

Bachelor Thesis

# Thermodynamic properties of quantum gases obeying fractional particle statistics

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# 1 Introduction to fractional particle statistics

In quantum mechanics, the indistinguishability of identical particles has deep consequences for the behaviour of many-particle quantum systems: Particles obey different particle statistics, that determine the occupation number of energy eigenstates at a specific thermodynamic temperature. In conventional quantum statistical mechanics (QSM) [1], we find two different categories of particles - bosons and fermions - obeying either Bose-Einstein (BE) statistics and Fermi-Dirac (FD) statistics, respectively. The deviations of these statistics from the Maxwell-Boltzmann statistics (MB) in the classical limit cause quantum phenomena like Bose-Einstein condensation, superfluidity, the Pauli exclusion principle being responsible for the structure of all visible matter and many more.

The indistinguishability of identical particles implies, that the physical state of a many-particle quantum system must *not* be changed by exchange of two or more identical particles. Regarding an  $N$ -particle wave function, we can only receive a phase factor after particle exchange.

Usually, one considered permutation of particle coordinates giving rise to particle exchange and received possible phase factors  $\lambda = \pm 1$  which are eigenvalues of a permutation operator acting on the many-particle wave function. From the mathematical point of this approach, there is no rule or restriction on which sign has to be taken. However, the so-called *symmetrization postulate* has been established: A many-particle state may be constructed by arbitrary linear combinations of direct product states of the different one-particle states. As an empirical result of numerous experiments, all many-particle wave functions are constructed in a way, that an arbitrary permutation of particles, i.e. a sequence of many single permutations of two particles, can only effect the wave-function in two ways:

Firstly  $\lambda = 1$  for all permutations, i.e. the initial many-particle wave function is totally symmetric. Secondly  $\lambda = \pm 1$ , depending on the permutation being even or odd, where the initial wave function is totally antisymmetric. Particles, that show first (second) kind of exchange behaviour in a large system of identical particles are being called bosons (fermions). The exchange behaviour directly determines the counting of accessible single-particle states: The fermionic particle exchange behaviour mathematically implies the Pauli principle: Each single-particle state can only be occupied by one particle, if we neglect internal quantum numbers. For the bosonic exchange behaviour we cannot derive any restriction of occupation numbers whatsoever. With this information, one can further derive the two kinds of particles statistics, namely FD and BE distribution functions.

In large contrast to the conventional theory of QSM, if one abandons the non-physical permutation of particles coordinates and instead correctly considers adiabatic transport giving rise to particle exchange, one finds that the structure of the phase factors is dependent on the spatial dimension and, therefore, the topology of the single-particle configuration space. It has been shown [2], that the dimension  $d$  of configuration space determines the allowed values for the phase factors: Again, the indistinguishability of identical particles is the root of causality. One has to exclude those points from the many-particles configuration space, that correspond to a coincidence of position of two particles. As a consequence, the group structure of particle trajectories is determining the effects of adiabatic transport giving rise to particle exchange.

In  $d \geq 3$  spatial dimensions, we find the permutation group  $S_N$  since all different paths for adiabatic exchange can be smoothly transformed into one another. This basically leads to the same characterization of particles being either bosons or fermions. It should be noted however, that with the correct arguments of Ref. [2], there is no symmetrization postulate needed in order to recover boson and fermion statistics.

The more ground-breaking result of this work was, that for systems of low spatial dimensionality, i.e.  $d = 1$  and  $d = 2$ , we find arbitrary exchange phase factors  $\chi(\alpha) = e^{i\alpha\pi}$  that in principle can lead to a continuum of statistics other than BE and FD, which we will call *fractional exchange statistics*.

In 2-dimensional systems, the group structure of paths of identical particles is found to be the braid group  $B_N$ , where we can find arbitrary phase factors other than  $\chi(0) = 1$  and  $\chi(1) = -1$  for bosons and fermions. Therefore, statistics other than FD and BE is expected and a wider category of particles is being labeled as *anyons*. One can imagine, that for each value of the statistical exchange parameter  $\alpha$  we find a different class of particles, including the special cases  $\alpha = 0$  and  $\alpha = 1$  for bosons and fermions, respectively.

From the arbitrary exchange factor, it is not directly possible to derive a generalized distribution function, similar to the procedure for FD and BE. Therefore, in  $d = 2$  the anyonic exchange phase factor  $\alpha$  was implemented in a dynamical model [3, 4]. Anyons are represented by a Hamiltonian for Newtonian particles of fictitious charge  $q$  with a delta-function flux tube vector potential attached. The phase factors arise due to interactions between particles and the **gauge field** of the flux tube of other particles while moving around them, known from the Aharonov-Bohm effect.

Since then, those model particles are used as synonym for anyons in the narrow sense. The flux tube model allows one to develop quantum mechanics of anyon systems. However, it induces non-trivial particle interactions, that so far made it impossible to receive the spectrum of an  $N$ -anyon problem for  $N > 2$ . As a consequence, statistical mechanics of a many-particle system of anyons in the narrow sense is very difficult to handle. An ideal gas of anyons can only be described so far by a virial expansion to the second order [5].

The ideal anyon gas was successfully applied to describe the fractional quantum Hall effect (FQHE), where a  $2d$  electron gas in a very strong magnetic field condenses to an incompressible quantum fluid with fractionally elementary charged excitations [6, 7]. The Laughlin wave-functions [8] describing quasiparticle and quasi-hole excitations in the FQHE ground state were found to obey anyonic exchange behaviour. The anyon parameter  $\alpha$  is fixed by the Landau level filling factor (LLFF), i.e. the number of electrons per number of Landau levels available. While at first only the notion of fractional electric charge has been experimentally observed [9], the direct evidence of the fractional exchange phase factors was seen in 2005 [10].

For a long time, *fractional exchange statistics* in  $1d$ , as predicted by the theory [2], were unexplored. This is because particle exchange in  $1d$  always goes along with a scattering process, which cannot easily be un-tangled. Still, based on the theory of exactly solvable  $1d$  interacting systems, models of anyons in  $1d$  have been proposed and investigated [11, 12, 13], which will be discussed in Section 6.

As seen from the boson and fermion case, the different exchange statistics  $\alpha$  of particles determine a possible limitation of the number of accessible particle states. Therefore, one must conjecture that fractional  $\alpha$  will have a non-trivial influence on the exclusion proper-

ties of particle states.

Motivated by his work on the FQHE hierarchy states [14], Haldane proposed a different theory to recover fractional particle statistics. Without specification of dimensionality, he introduced a **statistical interaction** leading to a generalized exclusion principle for quasiparticles as elementary excitations of condensed matter systems [15]. This is carried out through an interpolation in the combinatorial approach to QSM between boson and fermion borders. The physical idea is, that in a system of very strong or singular interaction between particles, the number of available slots in one quantum level depends on the number of particles already occupying this level. The category of particles obeying this generalized exclusion principle is called *exclusons* with *fractional exclusion statistics*  $g$ . Haldane claimed, that the exclusion parameter  $g$  for FQHE quasiparticles is equal to the anyonic exchange parameter  $\alpha$ . This claim was confirmed by numerical [16] as well as analytical means [17]. From Haldane's intermediate counting of states, it has been achieved [18] to develop QSM and, as a consequence, thermodynamics of a gas of *exclusion particles*. Since then, many authors have contributed to this research field and applied the ideal *exclusion gas* (IEG) model to several problems, as shown in Section 6.

Mathematically, the concept of *fractional exclusion statistics*  $g$  in arbitrary dimensions seems unrelated to the *fractional exchange statistics*  $\alpha$  resulting from braiding properties of particle trajectories in low spatial dimensions. Therefore, particles that each follow one of those statistics belong to *different classes of particles*. One main difference is that anyon statistics are usually assigned to Newtonian point particles, while *exclusion statistics* are assigned to elementary excitations of condensed matter system. More specifically, it is known that the ideal anyon gas is not equivalent to the IEG. In a couple of systems including FQHE systems, however, a coincidence of the two concepts was shown [13, 17]. One can imagine that the Haldane statistical interaction inducing fractional exclusion statistics of excitations in condensed matter systems is resulting from the anyon gauge field of real particles that form the system. We must not be confused by the fact, that the term anyon is widely used in the literature for any class of particles obeying fractional statistics, **including exclusons**.

In this work I will discuss the distribution functions and thermodynamic properties of an ideal quantum gas of identical particles obeying Haldane fractional exclusion statistics in  $d$  spatial dimensions, both for a free and a harmonically trapped ideal gas. We will start by developing QSM of *exclusion particles* by the state counting approach of Haldane in Section 2. An expression for the distribution function interpolating between FD and BE distributions is derived and further investigated. Also, the grand-canonical partition function is derived, which enables us to establish thermodynamics of an ideal gas of *exclusion particles* in Section 3. It is shown, that all thermodynamic functions can be expressed by evaluating a single type of integral function, which is then calculated numerically and by series representation in low-temperature/high-density and high-temperature/low-density limits. In Section 4, the theory of the ideal gas is transferred to the harmonically trapped gas. Section 5 will feature some slightly different results for *exclusion gases* with a state density, that is constant in energy. The last Section will feature some applications of the model of the IEG to physical problems.

## 2 Intermediate quantum statistical mechanics

From the starting point of a generalized exclusion principle, this Section will feature the development of QSM of an exclusion many-particle system by deriving the grand-canonical partition function as well as implicit and explicit expressions for the intermediate distribution function.

### 2.1 Haldane exclusion statistics

Haldane [15] implemented fractional statistics in the state counting approach to statistical mechanics, which is formulated as follows [1]: Regarding a system with an extensively large number  $N$  of identical particles, we divide all possible one-particle states into a big number of cells, each containing a large number  $p_i$  of states. By counting the number of all configurations, i.e. ways to put  $N_i$  particles into cell  $i$ , we receive the total number of many-particle states of  $N$  particles occupying a group of each  $p_i$  states

$$W = \prod_i \frac{(d(\{N_i\}) + N_i - 1)!}{N_i!(d(\{N_i\}) - 1)!}. \quad (1)$$

using the dimension  $d(\{N_i\})$  of the subspace  $H_{N_i}$  of the  $i$ -th cell many-particle Hilbert space with coordinates of  $N_i - 1$  particles held fixed.

Since for bosons we can have an arbitrary number of particles occupying a single state, the dimension of the Hilbert space does not reduce as particles are being added, i.e. we have  $d(\{N_i + \Delta N_i\}) = d(\{N_i\})$ . For fermions, the dimension is always reduced by 1 for each identical particle added, as a consequence of the common Pauli principle. This leads to the exclusion statistics parameter  $g = 0$  for bosons and  $g = 1$  for fermions.

The key argument of Haldane is, that the dimension  $d(\{N_i\})$  changes as we add new particles  $\Delta N_i$  to the system with fixed size and boundary conditions. Haldane defined the statistical interaction  $g$  through the linear dependence

$$g = -\frac{d(\{N_i + \Delta N_i\}) - d(\{N_i\})}{\Delta N_i}, \quad (2)$$

which can be expressed as a differential relation in the limit of large  $N_i$ :

$$g = -\frac{\partial d(\{N_i\})}{\partial N_i}. \quad (3)$$

An expression for the dimension  $d(\{N_i\})$  is found by integrating Eq. (3):

$$d(\{N_i\}) = p_i - g(N_i - 1) \quad (4)$$

The integration constant  $p_i$  may be regarded as the total number of possible single-particle states when one particle is present. From the last equation, we can see that the generalized dependence of the number of accessible states in the  $i$ -th cell on the number of particles  $N_i$  already present in the cell interpolates between the two border cases: For bosons, we have a complete *independence* on  $N_i$ , while for fermions *each* identical particle locally present reduces the number of available states by one.

Inserting (4) in the number of many-body states (1) yields

$$W = e^S = \prod_i \frac{[p_i + (N_i - 1)(1 - g)]!}{N_i! [p_i - gN_i - (1 - g)]!}. \quad (5)$$

We identify  $W$  with  $e^S$ , since the quantity  $S$ , which is proportional to the logarithm of the number of accessible states, is identical to the entropy if we calculate  $p_i$  by extremizing  $W$  with variation of  $N_i$ .

