad (255)$$

With the substitutions

$$\boldsymbol{\kappa} = \mathbf{k}\Lambda \wedge \boldsymbol{\kappa}' = \mathbf{k}'\Lambda, \quad (256)$$

we obtain

$$\begin{aligned}
\overline{\psi_{01}(\mathbf{x})\psi_{11}(\mathbf{x})} &= -\sqrt{V} \frac{5}{96\pi^5} \frac{m^2}{\hbar^4\Lambda^2} \int d^3\boldsymbol{\kappa} R\left(\frac{\boldsymbol{\kappa}}{\Lambda}\right) \frac{\boldsymbol{\kappa}^2 - 1}{(\boldsymbol{\kappa}^2 + 1)^3} - \sqrt{V} \frac{2}{(2\pi)^6} \frac{m^2}{\hbar^4\Lambda^2} \int d^3\boldsymbol{\kappa} \int d^3\boldsymbol{\kappa}' R\left(\frac{\boldsymbol{\kappa}}{\Lambda}\right) \\
&\times \frac{1}{(\boldsymbol{\kappa}^2 + 1)^2} \frac{1}{\sqrt{\boldsymbol{\kappa}'^2} \sqrt{\boldsymbol{\kappa}'^2 + 1} + \sqrt{(\boldsymbol{\kappa} - \boldsymbol{\kappa}')^2} \sqrt{(\boldsymbol{\kappa} - \boldsymbol{\kappa}')^2 + 1}} \frac{1}{\sqrt{\boldsymbol{\kappa}'^2} \sqrt{\boldsymbol{\kappa}'^2 + 1}} \\
&\times \frac{1}{\sqrt{(\boldsymbol{\kappa} - \boldsymbol{\kappa}')^2} \sqrt{(\boldsymbol{\kappa} - \boldsymbol{\kappa}')^2 + 1}} \left(\left\{ \frac{1}{4} + \boldsymbol{\kappa}'^2 (\boldsymbol{\kappa} - \boldsymbol{\kappa}')^2 + \frac{1}{4} [\boldsymbol{\kappa}'^2 + (\boldsymbol{\kappa} - \boldsymbol{\kappa}')^2] \right. \right. \\
&- \left. \left. \sqrt{\boldsymbol{\kappa}'^2} \sqrt{\boldsymbol{\kappa}'^2 + 1} \sqrt{(\boldsymbol{\kappa} - \boldsymbol{\kappa}')^2} \sqrt{(\boldsymbol{\kappa} - \boldsymbol{\kappa}')^2 + 1} \right\} - \frac{1}{\boldsymbol{\kappa}^2 + 1} \left\{ \frac{1}{4} + \frac{5}{2} \boldsymbol{\kappa}'^2 (\boldsymbol{\kappa} - \boldsymbol{\kappa}')^2 \right. \right. \\
&+ \left. \left. \frac{1}{4} [\boldsymbol{\kappa}'^2 + (\boldsymbol{\kappa} - \boldsymbol{\kappa}')^2] - \frac{3}{2} \sqrt{\boldsymbol{\kappa}'^2} \sqrt{\boldsymbol{\kappa}'^2 + 1} \sqrt{(\boldsymbol{\kappa} - \boldsymbol{\kappa}')^2} \sqrt{(\boldsymbol{\kappa} - \boldsymbol{\kappa}')^2 + 1} \right\} \right). \quad (257)
\end{aligned}$$

Using the definitions and expressions in (90), (96), (100), (101), (106)–(109), (113), (210) (214), (220),

(225) and the substitutions in (256) we obtain in the thermodynamic limit (211) and (212)

$$\begin{aligned}
\left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_{U^2, T=0} &= \sqrt{V} \frac{4}{(2\pi)^6} \frac{m^2}{\hbar^4 \Lambda^2} \int d^3\kappa \int d^3\kappa' \left(-\frac{R\left(\frac{\kappa-\kappa'}{\Lambda}\right)}{2\sqrt{\kappa^2}^3 \sqrt{\kappa^2+1}^3 \sqrt{\kappa'^2} \sqrt{\kappa'^2+1}} \right. \\
&\times \frac{1}{(\sqrt{\kappa^2} \sqrt{\kappa^2+1} + \sqrt{\kappa'^2} \sqrt{\kappa'^2+1})} \left\{ \frac{1}{4}(\kappa^2 + \kappa'^2) + \frac{1}{4} \right. \\
&- \frac{\kappa^2 \kappa'^2 - \frac{1}{2}(\kappa^2 + \kappa'^2) - \frac{1}{2}}{(\kappa - \kappa')^2 + 1} + \frac{2\kappa^2 \kappa'^2 - \frac{1}{4}(\kappa^2 + \kappa'^2) - \frac{1}{4}}{[(\kappa - \kappa')^2 + 1]^2} \left. \right\} \\
&+ \frac{R\left(\frac{\kappa-\kappa'}{\Lambda}\right) \left(\kappa^2 + \frac{1}{2}\right)}{\kappa^2(\kappa^2+1) \sqrt{\kappa'^2} \sqrt{\kappa'^2+1} (\sqrt{\kappa^2} \sqrt{\kappa^2+1} + \sqrt{\kappa'^2} \sqrt{\kappa'^2+1})^2} \\
&\times \left\{ \frac{1}{2} \kappa^2 \kappa'^2 + \frac{1}{4}(\kappa^2 + \kappa'^2) - \frac{1}{2} \sqrt{\kappa^2} \sqrt{\kappa^2+1} \sqrt{\kappa'^2} \sqrt{\kappa'^2+1} + \frac{1}{4} \right. \\
&- \frac{2\kappa^2 \kappa'^2 + \frac{1}{2}(\kappa^2 + \kappa'^2) - 2\sqrt{\kappa^2} \sqrt{\kappa^2+1} \sqrt{\kappa'^2} \sqrt{\kappa'^2+1} + \frac{1}{2}}{(\kappa - \kappa')^2 + 1} \\
&+ \left. \frac{\frac{5}{2} \kappa^2 \kappa'^2 + \frac{1}{4}(\kappa^2 + \kappa'^2) - \frac{3}{2} \sqrt{\kappa^2} \sqrt{\kappa^2+1} \sqrt{\kappa'^2} \sqrt{\kappa'^2+1} + \frac{1}{4}}{[(\kappa - \kappa')^2 + 1]^2} \right\} \\
&+ R\left(\frac{\kappa}{\Lambda}\right) \frac{\kappa^2 - \frac{1}{2}}{2\sqrt{\kappa^2}^3 \sqrt{\kappa^2+1}^3} \frac{\kappa'^2}{2(\kappa'^2+1)^2} \left. \right). \tag{258}
\end{aligned}$$

In the limit of the model given in (239) we obtain with the definition of the characteristic length Λ in (255)

$$2\overline{\psi_{01}(\mathbf{x})\psi_{11}(\mathbf{x})} + \left\langle \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \right\rangle_{U^2, T=0} = c \frac{m^2}{\hbar^4} naR, \tag{259}$$

where the dimensionless number c is defined by the following integrals:

$$\begin{aligned}
c &= -\frac{5}{3\pi^4} \int d^3\kappa \frac{\kappa^2 - 1}{(\kappa^2 + 1)^3} + \frac{1}{\pi^5} \int d^3\kappa \int d^3\kappa' \left[-\frac{1}{\sqrt{\kappa^2} \sqrt{\kappa'^2+1} \sqrt{(\kappa - \kappa')^2} \sqrt{(\kappa - \kappa')^2+1}} \right. \\
&\times \frac{1}{(\kappa^2 + 1)^2} \frac{1}{\sqrt{\kappa'^2} \sqrt{\kappa'^2+1} + \sqrt{(\kappa - \kappa')^2} \sqrt{(\kappa - \kappa')^2+1}} \left(\left\{ \kappa'^2(\kappa - \kappa')^2 + \frac{1}{4}[\kappa'^2 + (\kappa - \kappa')^2] \right. \right. \\
&- \left. \left. \sqrt{\kappa'^2} \sqrt{\kappa'^2+1} \sqrt{(\kappa - \kappa')^2} \sqrt{(\kappa - \kappa')^2+1} + \frac{1}{4} \right\} - \frac{1}{\kappa^2 + 1} \left\{ \frac{5}{2} \kappa'^2(\kappa - \kappa')^2 \right. \right. \\
&+ \left. \left. \frac{1}{4}[\kappa'^2 + (\kappa - \kappa')^2] - \frac{3}{2} \sqrt{\kappa'^2} \sqrt{\kappa'^2+1} \sqrt{(\kappa - \kappa')^2} \sqrt{(\kappa - \kappa')^2+1} + \frac{1}{4} \right\} \right) - \frac{1}{2\sqrt{\kappa^2}^3 \sqrt{\kappa^2+1}^3} \\
&\times \frac{1}{\sqrt{\kappa'^2} \sqrt{\kappa'^2+1} (\sqrt{\kappa^2} \sqrt{\kappa^2+1} + \sqrt{\kappa'^2} \sqrt{\kappa'^2+1})} \left\{ \frac{1}{4}(\kappa^2 + \kappa'^2) + \frac{1}{4} - \frac{\kappa^2 \kappa'^2 - \frac{1}{2}(\kappa^2 + \kappa'^2) - \frac{1}{2}}{(\kappa - \kappa')^2 + 1} \right. \\
&+ \left. \frac{2\kappa^2 \kappa'^2 - \frac{1}{4}(\kappa^2 + \kappa'^2) - \frac{1}{4}}{[(\kappa - \kappa')^2 + 1]^2} \right\} + \frac{\kappa^2 + \frac{1}{2}}{\kappa^2(\kappa^2+1) \sqrt{\kappa'^2} \sqrt{\kappa'^2+1} (\sqrt{\kappa^2} \sqrt{\kappa^2+1} + \sqrt{\kappa'^2} \sqrt{\kappa'^2+1})^2} \\
&\times \left\{ \frac{1}{2} \kappa^2 \kappa'^2 + \frac{1}{4}(\kappa^2 + \kappa'^2) - \frac{1}{2} \sqrt{\kappa^2} \sqrt{\kappa^2+1} \sqrt{\kappa'^2} \sqrt{\kappa'^2+1} + \frac{1}{4} - \frac{1}{(\kappa - \kappa')^2 + 1} \left[2\kappa^2 \kappa'^2 \right. \right. \\
&+ \left. \left. \frac{1}{2}(\kappa^2 + \kappa'^2) - 2\sqrt{\kappa^2} \sqrt{\kappa^2+1} \sqrt{\kappa'^2} \sqrt{\kappa'^2+1} + \frac{1}{2} \right] + \frac{1}{[(\kappa - \kappa')^2 + 1]^2} \left[\frac{5}{2} \kappa^2 \kappa'^2 \right. \right. \\
&+ \left. \left. \frac{1}{4}(\kappa^2 + \kappa'^2) - \frac{3}{2} \sqrt{\kappa^2} \sqrt{\kappa^2+1} \sqrt{\kappa'^2} \sqrt{\kappa'^2+1} + \frac{1}{4} \right] \right\} + \frac{\kappa^2 - \frac{1}{2}}{2\sqrt{\kappa^2}^3 \sqrt{\kappa^2+1}^3} \frac{\kappa'^2}{2(\kappa'^2+1)^2} \left. \right]. \tag{260}
\end{aligned}$$