In the thermodynamic limit, both  $N_i$  and  $p_i$  will become infinite. However, the average occupation number  $\bar{n}_i = \frac{N_i}{p_i}$  of cell  $i$  stays finite. Considering a grand-canonical ensemble with thermodynamic temperature  $T$  and chemical potential  $\mu$ , the grand-canonical partition function is given by

$$Z = \sum_{\{N_i\}} W(\{N_i\}) e^{-\beta \sum_i N_i (\epsilon_i - \mu)} \quad (6)$$

with  $\beta = \frac{1}{k_B T}$ . The sum is to be taken over all sets of particle numbers  $\{N_i\}$  that satisfy the constraints of fixed energy

$$E = \sum_i \epsilon_i N_i, \quad (7)$$

and total particle number

$$N = \sum_i N_i. \quad (8)$$

The first equation expresses that the simple sum of the single-state eigenenergies gives us the total energy of the microsystem, which determines the class of systems regarded by this theory as *ideal gases*. In order to calculate the grand-canonical partition function, we can just consider the summand with the most probable set of particle numbers, i.e. the set  $\{N_i\}$  that maximizes  $W(\{N_i\})$  under the constraints (7) and (8). Equivalently, we maximize the quantity  $S = \ln W(\{N_i\})$  by variation of  $\{N_i\}$  to find the most probable average occupation number  $\bar{n}_i$  and then approximate  $\sum_{\{N_i\}} W(\{N_i\}) \approx W(\{N_i = p_i \bar{n}_i\})$ . Starting with

$$S = \ln \left( \prod_i \frac{[p + (N_i - 1)(1 - g)]!}{N_i! [p - gN_i - (1 - g)]!} \right), \quad (9)$$

we can carry out the logarithm of this product by using the Stirling expansion to first order, i.e.  $\ln N! \approx N \ln N$ , yielding

$$S = \sum_i \left\{ [p + (N_i - 1)(1 - g)] \ln [p + (N_i - 1)(1 - g)] - N_i \ln(N_i) - [p - gN_i - (1 - g)] \ln [p - gN_i - (1 - g)] \right\}. \quad (10)$$

The constraints (7) and (8) are implemented by the method of Lagrangian multipliers. The Lagrangian multipliers  $\beta$  for the fixed energy and  $\beta\mu$  for the particle number are added to the function  $S$  to give the new function

$$\tilde{S}(\{N_i\}, \beta, \mu) = S + \beta \left( E - \sum_i \epsilon_i N_i \right) + \beta\mu \left( \sum_i N_i - N \right), \quad (11)$$

which is extremized yielding

$$\frac{\partial \tilde{S}}{\partial N_i} = 0 = (1-g) \ln[p_i + N_i(1-g)] - \ln(n_i) + g \ln(p_i - gN_i) + \beta(\mu - \epsilon_i). \quad (12)$$

After taking the derivative, terms of  $O(1)$ , as 1 or  $g$  for instance, can be neglected compared to  $\ln(p)$ . We can now separate the terms and take the exponential of the equation yielding

$$e^{\beta(\epsilon_i - \mu) + \ln(N_i)} = e^{(1-g) \ln[p_i + N_i(1-g)]} e^{g \ln[p_i - gN_i]}. \quad (13)$$

Using the average occupation number  $\bar{n}_i$  we get

$$e^{\beta(\epsilon_i - \mu)} = \left[1 + \frac{1}{\bar{n}_i} - g\right]^{1-g} \left[\frac{1}{\bar{n}_i} - g\right]^g. \quad (14)$$

We now set  $e^{\beta(\epsilon_i - \mu)} = \xi_i$  and  $w(\xi_i) = \frac{1}{\bar{n}_i} - g$  to get the implicit equation

$$\xi_i = w(\xi_i)^g [1 + w(\xi_i)]^{1-g}. \quad (15)$$

The result  $w(\xi)$  of this equation gives us the most probable average occupation number

$$\bar{n}_i = \frac{1}{w(\xi_i) + g}. \quad (16)$$

One can directly see that the special cases of  $g = 0$  (bosons) and  $g = 1$  (fermions) reduce equation (15) to  $w = \xi - 1$  and  $w = \xi$ , respectively. The generalized occupation number function (16) is then identified with the well-known BE and FD distribution.

In the MB limit, with  $e^{\frac{\epsilon_i - \mu}{k_B T}} = w^g (1 + w)^{1-g}$  being very large, we can furthermore see, that also  $w^g$  and therefore  $w$  has to be very large, allowing us to perform the approximation  $\xi \approx w^g w^{1-g} = w$ . Neglecting  $g$  for large  $w$ , our distribution function (16) reduces to the well-known MB distribution  $n = e^{-\beta(\epsilon_i - \mu)}$  irrespective of  $g$ , which is also valid for FD and BE statistics with sufficiently low particle density and/or high temperature.

We will now regard properties of Eq. (15) at  $T \rightarrow 0$ . First of all,  $\xi$  is clearly non-negative. It is claimed [5, 18] that then also  $w(\xi)$  is non-negative, limiting the average occupation number to  $n_i \leq 1/g$  and therefore showing us pseudo-fermion behaviour with generalized exclusion for  $g \neq 0$ .

Especially at  $T = 0$ , for  $\epsilon_i < \mu$  we find  $w(\xi) \rightarrow 0$  and therefore  $n_i = \frac{1}{g}$ , because  $w(\xi) > 0$ . For  $\epsilon_i > \mu$  we find  $w(\xi) \rightarrow \infty$  and  $n_i = 0$ , from the argument above. This nicely illustrates a generalized Pauli exclusion principle, with exclusions of  $g \neq 0$  having a Fermi surface at  $T = 0$ , which separates occupied states of average occupation  $n_i = \frac{1}{g}$  from unoccupied states.

The border is found in momentum space, where all eigenstates are occupied with a mean number of  $\frac{1}{g}$  particles, up to the generalized Fermi momentum  $p_F$  which in  $d$  spatial dimensions is defined by the following equation:

$$N = \sum_{s_z} \sum_{|\mathbf{p}| \leq p_F} \frac{1}{g} = \frac{g_s V}{(2\pi\hbar)^d} \int_{|\mathbf{p}| \leq p_F} \frac{1}{g}. \quad (17)$$

The sum  $s_z$  is taken over different spin projections, yielding a degeneracy factor  $g_s = 2s + 1$  for spin  $s$  particles. The integral is calculated using the volume of the  $d$ -dimensional unit sphere  $K_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$  yielding

$$N = \frac{g_s V}{(2\pi\hbar)^d} \frac{1}{g} \frac{\pi^{d/2}}{\Gamma(d/2+1)} p_F^d \quad (18)$$

and finally

$$p_F = \frac{h}{\pi^{1/2}} \left[ \Gamma(d/2+1) \frac{g}{g_s} \frac{N}{V} \right]^{\frac{1}{d}}. \quad (19)$$

With the usual energy momentum relation of free mass  $m$  particles  $\epsilon(\mathbf{p}) = \frac{\mathbf{p}^2}{2m}$  we get the generalized Fermi energy

$$\epsilon_F = \left( \frac{\hbar^2}{2\pi m} \right) \left[ \Gamma(d/2+1) \frac{g}{g_s} \frac{N}{V} \right]^{2/d}. \quad (20)$$

This result can be reduced to the fermionic case, when setting  $g = 1$  and  $g_s = 2$ . In  $d = 3$  dimensions we receive the fermionic Fermi energy  $\bar{\epsilon}_F$

$$\bar{\epsilon}_F(d=3) = \frac{\hbar^2}{2m} \left( 3\pi^2 \frac{N}{V} \right)^{2/3} \quad (21)$$

with  $\Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi}$ . Leaving  $g_s$  as an unknown parameter, we get the following relation

$$\epsilon_F(g) = g^{2/d} \bar{\epsilon}_F \quad (22)$$

for free particles, which will be used in later results.

In order to connect statistical mechanics to the thermodynamics of an ideal gas of particles, we shall use the result for the most probable average occupation number  $\bar{n}_i$ , i.e. the distribution function (16), to get the grand-canonical partition function (6)

$$Z = W(\{n_i = p_i \bar{n}_i\}) e^{-\beta \sum_i n_i (\epsilon_i - \mu)} \quad (23)$$

and then derive the grand-canonical thermodynamic potential  $\Omega = -k_B T \ln Z$ :

$$\Omega = -k_B T \left[ \ln W(\{n_i = p_i \bar{n}_i\}) - \beta \sum_i n_i (\epsilon_i - \mu) \right]. \quad (24)$$

While the first summand is known from the calculation of the entropy in Eq. (10), the second summand can be rewritten using Eq. (15) giving the simple result

$$\Omega = -k_B T \sum_i p_i \ln \left[ \frac{1 + (1-g)\bar{n}_i}{1 - g\bar{n}_i} \right]. \quad (25)$$

Using the result for the distribution function  $\bar{n}_i$  in Eq. (16) we get

$$\Omega = -k_B T \sum_i p_i \left( 1 + \frac{1}{w_i} \right) \quad (26)$$

with  $w_i$  according to Eq. (15).

## 2.2 Explicit expressions for the distribution function

As a second part of this section, we will shed more light on possibilities to get explicit expressions of the intermediate distribution function. In order to illustrate how the interpolation between FD and BE distribution functions looks like, we will calculate the distribution function for special values of  $g$ . Also, one can give useful series representations for the distribution function  $n_g(\epsilon)$  and  $w_g(\epsilon)$ .

### 2.2.1 Evaluation in special cases

If we look at the implicit equation (15), one can see that for the values  $g = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$ , we will receive an algebraic equation of order  $n \leq 4$ , which can generally be solved analytically. To this end have to obtain solutions  $w(\xi)$  of the following equations:

$$\begin{aligned} w^2 + w - \xi^2 &= 0 & ; g = \frac{1}{2}, \\ w^3 + w^2 - \xi^3 &= 0 & ; g = \frac{2}{3}, \\ w^3 + 2w^2 + w - \xi^3 &= 0 & ; g = \frac{1}{3}, \\ w^4 + w^3 - \xi^4 &= 0 & ; g = \frac{3}{4}, \\ w^4 + 3w^3 + 3w^2 + w - \xi^4 &= 0 & ; g = \frac{1}{4}. \end{aligned} \quad (27)$$

Solving the equation for  $g = \frac{1}{2}$  is rather easy, yielding one positive solution

$$n(\epsilon) = \frac{2}{\sqrt{1 + 4e^{2\beta(\epsilon - \mu)}}}. \quad (28)$$

The equations of order  $n = 3$  can be solved by Cardano's method. However, a calculation via Mathematica for  $n \geq 3$  is more comfortable. For both values  $g = \frac{1}{3}, \frac{2}{3}$  we find one real solution which can be written down in closed form using the abbreviation  $a_{1,2}(\xi) = 27\xi^3 + 3\sqrt{3}\sqrt{27\xi^6 \pm 4\xi^3} \pm 2$ :

$$n_{1,2}(\epsilon) = \frac{3}{\left(\frac{2}{a_{1,2}(\xi)}\right)^{1/3} + \left(\frac{a_{1,2}(\xi)}{2}\right)^{1/3} \mp 1}. \quad (29)$$

The corresponding expressions for  $g = \frac{1}{4}, \frac{3}{4}$  are more complicated and will not be given here. In Fig. 1, we can see the distribution functions  $n(\epsilon)$  for various values of  $g$ , compared to the FD ( $g = 1$ ) and BE ( $g = 0$ ) distributions.

### 2.2.2 Lagrange reversion theorem

While at first it was thought that an analytic expression for the distribution function could only be given for the special cases above, it has been achieved [19, 20] to solve the implicit equation (15) for any value of the parameter  $g$  using the Lagrange reversion theorem [21], which will be illustrated in the following: In general, a solution  $y$  of an implicit equation of the type

$$y = a + x\phi(y) \quad (30)$$

can be obtained by introducing an arbitrary function  $g$ , which can be expressed as

$$g(p_i) = \int \delta(f(p))g(p)|f'(p_i)|dp, \quad (31)$$

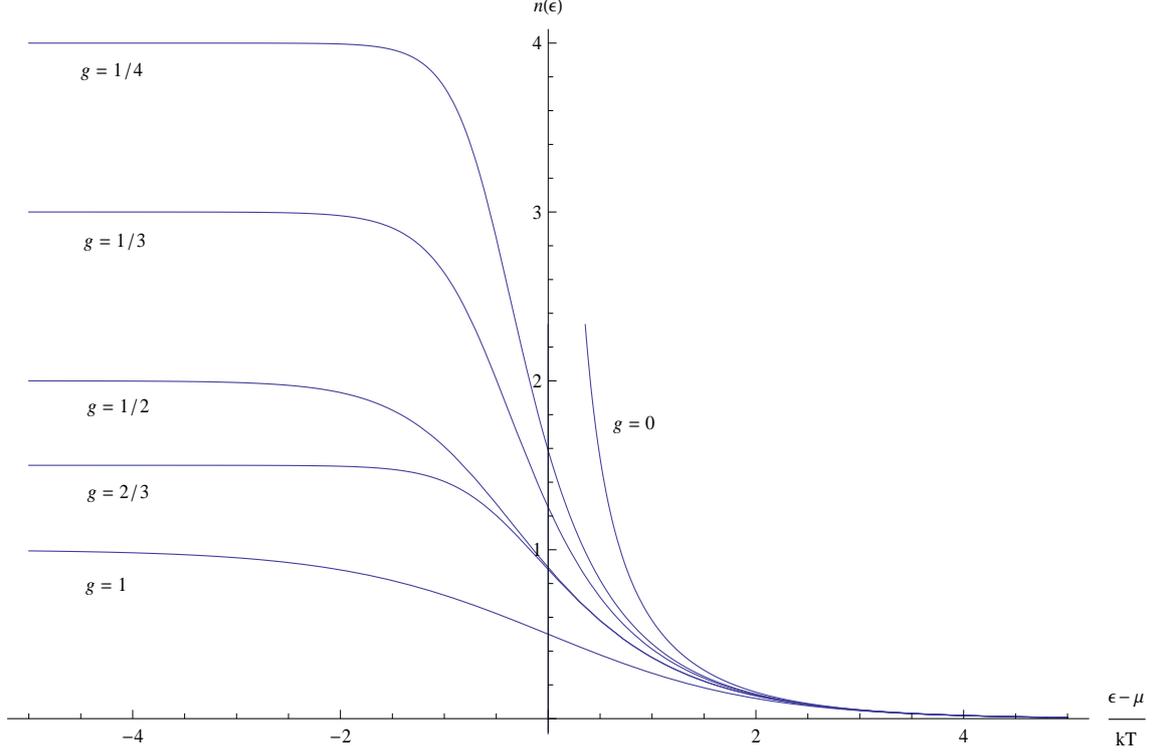


Figure 1: Distribution functions  $n_g(\epsilon)$  for several values of  $g$ .

using the definition of the delta distribution and  $\delta(f(p)) = \sum_i \frac{1}{|f'(p_i)|} \delta(p - p_i)$  for  $f(p_i) = 0$ . With the implicit equation (30), we can write  $f(p) = a + x\phi(p) - p$  with  $f(y) = 0$  yielding

$$g(y) = \int \delta(x\phi(p) - p + a) g(p) [1 - x\phi'(p)] dp \quad (32)$$

for  $x\phi'(p) < 1$ , which will limit the parameter range. With the integral representation of the Dirac  $\delta$ -function we get:

$$g(y) = \int \int dp \frac{dk}{2\pi} e^{\{ik[x\phi(p) - p + a]\}} g(p) [1 - x\phi'(p)]. \quad (33)$$

Splitting the exponential function and using the exponential row yields

$$g(y) = \sum_{n=0}^{\infty} \int \int dp \frac{dk}{2\pi} \frac{[ikx\phi(p)]^n}{n!} g(p) [1 - x\phi'(p)] e^{ik(a-p)}. \quad (34)$$

We can use the  $n$ -th derivation of  $e^{ik(a-p)}$  with respect to  $a$  to rewrite the factor  $(ik)^n$ :

$$g(y) = \sum_{n=0}^{\infty} \left( \frac{\partial}{\partial a} \right)^n \int \int dp \frac{dk}{2\pi} \frac{[x\phi(p)]^n}{n!} g(p) [1 - x\phi'(p)] e^{ik(a-p)}. \quad (35)$$

The integral over  $k$  is then identified with  $\delta(a - p)$ , allowing us to also perform the integral over  $p$  yielding

$$g(y) = \sum_{n=0}^{\infty} \left( \frac{\partial}{\partial a} \right)^n \left\{ \frac{[x\phi(a)]^n}{n!} g(a) [1 - x\phi'(a)] \right\}. \quad (36)$$

The sum is now being split into two parts

$$g(y) = \sum_{n=0}^{\infty} \left( \frac{\partial}{\partial a} \right)^n \frac{x^n}{n!} \phi(a)^n g(a) - \sum_{n=0}^{\infty} \left( \frac{\partial}{\partial a} \right)^n \frac{x^{n+1}}{n!} \phi(a)^n \phi'(a) g(a). \quad (37)$$

For the first sum, we will split the term for  $n = 0$  which is equal to  $g(a)$ , and for the second sum we will use the identity  $\phi(a)^n \phi'(a) g(a) = \frac{1}{n+1} (\phi^{n+1})' g(a)$  and shift the indices from  $n$  to  $n + 1$  yielding

$$g(y) = g(a) + \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial a} \right)^n \frac{x^n}{n!} \phi(a)^n g(a) - \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial a} \right)^{n-1} \frac{x^n}{n!} (\phi(a)^n)' g(a). \quad (38)$$

By using  $\left( \frac{\partial}{\partial a} \right)^n (\dots) = \left( \frac{\partial}{\partial a} \right)^{n-1} (\dots)'$  in the first sum, we can reassemble the two sums:

$$g(y) = g(a) + \sum_{n=1}^{\infty} \frac{x^n}{n!} \left( \frac{\partial}{\partial a} \right)^{n-1} \left\{ [\phi(a)^n g(a)]' - [\phi(a)^n]' g(a) \right\}. \quad (39)$$

With  $[\phi(a)^n g(a)]' = \phi(a)^n g'(a) + [\phi(a)^n]' g(a)$  we get the final result:

$$g(y) = g(a) + \sum_{n=1}^{\infty} \frac{x^n}{n!} \left( \frac{\partial}{\partial a} \right)^{n-1} [\phi(a)^n g'(a)]. \quad (40)$$

For  $g$  being the identity function, i.e.  $g(y) = y$ , we finally receive a general solution  $y$  of the implicit equation (30):

$$y = a + \sum_{n=1}^{\infty} \frac{x^n}{n!} \frac{d^{n-1} \phi(y)^n}{dy^{n-1}} \Big|_{y=a}. \quad (41)$$

### 2.2.3 Series representations

With the Lagrange reversion theorem we now have a tool to obtain explicit series representations of the distribution function, which is only given implicitly so far. The first application of (41) is to expand  $w_i(\xi)$  by rewriting the implicit formula (15) alternatively with introducing  $\Theta = \xi^{1/g}$  and  $\nu = \frac{1-g}{g}$ :

$$w = \Theta(1 + w)^{-\nu}. \quad (42)$$

We now set  $y = w$ ,  $a = 0$ ,  $x = \Theta$  and  $\phi(w) = (1 + w)^{-\nu}$ . Because of the restriction  $x\phi'(p) < 1$  in the derivation above, the following expansion is valid for *small values* of  $\xi$ :

$$w = \sum_{n=1}^{\infty} \frac{\Theta^n}{n!} \frac{d^{n-1} \phi^n(w)}{dw^{n-1}} \Big|_{w=0}. \quad (43)$$

One can find a closed expression for the  $n$ -th derivation of  $\phi^n(w) = (1 + w)^{-\nu n}$ , using a generalized faculty, the Pochhammer symbol  $(z)_m = \frac{\Gamma(z+m)}{\Gamma(z)}$  and therefore we get [19, 20]:

$$w = \Theta \left( 1 + \nu \sum_{m=1}^{\infty} (-1)^m \frac{\Theta^m}{m!} (m\nu + \nu + 1)_{m-1} \right). \quad (44)$$

In order to validate this expansion, we can reduce it to the results for the distribution function of the previous chapter for specific values of  $g$ . As an example we consider  $g = \frac{1}{2}$ , where the corresponding function  $w(\xi)$  to the distribution function (28) is

$$w_{1/2}^s = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\xi^2}. \quad (45)$$

This function will be expressed by its Taylor expansion, using the expansion  $f(x) = \sqrt{1+x}$  around the point  $x = 0$ :

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^n}{(1-2n)4^n} x^n. \quad (46)$$

With  $x = 4\xi^2$  we find:

$$w_{1/2}^s = -\frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \frac{(-1)^n}{(1-2n)} \xi^{2n}. \quad (47)$$

We can split the term for  $n = 0$ , which is equal to 1, from the sum yielding

$$w_{1/2}^s = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} \frac{(-1)^n}{(1-2n)} \xi^{2n}. \quad (48)$$

This expansion is now compared to the result of the Lagrange reversion (44) for  $g = \frac{1}{2}$ . We can see that  $\nu = 1$  and yield

$$w_{1/2}^L = \xi^2 \left[ 1 + \sum_{m=1}^{\infty} (-1)^m \frac{(m+2)_{m-1}}{m!} \xi^{2m} \right]. \quad (49)$$

With the definition of the Pochhammer symbol applied to integer numbers, we find  $(m+2)_{m-1} = \frac{(2m)!}{(m+1)!}$ . We can now include the term  $m = 0$ , which is equal to 1, in the sum:

$$w_{1/2}^L = \sum_{m=0}^{\infty} \frac{(2m)!}{m!(m+1)!} (-1)^m \xi^{2(m+1)}. \quad (50)$$

We now switch indices to  $n = m + 1$  yielding:

$$w_{1/2}^L = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(2n-2)!}{(n-1)!n!} \xi^{2n} = \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!n}{(n!)^2(-1)(2n-1)(2n)} \xi^{2n}. \quad (51)$$

We immediately see, that this is the same expression as the Taylor expansion (48) of our result  $w_{1/2}^s$ :

$$w_{1/2}^L = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} \frac{(-1)^n}{(1-2n)} \xi^{2n}. \quad (52)$$

A second application of the Lagrange reversion theorem is possible for *large values* of  $\xi$  [20, 22]. We can see from Eq. (14), that the distribution function can be defined through the following implicit equation [22]

$$\bar{n} = \frac{1}{\xi} \psi(g, \bar{n}) \quad (53)$$

with  $\psi(g, \bar{n}) = (1 - g\bar{n})^g [1 + (1 - g)\bar{n}]^{1-g}$ . This implicit equation can be solved by applying Eq. (41) with  $x = \frac{1}{\xi}$ ,  $a = 0$  and  $\phi(y) = \psi(g, \bar{n})$ . We find

$$n(g, \xi) = \sum_{m=0}^{\infty} \frac{1}{(m+1)! \xi^{m+1}} \left( \frac{d^m}{dn^m} \phi^{m+1}(g, n) \right) \Big|_{n=0}. \quad (54)$$

Again, we find a closed expression for the  $m$ -th derivation of  $\phi^{m+1}(g, n)$  yielding

$$n(g, \xi) = \sum_{m=0}^{\infty} \frac{(gm + g - m)_m}{m!} \frac{(-1)^m}{\xi^{m+1}}. \quad (55)$$

### 2.3 Mutual statistics

In some physical problems, we will have to regard multiple particle species, with the single particle Hilbert space dimension depending on the number of particles  $\{N_i\}$  of all particle species  $i$ . The term particle species is ment in a general way, distinguishing also identical particles with different momenta. Following Wu [18], we regard more than one particle species  $i, j$ , and alternatively write the change of dimension of the single particle Hilbert space of particle species  $i$  as

$$\Delta d_i = - \sum_j g_{ij} \Delta N_j \quad (56)$$

which defines the *mutual statistics*  $g_{ij}$ . For the total number of  $N_i$ -particle states with  $N_i$  particles occupying  $p_i$  single particle states we receive:

$$W = \prod_i \frac{[p_i + N_i - 1 - \sum_j g_{ij} (N_j - \delta_{ij})]!}{(N_i)! [p_i - 1 - \sum_j g_{ij} (N_j - \delta_{ij})]!}. \quad (57)$$

This generalized statistics becomes important in real physical problems being described by fractional exclusion statistics. It is not possible to derive an expression for the distribution function as in the non-mutual case. However, one can find [18] implicit equations for the occupation numbers  $n_i$  and give an expression similar to (25) for the grand-canonical thermodynamic potential, that in principle allows developing all other thermodynamic functions.