5.4 Result for the Depletion at $T = 0$ in Case of Delta-Correlated Disorder Potentials

In this section we derived for (44) in the case of a delta-correlated random potential (238) at $T = 0$ with (249), (250), (259) and (260) the result

$$n_0 = n - c_{\text{Bog}} \sqrt{an}^3 - c_{\text{GP}} \frac{m^2}{\hbar^4} \sqrt{\frac{n}{a}} R - c_{\text{BogU}} \frac{m^2}{\hbar^4} anR, \tag{261}$$

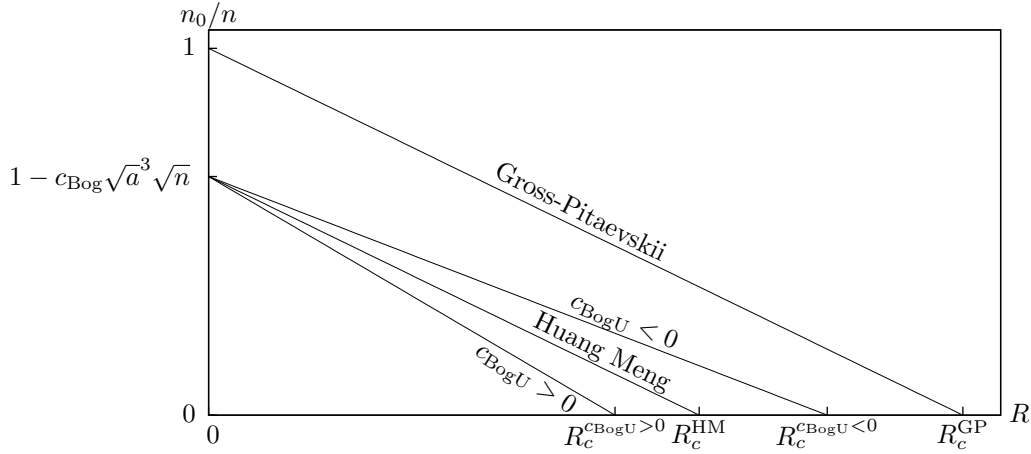


Figure 1: Plot of the ratio of the global condensate and particle density as function of the disorder strength with constant s-wave scattering length where the cases of a pure Gross-Pitaevskii theory, of the calculation by Huang and Meng in [8] and of our calculation for both signs of c_{BogU} in (261) are shown, where the corresponding critical disorder strength for a vanishing condensate are indicated.

which is plotted in Fig. 1 with the constants

$$c_{\text{Bog}} = \frac{8}{3\sqrt{\pi}}, \quad (262)$$

$$c_{\text{GP}} = \frac{1}{8\sqrt{\pi^3}}, \quad (263)$$

$$c_{\text{BogU}} = \frac{11}{3\pi^2} + c. \quad (264)$$

Note that c_{BogU} is composed out of the contribution due to the expansion of the chemical potential in (249) and (250) and c in (259). As the explicit value of c is not known, we have taken into account both signs of c_{BogU} in Fig. 1.

The case of a vanishing global condensate corresponds to a quantum phase transition from the condensate phase into a Bose glass phase [29] which was recently studied in a non-perturbative approach in Ref. [21]. Depending on the underlying theory as indicated in Fig. 1 we obtain from (261)

$$R_c^{\text{GP}} = \frac{\hbar^4 \sqrt{an}}{m^2 c_{\text{GP}}}, \quad (265)$$

$$R_c^{\text{HM}} = \frac{\hbar^4 \sqrt{an}}{m^2 c_{\text{GP}}} - \frac{\hbar^4 c_{\text{Bog}} a^2 n}{m^2 c_{\text{GP}}}, \quad (266)$$

$$R_c^{c_{\text{BogU}} > 0} = \frac{\hbar^4 \sqrt{an} - c_{\text{Bog}} a^2 n}{m^2 c_{\text{GP}} + c_{\text{BogU}} \sqrt{a^3 n}}. \quad (267)$$

5.5 Validity of the Expansion

The result given in (261) is valid as long as the depletion terms containing c_{Bog} , c_{GP} and c_{BogU} are small.