## 3 Thermodynamics of the ideal exclusion gas

We regard the thermodynamics of an ideal gas of exclusion particles with no mutual statistics, being held in a  $d$ -dimensional rigid box of volume  $V$ . In this section, we will show that all thermodynamic functions of interest can be expressed and calculated by evaluating a single type of integral function. The connection between the statistical mechanics of the previous chapter and the thermodynamics is being established by further evaluating the thermodynamic potential (26). In the thermodynamic limit, when the eigenenergies lie very close together, one can transform the sum over the energy levels  $\epsilon_i$  into an integral yielding

$$\Omega = -k_B T \int_0^{\infty} d\epsilon D(\epsilon) \ln \left( 1 + \frac{1}{w} \right) \quad (58)$$

with a specific state density  $D(\epsilon)$  that results from the energy levels  $\epsilon_{\mathbf{p}}$  of different problems. For an ideal gas of particles we find the  $d$ -dimensional state density as

$$D(\epsilon) = c\epsilon^{d/2-1}, \quad (59)$$

with the constant

$$c = g_s \left( \frac{2\pi m}{\hbar^2} \right)^{d/2} \frac{V}{\Gamma(d/2)}. \quad (60)$$

Inserting the ideal state density into the thermodynamic potential and integrating by parts, we receive

$$\Omega = -k_B T c \left[ \frac{1}{d/2} \epsilon^{d/2} \ln \left( 1 + \frac{1}{w} \right) \Big|_0^\infty - \int_0^\infty \frac{1}{d/2} \epsilon^{d/2} \frac{1}{1+1/w} \frac{\partial(1/w)}{\partial \epsilon} d\epsilon \right]. \quad (61)$$

The first term vanishes, regarding  $w \rightarrow \infty$  for  $\epsilon \rightarrow \infty$ . We can evaluate the derivative using  $\frac{\partial \epsilon}{\partial w} = \frac{w(1+w)}{g+w} k_B T$  from Eq. (15), yielding

$$\Omega = -\frac{c}{d/2} \int_0^\infty \frac{\epsilon^{d/2}}{w+g} d\epsilon. \quad (62)$$

Introducing the dimensionless energy  $t = \beta \epsilon$ , the dimensionless chemical potential  $\eta = \beta \mu$  and the dimensionless parameter  $x = \frac{\lambda_T^d}{g_s} \frac{N}{V}$  with the thermal wavelength  $\lambda_T = \sqrt{\frac{\beta \hbar^2}{2\pi m}}$  we find

$$\Omega = -\frac{k_B T}{x} I_{d/2}(\eta, g) \quad (63)$$

with the Calogero-Sutherland integral function

$$I_n(\eta, g) = \frac{1}{\Gamma(n+1)} \int_0^\infty dt \frac{t^n}{w(e^{t-\eta}) + g}, \quad (64)$$

for which we find

$$\frac{\partial}{\partial \eta} I_n(\eta, g) = I_{n-1}(\eta, g). \quad (65)$$

From the grand-canonical thermodynamic potential given in Eq. (63) we can derive all thermodynamic functions of interest. For the equation of state  $\Omega = -pV$  we find

$$\frac{pV}{Nk_B T} = \frac{1}{x} I_{d/2}(\eta, g). \quad (66)$$

The total particle number  $N$  is defined by

$$N = -\frac{\partial \Omega}{\partial \mu} \Big|_{T,V} = -\beta \frac{\partial \Omega}{\partial \eta} \Big|_{T,V}. \quad (67)$$

Using relation (65) we find

$$x = I_{d/2-1}(\eta, g). \quad (68)$$

The internal energy is obtained from the thermodynamic potential  $\Omega = -k_B T \ln Z$  as follows

$$E = -\frac{\partial \ln Z}{\partial \beta} \Big|_{T,V} \quad (69)$$

yielding

$$\frac{E}{k_B T N d/2} = \frac{1}{x} I_{d/2}(\eta, g). \quad (70)$$

The universal relation between pressure and internal energy is found to be  $pV = \frac{2}{d}E$ . We receive identical results, if we simply express the total energy  $E$  and particle number  $N$  of the system, given by Eqs. (7) and (8), as integrals in the thermodynamic limit:

$$N = \int_0^\infty \bar{n}(\epsilon) D(\epsilon) d\epsilon, \quad (71)$$

and

$$E = \int_0^\infty \epsilon \bar{n}(\epsilon) D(\epsilon) d\epsilon. \quad (72)$$

We now have to find methods for evaluating the integrals  $I_n$  either numerically for finite temperatures or analytically in special cases, for instance in the limit of high density/low temperature (large values of parameter  $x$ ) or low density/high temperature (small values of parameter  $x$ ).

In the integrand of (64), we recognize the distribution function  $\bar{n}_i = \frac{1}{w+g}$ . In the previous Section we have seen, that, in principle, it is possible to obtain an expression for the distribution function of exclusion states with energy  $\epsilon$  for any value of the statistical parameter  $g$ . However, regarding the expressions for  $g = \frac{1}{3}, \frac{2}{3}$ , we can also imagine the distribution functions becoming arbitrarily complicated for most values of  $g$ . Therefore, the finite and low-temperature thermodynamics are evaluated without explicit specification of the function  $w(\xi)$ . Afterwards, the implicit equation (15) is being used to re-express the Calogero-Sutherland integral functions as integrals over the variable  $w$ : We find  $t = \ln \left[ \left( \frac{w}{w_0} \right)^g \left( \frac{1+w}{1+w_0} \right)^{1-g} \right]$ , while  $w_0$  is defined by the equation  $e^{-\beta\mu} = w_0^g (1+w_0)^{1-g}$ , and  $\frac{dt}{dw} = \frac{w+g}{w(w+1)}$  yielding

$$I_n(w_0, g) = \frac{1}{\Gamma(n+1)} \int_{w_0}^\infty \frac{\left\{ \ln \left[ \left( \frac{w}{w_0} \right)^g \left( \frac{1+w}{1+w_0} \right)^{1-g} \right] \right\}^n}{w(w+1)} dw. \quad (73)$$

### 3.1 Finite temperature properties

In order to calculate the Integrals  $I_n$  in Eq. (73) numerically, we will consider them as functions of the **fugacity**  $z = e^{\beta\mu}$ , by expressing the variable  $w_0$  through  $\frac{1}{z} = w_0^g (1+w_0)^{1-g}$ . For  $g = 0$  the fugacity lies in the range of  $0 \leq z \leq 1$ , since the chemical potential is strictly non-positive. For all other values of  $g$  we can find any positive value of the fugacity. Large values of  $z$  correspond to low-temperature or high-density limits, while small values of  $z$  resemble high-temperature or low-density properties. As  $z$  approaches zero, we receive MB behaviour.

The results of the numerical evaluation of the Calogero-Sutherland integral functions  $I_{1/2}$  and  $I_{3/2}$  over a range of the fugacity can be viewed in Fig. 2 and Fig. 3. It is the shape of these functions, that influences the variation of thermodynamic functions over different values of  $g$ . For  $g \geq \frac{1}{2}$  the functions are concave, while for  $g \leq \frac{1}{2}$  they are convex for small

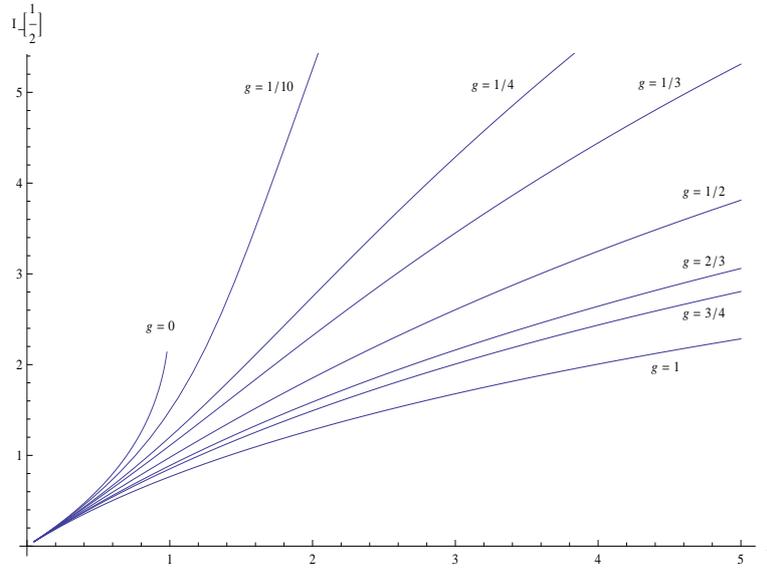


Figure 2: Calogero-Sutherland integral function  $I_{1/2}$  as a function of the fugacity  $z = e^{\beta\mu}$  for several values of  $g$ .

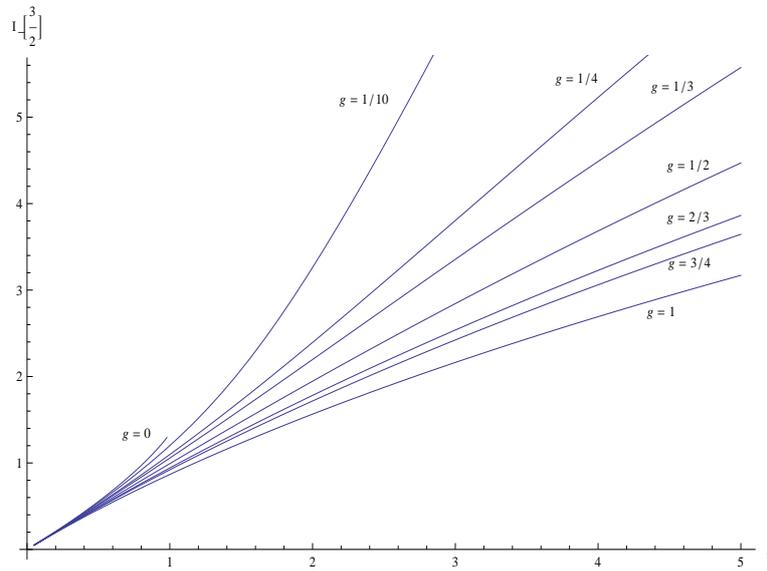


Figure 3: Calogero-Sutherland integral function  $I_{3/2}$  as a function of the fugacity  $z = e^{\beta\mu}$  for several values of  $g$ .

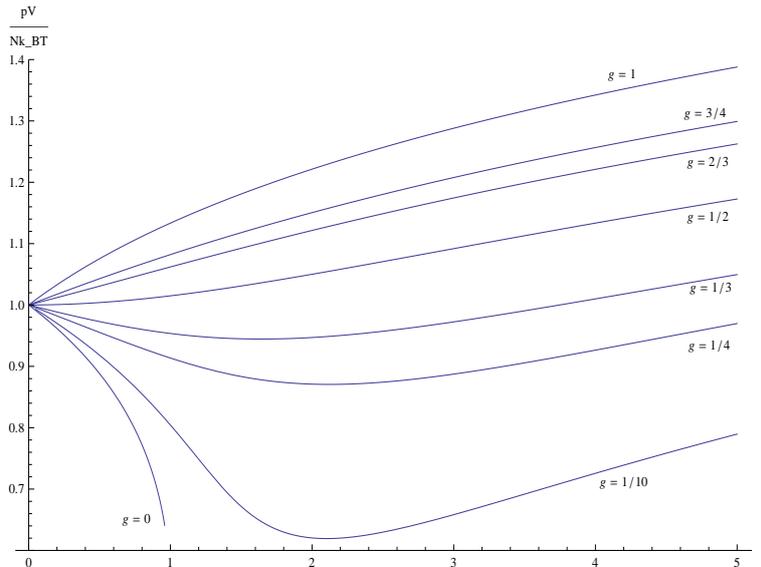


Figure 4: Thermodynamic equation of state in  $d = 3$  as a function of the fugacity  $z = e^{\beta\mu}$  for several values of  $g$ .

values of  $z$  and then change to concave behaviour.

The pressure as a function of  $z$  can be received from Eqs. (66) and (68):

$$\frac{pV}{Nk_B T} = \frac{I_{d/2}(z, g)}{I_{d/2-1}(z, g)}. \quad (74)$$

As an example, the case  $d = 3$  is given in Fig. 4. One can see, that all curves for  $g > \frac{1}{2}$  show a fermionic monotone increase. In contrast, for  $g < \frac{1}{2}$  we find a bosonic decrease of the pressure for small values of  $z$  until a minimum is reached and then the curve switches to fermionic behaviour. For all values of  $g$ , we obtain the classical result  $pV = Nk_B T$  for an ideal gas in the MB limit as  $z \rightarrow 0$ . One can imagine, that the statistical interactions of exclusions is divided into a long-ranged attractive part, i.e. for low-density limits and small fugacity, and a short-ranged repulsive part, i.e. for high-density limits and large fugacity. The attractive part, in other words the initial negative slope, is only found for  $g < \frac{1}{2}$ , while the repulsive part is found for all  $g > 0$ . One can see, that the repulsive interaction prevents all systems of  $g \neq 0$  from forming a Bose-Einstein condensate.

The isochoric specific heat  $C_V = \left(\frac{\partial E}{\partial T}\right)_V$  can be calculated from  $pV = \frac{2}{d}E$  and Eqs. (65) and (74) yielding [23]

$$\frac{C_V}{k_B N} = d/2 \left[ (d/2 + 1) \frac{I_{d/2}(z, g)}{I_{d/2-1}(z, g)} - d/2 \frac{I_{d/2-1}(z, g)}{I_{d/2-2}(z, g)} \right]. \quad (75)$$

In order to eliminate the fugacity and, hence, the chemical potential  $\mu$  from the thermodynamic formula we extract the temperature from Eq. (68) and identify all other constants with the Fermi energy  $\bar{\epsilon}_F$  (20) for  $g = 1$  yielding

$$\frac{T}{T_F} = \frac{\Gamma(d/2 + 1)}{[I_{d/2-1}(z, g)]^{2/d}} \quad (76)$$

using the characteristic Fermi temperature  $T_F = \frac{\bar{\epsilon}_F}{k_B}$ . From this we can obtain an explicit temperature dependence of thermodynamic quantities at constant particle density  $\rho = \frac{N}{V}$ . Further numerical results for thermodynamic functions of the IEG as well as other spatial dimensional cases are calculated in an analogue way, and may be viewed in the literature [23].

### 3.2 Low-temperature properties

The low-temperature properties of the IEG for  $g > 0$  can be established by a variation of the fermionic Sommerfeld expansion with our generalized exclusion statistics  $n_g(\epsilon)$ .