In the limit of a vanishing disorder potential, which is realized by $c_{\text{GP}} = c_{\text{BogU}} = 0$, this condition is obviously fulfilled for a small s-wave scattering length a .

A description of the system within a Gross-Pitaevskii theory would lead to the result given by (261) with $c_{\text{Bog}} = c_{\text{BogU}} = 0$ where the depletion term is small for small R or large a .

Taking into account both depletion processes our result (261) is valid for s-wave scattering lengths of the order of the s-wave scattering length $a(n, R)$ that minimizes the depletion.

Neglecting all depletion terms that are of first order in the fluctuations and of second order in the disorder potential yields the result (261) with $c_{\text{BogU}} = 0$ and corresponds to the result of Huang and Meng in [8] at $T = 0$. The corresponding depletion terms are minimal for the s-wave scattering length

$$a(n, R) = \frac{1}{8\sqrt{\pi}} \frac{m}{\hbar^2} \sqrt{\frac{R}{n}}, \quad (268)$$

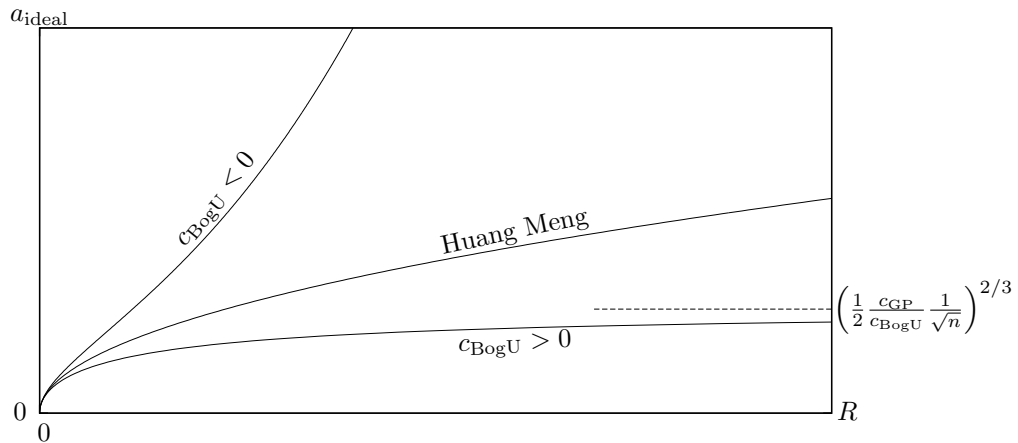


Figure 2: Plot of the ideal s-wave scattering length referring to the validity of our expansion for the result of Huang and Meng in (268) and for our result in (269), where both signs of c_{BogU} are considered, as a function of the disorder strength.

which can be easily seen by determining the zeroth of the first derivation of (261) with respect to a , realizing that a is in general positive, the depletion diverges to ∞ for $a \rightarrow 0$, $a \rightarrow \infty$ and inserting (262) and (263).

Considering the complete result (261) yields the condition

$$a^2 = \frac{1}{3} \frac{c_{\text{GP}}}{c_{\text{Bog}}} \frac{m^2 R}{\hbar^4 n} - \frac{2}{3} \frac{c_{\text{BogU}}}{c_{\text{Bog}}} \frac{m^2}{\hbar^4} R \frac{\sqrt{a}^3}{\sqrt{n}}. \quad (269)$$

Depending on the yet unknown sign of c_{BogU} , minimizing the depletion terms in (261) leads to a lowered result in case of $c_{\text{BogU}} > 0$, or a raised result in case of $c_{\text{BogU}} < 0$ for $a(n, R)$ as shown in Fig. 2. As Fig. 2 and (269) show we obtain in the case of $c_{\text{BogU}} < 0$ a rapidly divergent behavior and for $c_{\text{BogU}} > 0$ a minimizing s-wave scattering length with the upper boundary of $\left(\frac{1}{2} \frac{c_{\text{GP}}}{c_{\text{BogU}}} \frac{1}{\sqrt{n}}\right)^{2/3}$ because R is greater than zero due to (3) and (37).