#### 3.2.1 Generalized Sommerfeld expansion

We take a step back and consider general integrals of the type  $I = \int_0^\infty F(\epsilon)n(\epsilon)d\epsilon$  with any thermodynamic function  $F(\epsilon)$ , which will be left arbitrary, so that we can get results not only for an ideal gas of exclusions, but also for a modified density function for harmonically trapped exclusions. Following Ref. [19], we solve these integrals by a variation of the fermionic Sommerfeld expansion [24]: we first substitute  $y = \beta\epsilon - \beta\mu$  and then divide the resulting integral into parts yielding

$$I = \int_0^\infty F(\epsilon)n(\epsilon)d\epsilon = \frac{1}{\beta} \int_{-\beta\mu}^\infty dy F\left(\frac{y}{\beta} + \mu\right) n(y) = \frac{1}{\beta} \int_0^\infty dy F n + \frac{1}{\beta} \int_{-\beta\mu}^0 dy F n. \quad (77)$$

Next, we add and subtract  $\int_{-\beta\mu}^0 F a$  with  $a$  being an arbitrary constant, which we will later choose to be  $a = \frac{1}{g}$ , so that the expansion reduces to the known fermionic result for  $g = 1$ :

$$I\beta = \int_0^\infty dy F\left(\frac{y}{\beta} + \mu\right) n(y) + \int_{-\beta\mu}^0 dy F\left(\frac{y}{\beta} + \mu\right) [n(y) - a] + \int_{-\beta\mu}^0 dy F\left(\frac{y}{\beta} + \mu\right) a. \quad (78)$$

With substituting  $y$  by  $-y$  in the second integral and setting the lower border to  $-\infty$  since  $\beta\mu \gg 1$  for low enough temperatures, we find the two functions  $F_\pm = F(\mu \pm k_B T y)$  giving us:

$$I\beta = a \int_{-\beta\mu}^0 dy F + \int_0^\infty dy F_+ n(y) + \int_0^\infty dy F_- [n(-y) - a]. \quad (79)$$

The functions  $F_\pm$  can now be expanded around  $\mu$ . The similar Taylor expansions allow us to combine the two integrals yielding

$$I\beta = a \int_{-\beta\mu}^0 dy F + \int_0^\infty dy \left( \sum_{m=0}^\infty \frac{(k_B T)^m}{m!} \frac{d^m F(\mu)}{dy^m} y^m \right) \{n(y) + (-1)^m [n(-y) - a]\}. \quad (80)$$

Dividing by  $\beta = \frac{1}{k_B T}$  and resubstituting to  $\epsilon$  we find

$$I = a \int_0^\mu F(\epsilon)d\epsilon + \sum_{i=0}^\infty C_i(a) \frac{1}{i!} \frac{d^i F(\mu)}{d\epsilon^i} (k_B T)^{i+1} \quad (81)$$

with the coefficients

$$C_i(a) = \int_0^\infty dy y^i \{n(y) + (-1)^i [n(-y) - a]\}. \quad (82)$$

By the study of specific values of the statistical parameter  $g$ , it has been claimed [25] that  $C_0(a) = 0$ . However, there also exists a proof of this assumption [22] for all values of  $g$ . This is an important fact, since we will use the expansion to the first (or higher) order in all following calculations and neglect the zeroth order. For the purpose of the proof and for further evaluation, we will reexpress (82) as an integral over the variable  $w$  by using Eq. (15). Following Ref. [22] we will do this in multiple steps: First, we split the integral (82) into two parts:

$$C_i(g) = A_i(g) + B_i(g), \quad (83)$$

with the respective abbreviations

$$\begin{aligned} A_i(g) &= \int_0^\infty dy \frac{y^i}{w(g, e^y) + g} \\ B_i(g) &= (-1)^i \int_0^\infty dy y^i \left[ \frac{1}{w(g, e^{-y}) + g} - \frac{1}{g} \right]. \end{aligned} \quad (84)$$

Secondly, we substitute  $e^y = \xi$  in the first integral  $A_i(g)$  and  $e^{-y} = \xi$  in the second integral  $B_i(g)$  yielding

$$A_i(g) = \int_1^\infty \frac{\ln^i \xi}{\xi} \frac{1}{w + g} d\xi, \quad (85)$$

and

$$B_i(g) = - \int_1^0 \frac{1}{\xi} (-1)^i (-\ln \xi)^i \left( \frac{1}{w + g} - \frac{1}{g} \right) d\xi = \int_0^1 \frac{\ln^i \xi}{\xi} \left( \frac{1}{w + g} - \frac{1}{g} \right) d\xi. \quad (86)$$

In order to get to the variable  $w$ , we take the logarithm of Eq. (15) and find the substitution  $\ln \xi = g \ln w + (1 - g) \ln(1 + w)$ . We will not have to make a substitution for  $\xi$  itself, since it gets cancelled from the expression for the derivative:

$$\frac{d\xi}{dw} = gw^{g-1}(1+w)^{1-g} + (1-g)w^g(1+w)^{-g} = \frac{g}{w}\xi + \frac{1-g}{w+1}\xi = \frac{\xi(w+g)}{w(w+1)}. \quad (87)$$

The first integral  $A_i(g)$  in terms of  $w$  results in:

$$A_i(g) = \int_{w_1}^\infty dw \frac{[g \ln w + (1 - g) \ln(1 + w)]^i}{w(w + 1)} \quad (88)$$

with  $w_1 = w(\xi = 1, g)$ , which satisfies the equation

$$1 = w_1^g (1 + w_1)^{1-g}. \quad (89)$$

For the second integral  $B_i(g)$  we find similarly:

$$\begin{aligned} B_i(g) &= \int_0^{w_1} dw [g \ln w + (1 - g) \ln(1 + w)]^i \frac{\xi(w+g)}{w(w+1)} \frac{g-w-g}{\xi g(w+g)} \\ &= - \int_0^{w_1} dw \frac{[g \ln w + (1 - g) \ln(1 + w)]^i}{g(w+1)}. \end{aligned} \quad (90)$$

The borders are found according to the asymptotic behaviour of the function  $w(\xi)$ , as investigated in Section 2. For proving  $C_0(g) = 0$ , we will calculate  $A_0(g)$  and  $B_0(g)$  and use the following identity

$$\ln w_1 - \ln(1 + w_1) = -\frac{1}{g} \ln(1 + w_1), \quad (91)$$

which is obtained by taking the logarithm of Eq. (89). For  $B_0(g)$  we get

$$B_0(g) = -\frac{1}{g} \int_0^{w_1} dw \frac{1}{w+1} = -\frac{1}{g} \ln(1+w_1), \quad (92)$$

and for  $A_0(g)$

$$A_0(g) = \int_{w_1}^{\infty} dw \frac{1}{w(w+1)} = \ln(w_1+1) - \ln w_1. \quad (93)$$

With Eqs. (83) and (91) we immediately receive  $C_0(g) = 0$  for all values of  $g$ .

Having shown  $C_0(g) = 0$ , we can expand the results of the generalized Sommerfeld expansion to the first order yielding:

$$\int_0^{\infty} d\epsilon n(\epsilon) F(\epsilon) = a \int_0^{\mu} F(\epsilon) + C_1(a) \left. \frac{dF(\epsilon)}{d\epsilon} \right|_{\epsilon=\mu} (k_B T)^2 + O(k_B T)^3. \quad (94)$$

This result can be reduced to the results of the fermionic Sommerfeld expansion when setting  $g = 1$ . The first order of the fermionic Sommerfeld expansion [24] is in principle:

$$\int_0^{\infty} d\epsilon F(\epsilon) n_{\text{FD}}(\epsilon) = \int_0^{\mu} d\epsilon F(\epsilon) + \frac{F'(\mu)}{\beta^2} \int_{-\infty}^{\infty} dy y \eta(y) + O(T^4) \quad (95)$$

with  $y = \beta(\epsilon - \mu)$  and  $\eta(y) = n_{\text{FD}}(\epsilon) - \Theta(\mu - \epsilon)$ . We now regard the generalized expansion to the first order, setting  $g = \frac{1}{a} = 1$ . As shown earlier, our distribution function  $n(\epsilon)$  becomes the Fermi-Dirac distribution  $n_{\text{FD}}(\epsilon)$  and the expansion becomes

$$\int_0^{\infty} d\epsilon F(\epsilon) n_{\text{FD}}(\epsilon) = \int_0^{\mu} d\epsilon F(\epsilon) + \frac{F'(\mu)}{\beta^2} \int_0^{\infty} dy \{n_{\text{FD}}(y) - [n_{\text{FD}}(-y) - 1]\} y. \quad (96)$$

To show, that the second integral is equal to the fermionic one, we substitute  $y$  by  $-y$  in the second part of the integral. This allows us to combine both parts of the integral using the stepfunction  $\Theta(-y) = \Theta(\mu - \epsilon)$ :

$$\int_0^{\infty} dy n_{\text{FD}}(y) y + \int_{-\infty}^0 dy [n_{\text{FD}}(y) - 1] y = \int_{-\infty}^{\infty} dy [n_{\text{FD}}(y) - \Theta(-y)] y. \quad (97)$$

As a result, our generalized Sommerfeld expansion reduces to the fermionic expansion for  $g = 1$ :

$$\int_0^{\infty} d\epsilon F(\epsilon) n_{\text{FD}}(\epsilon) = \int_0^{\mu} d\epsilon F(\epsilon) + \frac{F'(\mu)}{\beta^2} \int_{-\infty}^{\infty} dy [n_{\text{FD}}(y) - \Theta(\mu - \epsilon)] y + O(k_B T)^3. \quad (98)$$

### 3.2.2 Expansion of thermodynamic functions

The Sommerfeld expansion can now be applied to give low-temperature expansions for the Calogero-Sutherland integral functions  $I_n(\eta, g)$  in Eq. (64). This allows us to get expressions for the chemical potential  $\mu$ , the internal energy  $E$  and the specific heat  $C_V$  of an ideal gas of exclusons. For now it is sufficient to use the expansion to the first order to show the principle. For the final results of the thermodynamic functions, terms of higher order in Eq. (81) are not altered by the following procedure and can later on easily be added to the

results for the expansion of order one. We will now choose  $a = \frac{1}{g}$  and leave the coefficient  $C_1(g)$

$$C_1(g) = \int_0^\infty dy \left\{ n_g(y) - \left[ n_g(-y) - \frac{1}{g} \right] \right\} y \quad (99)$$

as a constant. A concrete evaluation of the coefficients will follow later. Writing equation (71) alternatively as

$$\rho = \frac{N}{V} = K \int_0^\infty d\epsilon F(\epsilon) n(\epsilon) \quad (100)$$

with  $K = \frac{1}{(2\pi)^{d/2} \Gamma(d/2)} \left( \frac{m}{\hbar^2} \right)$  and expanding it to the second power yields

$$\rho = K \left[ \frac{1}{g} \int_0^\mu d\epsilon F(\epsilon) + C_1(g) \frac{dF(\epsilon)}{d\epsilon} \Big|_{\epsilon=\mu} (k_B T)^2 + O(k_B T)^3 \right]. \quad (101)$$

We split the integral as follows

$$a \int_0^\mu d\epsilon F(\epsilon) = \frac{1}{g} \int_0^{\epsilon_F} d\epsilon F(\epsilon) + \frac{1}{g} \int_{\epsilon_F}^\mu d\epsilon F(\epsilon). \quad (102)$$

The first integral is equal to  $\frac{\rho}{K}$ , since it equals Eq. (101) at  $T = 0$ , where  $\mu = \epsilon_F$ . For the second integral, we use the mean value theorem for integration with  $\tilde{\epsilon}$  being a value between  $\mu$  and  $\epsilon_F$ :

$$\frac{1}{g} \int_0^\mu d\epsilon F(\epsilon) = \frac{\rho}{K} + \frac{1}{g} (\mu - \epsilon_F) F(\tilde{\epsilon}). \quad (103)$$

Since we are approximating for low temperatures, we can assume that  $\tilde{\epsilon}$  and the chemical potential are very close to the Fermi energy  $\epsilon_F$ . Inserting the previous equation into Eq. (101) yields

$$\rho = K \left[ \frac{\rho}{K} + \frac{1}{g} (\mu - \epsilon_F) F(\epsilon_F) + C_1(g) F'(\epsilon_F) (k_B T)^2 + O(k_B T)^3 \right]. \quad (104)$$

This allows us to give an expression for the chemical potential with arbitrary state density  $F(\epsilon)$ , additionally considering terms up to  $O(k_B T)^4$ , yielding

$$\begin{aligned} \frac{\mu}{\epsilon_F} &= 1 - g C_1(g) \epsilon_F \frac{F'(\epsilon_F)}{F(\epsilon_F)} \left( \frac{k_B T}{\epsilon_F} \right)^2 - g C_2(g) (\epsilon_F)^2 \frac{F''(\epsilon_F)}{F(\epsilon_F)} \left( \frac{k_B T}{\epsilon_F} \right)^3 \\ &\quad - g C_3(g) (\epsilon_F)^3 \frac{F'''(\epsilon_F)}{F(\epsilon_F)} \left( \frac{k_B T}{\epsilon_F} \right)^4 - \frac{g}{\epsilon_F} O(k_B T)^5. \end{aligned} \quad (105)$$

Next, we can evaluate the energy equation (72) with the Sommerfeld expansion:

$$E = \frac{1}{g} \int_0^\mu d\epsilon D(\epsilon) \epsilon + C_1(g) \frac{d(D(\epsilon)\epsilon)}{d\epsilon} \Big|_{\epsilon=\mu} (k_B T)^2 + O(k_B T)^3. \quad (106)$$

Again, we split the integral yielding

$$E = \frac{1}{g} \int_0^{\epsilon_F} d\epsilon D(\epsilon) \epsilon + a (\mu - \epsilon_F) D(\tilde{\epsilon}) \tilde{\epsilon} + C_1(a) (k_B T)^2 [D'(\mu) \mu + D(\mu)] + O(k_B T)^3. \quad (107)$$

We will label the first integral as  $\int_0^{\epsilon_F} d\epsilon D(\epsilon)\epsilon = E_0$ , since it is temperature independent. Next, we approximate  $\tilde{\epsilon} \approx \epsilon_F$  and  $\mu \approx \epsilon_F$  and replace  $(\mu - \epsilon_F)$  with Eq. (105):

$$E = \frac{E_0}{g} - D(\epsilon_F)\epsilon_F g C_1(g) \frac{D'(\epsilon_F)}{D(\epsilon_F)} (k_B T)^2 \frac{1}{g} + C_1(g) (k_B T)^2 (D'(\epsilon_F)\epsilon_F + D(\epsilon_F)) + O(k_B T)^3. \quad (108)$$

By adding the terms of higher order in  $k_B T$  the internal energy is expressed by

$$E_g(T) = \frac{E_0}{g} + C_1(g) D(\epsilon_F) (k_B T)^2 + C_2(g) [D''(\epsilon_F)\epsilon_F + 2D'(\epsilon_F)] (k_B T)^3 + C_3(g) [D'''(\epsilon_F)\epsilon_F + 3D''(\epsilon_F)] (k_B T)^4 + O(k_B T)^5. \quad (109)$$

For the specific heat  $C_V = \frac{\partial E}{\partial T}$  we receive

$$C_{g,V}(T) = 2C_1(g) D(\epsilon_F) k_B^2 T + 3C_2(g) [D''(\epsilon_F)\epsilon_F + 2D'(\epsilon_F)] k_B^3 T^2 + 4C_3(g) [D'''(\epsilon_F)\epsilon_F + 3D''(\epsilon_F)] k_B^4 T^3 + O(T^4). \quad (110)$$

### 3.2.3 Evaluation of Sommerfeld coefficients

To fully evaluate the thermodynamic expressions of the ideal exclusion gas, we must at least be able to calculate the coefficient  $C_1(g)$ . With the explicit expressions for the distribution function  $n(\epsilon)$  derived in Section 2, this is not possible. However, as seen above, we can use Eq. (15) to re-express the integral  $C_i(g)$  as two integrals  $A_i(g)$  in (88) and  $B_i(g)$  in (90) over the variable  $w$ . These integrals shall now be further evaluated. Following Ref. [22], we start by rewriting the integrand in  $A_i(g)$

$$A_i(g) = \int_{w_1}^{\infty} dw \frac{\{g[\ln w - \ln(1+w)] + \ln(1+w)\}^i}{w(w+1)} \quad (111)$$

with the binomial theorem:

$$A_i(g) = \sum_{m=0}^i \binom{i}{m} g^m T_{i,m}, \quad (112)$$

where we have introduced the abbreviation

$$T_{i,m} = \int_{w_1}^{\infty} dw \frac{[\ln w - \ln(1+w)]^m [\ln(1+w)]^{i-m}}{w(w+1)}. \quad (113)$$

The summand  $T_{i,i}$  for  $m = i$  can be evaluated if we see that the denominator is the inner derivative of the numerator:

$$\int_{w_1}^{\infty} dw \frac{[\ln w - \ln(1+w)]^i}{w(w+1)} = -\frac{[\ln(w_1) - \ln(1+w_1)]^{i+1}}{i+1} = \frac{(-1)^i}{g^{i+1}(i+1)} [\ln(1+w_1)]^{i+1}. \quad (114)$$

Identity (91) has been used in the second step. Since we now know the stem function of  $f(w) = \frac{[\ln w - \ln(1+w)]^i}{w(w+1)}$ , we can partially integrate  $T_{i,m}$  in order to adjust the denominator of the integrand in  $A_i(g)$  to  $\frac{1}{w+1}$ , as in  $B_i(g)$  yielding

$$T_{i,m} = \frac{(-1)^m}{g^{m+1}(m+1)} [\ln(1+w_1)]^{i+1} + \frac{m-i}{m+1} U_{i,m+1}(w_1, \infty) \quad (115)$$

with the integral

$$U_{i,m+1}(w_1, \infty) = \int_{w_1}^{\infty} \frac{[\ln w - \ln(1+w)]^{m+1} [\ln(1+w)]^{i-m-1}}{w+1} dw. \quad (116)$$

Expressing  $A_i(g)$  in (112) with the two previous results yields

$$A_i(g) = g^i T_{i,i} + \sum_{m=0}^{i-1} \binom{i}{m} \frac{g^m}{m+1} \left\{ \frac{(-1)^m}{g^{m+1}} [\ln(1+w_1)]^{i+1} + (m-i) U_{i,m+1}(w_1, \infty) \right\}. \quad (117)$$

By using the following two identities

$$\sum_{m=0}^i \binom{i}{m} \frac{(-1)^m}{m+1} = \frac{1}{i+1}; \quad \binom{i}{m} \frac{i-m}{m+1} = \binom{i}{m+1}, \quad (118)$$

we find

$$A_i(g) = \frac{1}{g(i+1)} [\ln(1+w_1)]^{i+1} - \sum_{m=0}^{i-1} \binom{i}{m+1} g^m U_{i,m+1}(w_1, \infty). \quad (119)$$

We will now regard the integral  $B_i(g)$  in (90)

$$B_i(g) = - \int_0^{w_1} dw \frac{\{g[\ln w \ln(1+w)] + \ln(1+w)\}^i}{g(w+1)} \quad (120)$$

and expand the integrand according to the binomial theorem yielding

$$B_i(g) = - \sum_{m=0}^i \binom{i}{m} g^{m-1} U_{i,m}(0, w_1). \quad (121)$$

with  $U_{i,m}(0, w_1) = \int_0^{w_1} dw \frac{[\ln w \ln(1+w)]^m [\ln(1+w)]^{i-m}}{w+1}$ . We split the term  $U_{i,0}$  for  $m=0$  which is easily calculated as

$$U_{i,0} = \int_0^{w_1} \frac{[\ln(1+w)]^i}{w+1} dw = \frac{1}{i+1} [\ln(1+w_1)]^{i+1}, \quad (122)$$

and shift indices in the remaining sum from  $m$  to  $m+1$  yielding

$$B_i(g) = - \frac{1}{g(i+1)} [\ln(1+w_1)]^{i+1} - \sum_{m=0}^{i-1} \binom{i}{m+1} U_{i,m+1}(0, w_1). \quad (123)$$

The results for  $A_i(g)$  and  $B_i(g)$  can be added, cancelling all notion of  $w_1$ , to give a closed form for Eq. (83):

$$C_i(g) = - \sum_{m=0}^{i-1} \binom{i}{m+1} g^m \int_0^{\infty} dw \frac{[\ln w - \ln(1+w)]^{m+1} [\ln(1+w)]^{i-m-1}}{w+1}. \quad (124)$$

By substituting  $w = \frac{1-t}{t}$  we obtain:

$$C_i(g) = \sum_{m=0}^{i-1} \binom{i}{m+1} (-1)^{i-m-1} g^m \int_0^1 \frac{dt}{t} [\ln(1-t)]^{m+1} (\ln t)^{i-m-1}. \quad (125)$$

From this equation we can see, that  $C_i(g)$  is a polynomial in  $g$  of order  $i - 1$ . The general structure is as follows:

$$C_i(g) = \sum_{m=0}^{i-1} \binom{i}{m+1} g^m (-1)^{i-m-1} I_{i-m,m+1}. \quad (126)$$

The coefficients  $I_{i-m,m+1}$  are integrals of the type

$$I_{q,p} = \int_0^1 \frac{dt}{t} [\ln(1-t)]^p (\ln t)^{q-1}. \quad (127)$$

Those integrals have solutions that can be expressed [26] in terms of the Riemann zeta function. As a result, the first coefficients are as follows [5]:

$$\begin{aligned} C_1(g) &= \zeta(2) = \frac{\pi^2}{6}, \\ C_2(g) &= 2(1-g)\zeta(3), \\ C_3(g) &= \frac{3}{2}(4-g-4g^2)\zeta(4). \end{aligned} \quad (128)$$

Only in the strict fermion case  $g = 1$  the even Sommerfeld coefficients vanish.

### 3.2.4 Results

To conclude the low-temperature properties of the exclusons gas, we insert the results for the generalized Sommerfeld coefficients  $C_i(g)$  into the expressions for thermodynamic functions within the Sommerfeld expansion of Section 3.2.2. From Eq. (105), we get the actual dependence of the chemical potential  $\mu$  on  $g$  by considering that the Fermi energy  $\epsilon_F$  is also dependent on  $g$  according to  $\epsilon_F \propto g^{2/d}$ , for the IEG. We therefore use Eq. (22) to write the chemical potential  $\mu_g(T)$  in terms of the fermionic Fermi energy  $\bar{\epsilon}_F$  and the Fermi temperature  $T_F = \frac{\bar{\epsilon}_F}{k_B}$ , and see, that all arising powers of  $g$  cancel out:

$$\begin{aligned} \frac{\mu_g(T)}{\bar{\epsilon}_F} &= g^{2/d} - gC_1(g)\bar{\epsilon}_F \frac{F'(\epsilon_F)}{F(\epsilon_F)} \left(\frac{T}{T_F}\right)^2 - gC_2(g)(\bar{\epsilon}_F)^2 \frac{F''(\epsilon_F)}{F(\epsilon_F)} \left(\frac{T}{T_F}\right)^3 \\ &\quad - gC_3(g)(\bar{\epsilon}_F)^3 \frac{F'''(\epsilon_F)}{F(\epsilon_F)} \left(\frac{T}{T_F}\right)^4 + O\left(\frac{T}{T_F}\right)^5. \end{aligned} \quad (129)$$

Using the state density of  $d$ -dimensional ideal exclusons (59), we get the chemical potential  $\mu_g(T)$  of the  $d$ -dimensional ideal exclusion gas:

$$\begin{aligned} \frac{\mu_g(T)}{\bar{\epsilon}_F} &= g^{2/d} - g^{1-2/d}(d/2-1)\zeta(2) \left(\frac{T}{T_F}\right)^2 - g^{1-4/d}(1-g)(d/2-1)(d/2-2)2\zeta(3) \left(\frac{T}{T_F}\right)^3 \\ &\quad - g^{1-6/d}(4-g-4g^2)(d/2-1)(d/2-2)(d/2-3)\frac{3}{2}\zeta(4) \left(\frac{T}{T_F}\right)^4 + O\left(\frac{T}{T_F}\right)^5. \end{aligned} \quad (130)$$

From Eq. (110) we obtain the specific heat:

$$\begin{aligned} \frac{C_{g,V}}{\bar{\epsilon}_F^{d/2} k_B K V} &= g^{1-2/d} 2\zeta(2) \left(\frac{T}{T_F}\right) + g^{1-4/d} (1-g) d(d/2-1) 3\zeta(3) \left(\frac{T}{T_F}\right)^2 \\ &+ g^{1-6/d} (4-g-4g^2) d(d/2-1)(d/2-2) 3\zeta(4) \left(\frac{T}{T_F}\right)^3 \\ &+ O\left(\frac{T}{T_F}\right)^4. \end{aligned} \quad (131)$$

We can see, that for  $d = 2$  all terms of the expansions vanish. For the chemical potential we receive the temperature independent result  $\frac{\mu}{\bar{\epsilon}_F} = g$ , similar to the low-temperature behaviour of the ideal Fermi gas in  $2d$ , where we receive  $\mu = \bar{\epsilon}_F$  [5] for low but finite temperatures. For the heat capacity we obtain the linear temperature dependence as  $\frac{C_V}{Nk_B} = \frac{2}{x}\zeta(2)$ , *independent* on the statistical parameter  $g$ . This was known for a long time from the compliance of Fermi and Bose gases in  $d = 2$  [27]. Corrections to these asymptotic results in  $d = 2$  are further investigated in Section 5.