6 Conclusion and Outlook

In this thesis we studied the depletion of the global Bose-Einstein condensate in a disorder potential within a Bogoliubov theory. We described a generalized Bogoliubov transformation and obtained the general temperature dependent expansion of correlations of the Bogoliubov quasi particle operators in \mathbf{k} -space up to second order in the interaction terms of (105) in (156)–(159), (166)–(169) and (203)–(206).

The result of the disorder depletion of Huang and Meng in [8] was reproduced on the level of a Gross-Pitaevskii theory (242) and the general expressions of the disorder depletion on the level of a Bogoliubov theory (257) and (258) were derived, where an expansion of the chemical potential as described in Section 5.2 has to be taken into account. In the case of a delta correlated random potential we obtain for the condensate density n_0 in terms of the particle density n at $T = 0$ the result (261), where the term proportional to naR corresponds to the yet unknown qualitative form of an additional disorder depletion within a Bogoliubov theory, as c denotes a numerical constant.

The systematic difference to the calculations of Huang and Meng is the treatment of the disorder average and the chosen Bogoliubov transformation. They directly decompose the field operators into fluctuations and a disorder averaged background which corresponds with (39) to the square root of the condensate density. In the following they choose a Bogoliubov transformation which introduces fluctuation operators that diagonalizes the not disordered system and a function of space that is proportional to the disorder potential which correspond to our quasi particle operators in Section 3 and $\psi_{01}(\mathbf{x})$. This combination of averaged background and Bogoliubov transformation excludes terms of the order of a Bogoliubov disorder depletion, as the transformation separates the influence of the disorder potential and of the fluctuations. In order to take into account the Bogoliubov disorder depletion, terms that are proportional to the product of quasi particle operators and components of the disorder potential have to be considered within the transformation which in this case would not diagonalize the system anymore. Note that the power of one half of the Fourier component of the disorder potential in the Bogoliubov transformation in Ref. [8] is a typo as a consideration of units show.

The next obligatory step would be the analytic or numerical determination of the constant c for the delta correlated disorder potential in (260), where the definition of c contains ultraviolet divergences, which can be regularized or renormalized, as well as infrared divergences. The latter infrared divergences are well known for beyond Bogoliubov calculations, firstly treated in [30] and reviewed e. g. in Ref. [31], and they occur in (260) due to the expansion of the Bogoliubov theory in the disorder potential. They could be treated e. g. within a renormalisation group approach [32]. Supplementary, further calculations could include a numerical or perturbative treatment of non-zero temperatures. Furthermore it could be physically interesting to treat also other models of the disorder correlation $R(\mathbf{x})$, e. g. a Gaussian function as done e. g. in Ref. [33], which contains the result of Huang and Meng in (2) in the limit of a vanishing correlation length, or a confining potential within a Thomas-Fermi approximation which would be a better approximation to experimental setups. In addition, our results can be specified to a pure Hartree-Fock theory by dropping the Bogoliubov terms in (22) which would lead to

$$\lambda_{\mathbf{k}} = \epsilon_{\mathbf{k}}, \quad (270)$$

$$u_{\mathbf{k}}^2 = 1, \quad (271)$$

$$v_{\mathbf{k}} = 0, \quad (272)$$

$${}^2A_{\mathbf{k},\mathbf{k}'}^i = {}^3A_{\mathbf{k},\mathbf{k}'}^i = {}^4A_{\mathbf{k},\mathbf{k}'}^i = 0, \quad (273)$$

in Section 3, where $i = U, U^2$. The resulting simplification of the expressions in Section 5 could make it possible to obtain a general temperature dependent result which could then be compared with the corresponding ones of Ref. [34].

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Declaration

I hereby declare that I wrote this thesis on my own and listed all used sources of information in the references.

Berlin, 20.07.2010

Christian Krumnow