### 3.3 High-temperature properties

In this section, we will expand the equation of state in the virial form, i.e. in form of a power series with respect to powers of the particle density  $\rho = \frac{N}{V}$ . In Section 2.2.2 we derived a series expansion for the the distribution function  $n(\epsilon) = \frac{1}{w(\xi)+g}$  in (55) for large values of  $\xi$ :

$$n_g(\xi) = \sum_{m=0}^{\infty} b_m(g) \left(\frac{1}{\xi}\right)^{m+1} \quad (132)$$

with the coefficients  $b_m(g) = (-1)^m \frac{(gm+g-m)_m}{m!}$ , yielding  $b_0(g) = 1$  and  $b_1(g) = -(2g-1)$  etc. High-temperature/low density limits correspond to small values of the fugacity  $z = e^{\beta\mu}$  and to large values of  $\xi = \frac{1}{z}e^t$ . Therefore, the integrals  $I_n = \Gamma(n+1)^{-1} \int_0^{\infty} dt n_g(\xi) t^n$  in (64) can be expanded as:

$$I_n = \frac{1}{\Gamma(n+1)} \sum_{m=0}^{\infty} b_m(g) z^{m+1} \int_0^{\infty} t^n e^{-(m+1)t} dt. \quad (133)$$

Multiple integration by parts yields

$$I_n(g, z) = \sum_{m=0}^{\infty} c_{n,m}(g) z^{m+1} \quad (134)$$

with the coefficients

$$c_{n,m}(g) = \frac{b_m(g)}{(m+1)^{n+1}}. \quad (135)$$

To expand the equation of state in virial form, we have to eliminate the fugacity  $z$  from Eqs. (68) and (70). Applying the expansion for  $I_n$  to Eq. (68) gives us:

$$x = \sum_{m=0}^{\infty} c_{n,m}(g) z^{m+1} \quad (136)$$

with fixed  $n = d/2 - 1$ . This series is to be reverted in order to receive a function  $z = z(x)$ . The inverted series is of the following form:

$$z = r_1x + r_2x^2 + r_3x^3 + \dots \quad . \quad (137)$$

If this series is inserted in the series for  $x$  (136) we receive the equation

$$x = (c_{n,0}r_1)x + (c_{n,1}r_1^2 + c_{n,0}r_2)x^2 + (c_{n,2}r_1^3 + 2c_{n,1}r_1r_2 + c_{n,0}r_3)x^3 + \dots \quad , \quad (138)$$

which allows us to calculate the coefficients  $r_{n,m}$  of the inverted series step by step:

$$\begin{aligned} r_{n,1} &= \frac{1}{c_{n,0}}, \\ r_{n,2} &= -\frac{1}{c_{n,0}^3}c_{n,1}, \\ r_{n,3} &= \frac{1}{c_{n,0}^5}(2c_{n,1}^2 - c_{n,0}c_{n,2}), \\ &\dots \end{aligned} \quad (139)$$

The inverted series can be used to replace the series expansion of Eq. (70)

$$\frac{Ex}{Nk_B T d/2} = \sum_{m=0}^{\infty} c_{n+1,m} z^{m+1} \quad (140)$$

yielding

$$\frac{E}{Nk_B T d/2} = \sum_{m=0}^{\infty} r_{d/2,m+1} x^m. \quad (141)$$

With the definition of  $c_{n+1,m}$  in (135), the respective coefficients are easily calculated yielding

$$\begin{aligned} r_{d/2,1} &= 1, \\ r_{d/2,2} &= \frac{1}{2^{d/2}}(g - 1/2), \\ r_{d/2,3} &= \frac{1}{2^{d/2-2}}(g - 1/2)^2 - \frac{(3g - 2)(3g - 1)}{3^{d/2+1}}, \\ &\dots \end{aligned} \quad (142)$$

With  $pV = \frac{2}{d}E$  and the definition of the dimensionless parameter  $x$  from page 15 we see, that the equation of state has the form of a virial expansion. Expanded to the first power, the equation of state in the MB limit  $e^{\frac{\mu}{k_B T}} \ll 1$  becomes:

$$pV = Nk_B T \left[ 1 + (g - 1/2) \frac{\lambda_T^d}{2^{d/2}} \rho + O(\rho)^2 \right]. \quad (143)$$

From this result, we can confirm the numerical results of Section 3.1: In the high-temperature/low-density limits, the statistical interaction of an ideal gas of excludons of parameter  $g$  is attractive for  $g < 1/2$  and repulsive for  $g > 1/2$ . However, we have to note that from these results

it has also been shown in Ref. [28], that for  $g < 1/2$  the statistical interaction becomes repulsive in the high-density limit.

By differentiating the expansion of the internal energy in (141) with respect to the temperature, we receive a corresponding expansion for the heat capacity for small values of  $x$ :

$$\frac{C_V}{Nk_B d/2} = \sum_{m=0}^{\infty} r_{d/2, m+1} (1 - md/2) x^m. \quad (144)$$

## 4 Harmonically trapped exclusion gas

The proceedings in realizing experimental methods in order to create ultracold quantum gases have shown that it is possible to build magnetic particle traps, which show geometries that practically freeze out degrees of freedom of the particles perpendicular to certain axes (cigar-shaped trap) or planes (disk-shaped trap) [29]. At very low temperatures, one can then produce a quantum gas effectively residing in  $d = 1$  and  $d = 2$  spatial dimensions, allowing researchers to observe the effects of low dimensionality predicted by the theories. Therefore, we also regard a  $d$ -dimensional ideal exclusion gas in a harmonic trap with frequencies  $\omega_i$  for  $i = 1, \dots, d$ . In general, the Hamiltonian of a particle in a harmonic trap potential is given by:

$$H = \frac{\hat{p}^2}{2m} + \frac{m}{2} \sum_{i=1}^d \omega_i^2 \hat{x}_i^2. \quad (145)$$

From the energy spectrum, one can establish a thermodynamic limit for infinite particle number  $N$  and volume  $V$  and derive a state density function  $D(\epsilon)$ , which is defined using the geometric mean of the trap frequencies  $\bar{\omega} = (\prod_i \omega_i)^{1/d}$  [30]:

$$D(\epsilon) = \frac{\epsilon^{d-1}}{\Gamma(d)(\hbar\bar{\omega})^d}. \quad (146)$$

Due to the finite size, i.e. particle number and volume, of a harmonically trapped system one can also establish finite-size corrections to the state density above, which will be neglected however. With the given state density, we can use the same methods as in the ideal case. Starting from Eq. (58) for the grand-canonical thermodynamic potential with arbitrary state density, we also arrive at the Calogero-Sutherland integral functions  $I_n$ , for the values  $n = d, d-1$  and  $d-2$ . Additionally, we have to consider that the effective volume of the gas is equal to the harmonic volume  $V_h = \bar{\omega}^{-d}$ . In this section, we will just mention a couple of analytical results for the respective thermodynamic functions without any derivation. For finite temperature numerical findings as in Section 3.1 see Refs. [11, 31, 32].

### 4.1 Low-temperature properties

In Section 3.2.2 we derived low-temperature results for exclusion properties within the Sommerfeld expansion for arbitrary density functions. From Eq. (105), together with the results

for the coefficients  $C_i(g)$  in (128) and Eq. (146), we find

$$\begin{aligned} \frac{\mu_g(T)}{\bar{\epsilon}_F} &= g^{1/d} - g^{1-1/d}(d-1)\zeta(2) \left(\frac{T}{T_F}\right)^2 - g^{1-2/d}(1-g)(d-1)(d-2)2\zeta(3) \left(\frac{T}{T_F}\right)^3 \\ &\quad - g^{1-3/d}(4-g-4g^2)(d-1)(d-2)(d-3)\frac{3}{2}\zeta(4) \left(\frac{T}{T_F}\right)^4 + O(T/T_F)^5 \end{aligned} \quad (147)$$

with the trapped generalized Fermi energy  $\tilde{\epsilon}_F = g^{-1/d}\bar{\epsilon}_F$ . For the heat capacity we obtain from Eq. (110):

$$\begin{aligned} \frac{C_{g,V}}{\bar{\epsilon}_F^d k_B \tilde{K}} &= g^{1-1/d}2\zeta(2) \left(\frac{T}{T_F}\right) + g^{1-2/d}(1-g)d(d-1)6\zeta(3) \left(\frac{T}{T_F}\right)^2 \\ &\quad + g^{1-3/d}(4-g-4g^2)d(d-1)(d-2)6\zeta(4) \left(\frac{T}{T_F}\right)^3 + O(T/T_F)^4 \end{aligned} \quad (148)$$

with  $\tilde{K} = \frac{V_h}{\hbar^d \Gamma(d)}$ . We see that for the  $d = 1$  trapped exclusion gas the low-temperature expansions vanish, analogous to the  $d = 2$  ideal gas. This will be further discussed in Section 5.

## 4.2 High-temperature properties

A high-temperature expansion of the Calogero-Sutherland integral function completely analogue to Section 3.3 gives us the equation of state in the virial form:

$$\frac{E}{Nk_B T d} = \frac{pV}{Nk_B T} = \sum_{m=0}^{\infty} s_{d,m+1} x^m. \quad (149)$$

with coefficients

$$\begin{aligned} s_{d,1} &= 1, \\ s_{d,2} &= \frac{1}{2^d}(g-1/2), \\ s_{d,3} &= \frac{1}{2^{d-2}}(g-1/2)^2 - \frac{(3g-2)(3g-1)}{3^{d+1}}, \\ &\dots \end{aligned} \quad (150)$$

## 5 Gas with constant density of states

As seen from the low-temperature approximations of the chemical potential and the specific heat, in  $d = 2$  ( $d = 1$ ) the terms of the asymptotic expansions for the ideal (trapped) exclusion gas vanish. This behaviour is found, when we have a density of states that is constant in energy. Regarding a more general dispersion relation  $\epsilon(p) = ap^s$ , we find a constant density of states of ideal gases for  $d = s$ .

It is possible to obtain exact expressions for thermodynamic functions of the IEG with constant density of states [22, 33]. This is because we can solve the Calogero-Sutherland integral function  $I_0(g, \eta)$  from (73), which gives us the solution of Eq. (68):

$$x = I_0(g, \eta) = \int_{w_0}^{\infty} dw \frac{1}{w(w+1)} = \ln \left( 1 + \frac{1}{w_0} \right). \quad (151)$$

Now we can express  $w_0$  in terms of the dimensionless parameter  $x$ :

$$w_0 = \frac{1}{e^x - 1}. \quad (152)$$

This expression can be plugged into the defining equation for  $w_0 = w(\epsilon = 0)$ , namely  $e^{-\eta} = w_0^g(1 + w_0)^{1-g}$ , to give an exact result for the dimensionless chemical potential  $\eta$  of the exclusion gas with constant density of states:

$$\eta_g(x) = gx + \ln(1 - e^{-x}). \quad (153)$$

If we approach the high density/low-temperature limits, i.e. for large values of  $x$ , we recover the asymptotic result via Sommerfeld expansion  $\frac{\mu}{\epsilon_F} = g$ . Next we can calculate the internal energy from Eq.'s (70) and (65)

$$\frac{Ex}{Nk_B T} = I_1(g, \eta) = \int_{-\infty}^{\eta} I_0(g, \eta) d\eta = \int_{-\infty}^{\eta} x(\eta) d\eta. \quad (154)$$

Using the exact result for the dimensionless chemical potential, we find  $\frac{d\eta}{dx} = g + \frac{1}{e^x - 1}$  and can switch variables to  $x$  in the integral yielding the equation of state

$$\frac{E}{Nk_B T} = \frac{pV}{Nk_B T} = \frac{1}{x} \left( \frac{1}{2}gx^2 + \int_0^x \frac{x'}{e^{x'} - 1} dx' \right). \quad (155)$$

For increasingly large values of  $x$ , we see that the integral approaches the integral definition of the Riemann Zeta-function  $\zeta(s) = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx$ , giving us the low-temperature result

$$\frac{E}{Nk_B T} = \frac{pV}{Nk_B T} = \frac{1}{2}gx + \frac{1}{x}\zeta(2). \quad (156)$$

The isochoric specific heat is obtained from Eq. (155) by  $C_V = \frac{\partial E}{\partial T}$ , which will be done for the IEG in  $d = 2$  representative for all systems with constant state density. We can perform the derivative from the previous equation if we note that for the dimensionless parameter  $x$  in  $d = 2$  we find the temperature dependence  $x = \frac{T_F}{T}$  with fixed particle density  $\frac{N}{V}$ . Furthermore,  $\frac{dx}{dT} = -\frac{T_F}{T^2}$  yields

$$\frac{C_V}{Nk_B} = \frac{2}{x} \int_0^x \frac{x'}{e^{x'} - 1} dx' - \frac{x}{e^x - 1}. \quad (157)$$

Also this exact result for an exclusion gas with constant density of states does not depend on the parameter  $g$ . For very large values of  $x$ , we retain the low-temperature asymptotic result  $\frac{C_V}{Nk_B} = \frac{2}{x}\zeta(2)$ , from the definition of the Riemann zeta-function. In Fig. 5 we can see the specific heat as a function of the reduced temperature  $T/T_F$  for the  $d = 2$  IEG. We find linear behaviour  $\frac{C_V}{Nk_B} = 2\frac{T}{T_F}\zeta(2)$  for low temperatures, as well as the classical MB result for high temperatures. In the high-temperature/low density limits ( $x$  very small), we can obtain the virial expansion of the equation of state by expanding the integrand in Eq. (155) around the point  $x' = 0$ :

$$\frac{x'}{e^{x'} - 1} = 1 - \frac{2}{x} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!} B_n x'^{2n} \quad (158)$$

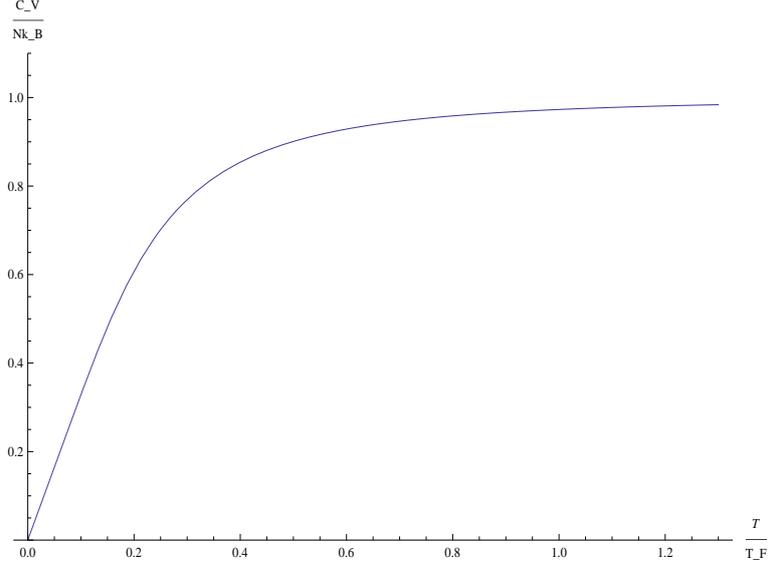


Figure 5: Isochoric heat capacity as a function of reduced temperature  $T/T_F$  for the  $d = 2$  IEG with any value of  $g$ .

with the Bernoulli numbers  $B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, \dots$ , which are defined as Taylor coefficients of this specific series. One can integrate each term yielding the virial expansion of the equation of state for constant density of states

$$\frac{pV}{Nk_B T} = 1 + a_2 x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} B_n x^{2n} \quad (159)$$

with the second virial coefficient  $a_2 = \frac{1}{2}(g - \frac{1}{2})$ . This virial expansion is included in the more general case in Section 3.3, as can be seen by comparison of the respective coefficients. We find that only the second virial coefficient  $a_2$  does depend on  $g$ . These results of the exclusion gas with constant density of states, when applied to the IEG in  $d = 2$ , are identical with calculations for the  $d = 2$  ideal Fermi and Bose gas [5] when setting  $g = 1$  and  $g = 0$  respectively: Only the second virial coefficient differs in  $d = 2$  Fermi and Bose gases. Hence, the specific heat is identical for  $d = 2$  ideal Fermi and Bose gases, as shown.

## 6 Applications of the ideal exclusion gas model

In the previous Sections, we have developed QSM and thermodynamics for a quantum gas of particles, that obey fractional exclusion statistics. This last Section is dedicated to the question, in which cases the simple interpolation (5) in the counting of states between fermionic and bosonic borders can be applied to physical problems.

Also, assumption (7), while developing the basic theory, has to be fulfilled for real particles. For instance, this is not the case for free anyons in the narrow sense since already the 2-anyon spectrum is unrelated to the single-state energies [5].

Murthy and Shankar showed, that one can derive generalized exclusion from gas models within the virial expansion [34]. In order to do so, they extended the Haldane exclusion statistics to infinite dimensional single-particle Hilbert spaces, i.e. particles in the continuum, and showed, that the statistical parameter  $g$  is completely determined by the high-temperature limit of the second virial coefficient  $a_2$  according to the following relation:

$$\frac{1}{2} - g = -2a_2. \quad (160)$$

This result is compatible with the second virial coefficient of the exclusion gas in Eqs. (143) and (159). In principle, the relation above can be applied to any system of interacting particles, for which a virial expansion in the high-temperature limit is possible. From a more phenomenological view, the main argument for the occurrence of generalized exclusion is that adding one particle to the system, which induces a phase shift of all other particles, results in an identical energy shift for all particles that can push one state above a fixed energy cutoff of the system. This will also happen for a continuous energy scale with high-energy cutoff.

The relation (160) was applied to show, that anyons confined to the LLL obey fractional exclusion statistics with the statistical parameter  $g$  being equal to the exchange statistics  $\alpha$ . Therefore, in agreement with the initial motivation for fractional exclusion statistics, the prominent application of  $2d$  anyons modelling quasiparticles in the FQHE is equivalent to fractional exclusion particles with statistical parameter  $g$  fixed by the LLFF and mutual statistics for quasi-holes and quasiparticles.

The IEG has been investigated as a promising alternate method to describe the highly non-trivial problem of interacting quantum gases, that are usually examined by approximations via low-order perturbation theory or mean-field theory. Especially, for the class of interacting  $1d$  many-body quantum systems that can be solved by the thermodynamic Bethe ansatz (TBA) [35], it has been established that, with fractional exclusion particles and mutual statistics, one can obtain equivalent results for the thermodynamic functions [13, 23, 36]. The dynamical interaction of the TBA, that has the form of a coupling term in the many-particle Hamiltonian, is completely transmuted into the Haldane statistical interaction that arise from the generalized exclusion statistics. TBA solvable systems include the Yang-Yang  $\delta$ -function gas [35] and the Calogero-Sutherland model (CSM) [37, 38]. Because of the equivalence of thermodynamic functions in  $d = 1$ , the IEG is seen as a valid method to generalize the CSM to higher spatial dimensions [23].

Also in the CSM context, a notion of fractional *exchange* statistics was introduced, that led to a second-quantized description of  $1d$  anyons [13]. The results were in agreement with the CSM and the application of fractional exclusion statistics. While for second-quantized bosons (fermions), the operators  $\hat{b}_j^\dagger$ ,  $\hat{b}_j$ , that create and annihilate a particle at spot  $j$ , commute (anticommute), the anyonic operators satisfy generalized commutation relations [11, 13, 39]:

$$\begin{aligned} [\hat{a}_j, \hat{a}_k^\dagger]_\alpha &= \hat{a}_j \hat{a}_k^\dagger - e^{-i\alpha\pi\text{sgn}(j-k)} \hat{a}_k^\dagger \hat{a}_j = \delta_{jk}, \\ [\hat{a}_j, \hat{a}_k]_\alpha &= \hat{a}_j \hat{a}_k - e^{i\alpha\pi\text{sgn}(j-k)} \hat{a}_k \hat{a}_j = 0. \end{aligned} \quad (161)$$

with  $\text{sgn}(j - k) = 0$  for  $j = k$ . Kundu found, that the problem of a  $1d$  multi- $\delta$ -function interacting Bose gas could be solved through the model of a  $\delta$ -function interacting anyon gas obeying the commutation relations as in Eq. (161) [12]. Moreover, the anyon statistical parameter and the dynamic coupling constant of this model were found to induce Haldane fractional exclusion statistics [11].

Quasiparticles in the  $1d$  Luttinger liquid, whose creation operators induce an anyonic exchange phase  $\alpha$ , were also considered in Ref. [34] and were shown to obey fractional exclusion with  $g = \alpha$  after Eq. (160). In accordance, it has been shown that the low-temperature properties of the Luttinger liquid are reproduced by the  $1d$  IEG with exclusion statistics  $g$  equal to the controlling parameter of the Luttinger liquid [40].

As a motivation for further studies, I will conclude this work by briefly presenting a recently proposed setup to create fractional statistics in  $1d$  optical lattices (OL) [39]. Optical particle traps provide a way of creating ultracold quantum gases, alternative to magnetical traps as in Section 4. One realization of an optical trap is the OL, where an interference pattern of overlapping laser beams provides a periodic potential for particles, because of the stark shift of the gas atoms. Interacting OL bosons may be described by the Bose-Hubbard model and, at zero temperature, exist in the three different quantum phases of superfluid, Mott insulator and, under influence of a disorder potential, Bose glass [41].

An Anyon-Hubbard model is now created from the Bose-Hubbard model with a variant of a Jordan-Wigner transformation [39]

$$\hat{a}_j = \hat{b}_j e^{i\alpha\pi \sum_{i=1}^{j-1} \hat{n}_i} \quad (162)$$

that maps the operators of the anyon system onto bosonic operators using the particle number operator  $\hat{n}_j = \hat{b}_j^\dagger \hat{b}_j$ , which is valid for both bosons and anyons. In Ref. [39] it is shown, that the mapped anyon operators satisfy the anyonic commutation relations (161). The Hamiltonian of the Anyon-Hubbard model

$$H^a = -J \sum_j^L (\hat{a}_j^\dagger \hat{a}_{j+1} + h.c.) + \frac{U}{2} \sum_j^L \hat{n}_j (\hat{n}_j - 1) \quad (163)$$

is thus mapped onto a bosonic Hamiltonian

$$H^b = -J \sum_j^L (\hat{b}_j^\dagger \hat{b}_{j+1} e^{i\alpha\pi \hat{n}_j} + h.c.) + \frac{U}{2} \sum_j^L \hat{n}_j (\hat{n}_j - 1) \quad (164)$$

with the tunneling amplitude  $J$  between two neighboring lattice sites and the on-site interaction energy  $U$ . This Hamiltonian  $H^b$  describes bosons with tunneling amplitude  $J e^{i\alpha\pi n_j}$ , depending on the occupation  $n_j$  of lattice site  $j$  and the exchange statistics angle  $\Theta = \alpha\pi$  as seen in Fig. 6a.

The anyonic exchange phase shall be realized by inducing a phase-shifted hopping term. In Fig. 6b the physical concept with an OL, that is tilted by the energy  $\Delta$ , is visualized. For the case of allowed occupancies  $n_j = 0, 1, 2$ , it is proposed to use four external radiation fields 1,2,3 and 4 with detuning  $\delta$  in order to establish a transition from the different occupational states to an excited state  $|e\rangle$ . This assisted tunneling can address each different hopping process from one occupational state to another and induce an occupation

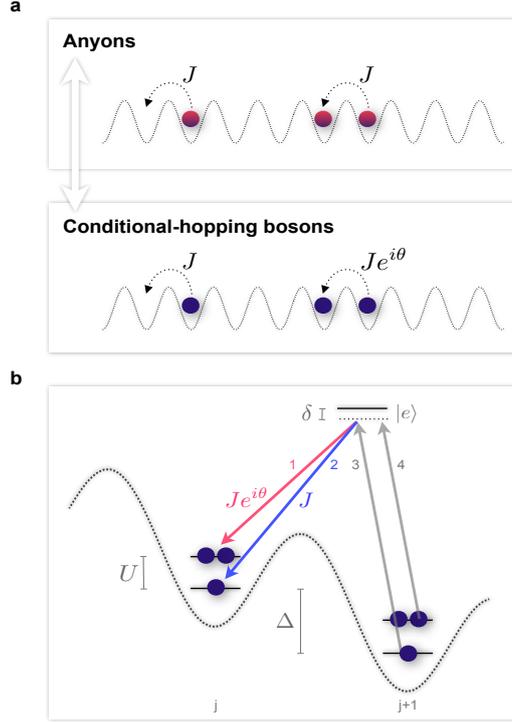


Figure 6: (a) Mapping between lattice anyons and bosons with occupation dependent hopping amplitude. (b) Assisted tunneling process inducing an occupation dependent, relative exchange phase (from Ref. [39]).

dependent relative phase. Most importantly, in the reference paper it is shown that the induced phase shift of the hopping amplitude, and thus the exchange statistics  $\alpha$ , can be directly controlled and tuned by variation of frequency, intensity, polarization, direction and detuning  $\delta$  of the external radiation fields.

Via examination of quantum phase diagrams, the possibility was shown to induce a quantum phase transition from a superfluid into a Mott insulating phase by simply increasing the statistical angle  $\Theta$  at fixed ratio  $J/U$  of tunneling amplitude and interaction energy. It is proposed to start the experiment by realizing a superfluid Bose gas at  $\Theta = 0$  and then to continuously increase the statistical angle to a critical value  $\Theta_c$  beyond which the gas will be in a Mott insulating phase. To summarize, the proposed experiment not only provides a method to *realize* fractional statistics in  $1d$ , but also to directly *tune* the exchange statistics establishing a quantum phase transition that could continuously transmute bosons into fermions, giving researchers further insight into fundamental quantum phenomena.

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## **Commitment**

This is to certify that I wrote this work on my own and that the references include all the sources of information I have utilised.

Berlin, May 10th 2012

Johannes Lohmann