

Bose-Einstein Condensation in Weak and Strong Disorder Potentials

Bachelor Thesis by
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1 Introduction

This thesis discusses the effects of an external random disorder potential on a Bose-Einstein condensate. The macroscopic quantum phenomenon of Bose-Einstein condensation was first predicted by Satyendra Nath Bose and Albert Einstein in 1924 [1, 2]. They postulated a new state of matter of weakly or non-interacting bosons at very low temperatures. In that case, a large number of atoms occupies the ground state, leading to a macroscopic quantum state. It took approximately 70 years until an experimental evidence was established in ultracold gases of alkali metals [3, 4] by Eric A. Cornell, Wolfgang Ketterle and Carl E. Wieman in 1995 which was rewarded with the Nobel price in 2001.

The problem of interacting disordered bosonic atoms, known as the 'dirty boson problem' [5], first came up in the context of superfluid Helium in Vycor [6]. Due to the discovery of new experimental techniques, researchers are currently able to investigate thoroughly this field. Theoretically, a random disorder potential comes along with Anderson localization [7]. For matter waves, this seminal problem was first proven in two experiments; the random or quasi-random disorder potential was either produced by laser speckles [8] or by an incommensurable optical lattice [9]. In the laser speckle realization the disorder potential is created by a laser beam which shines through a diffusive plate and yields a random interference pattern [10]. In the incommensurable optical lattice, however, it is created through two interfering laser beams with incommensurable wavelengths producing a quasi periodic potential. Figure 1 from Ref. [11] shows the experimental setup and a possible realization for such a random speckle potential.

Furthermore, in wire potential traps [12, 13], due to the conductor's roughness, a disorder potential is naturally created. In addition, imperfections of the wire itself can induce local disorder. The review [13] of József Fortágh and Claus Zimmermann discusses a resulting fragmentation of the atomic cloud, after having brought the atoms close enough to a current carrying conductor (see Figure 2).

Finally, according to a theoretical suggestion of Ref. [15], a specie of atoms is trapped in a deep optical lattice and serves as a frozen disorder potential for a second specie of different atoms. Although first experiments in this direction have already been performed [16], it has not yet been achieved to generate in this way an ideal frozen disorder potential.

One of the first important results of the dirty boson problem was introduced by K. Huang and H.-F. Meng in 1992 [17]. Within a Bogoliubov theory for a weakly interacting Bose-Einstein condensate it was found that the random disorder potential leads to a depletion of the global condensate density

$$n_0 = n - \frac{Rm^2}{8\pi^{\frac{3}{2}}\hbar^4} \sqrt{\frac{n}{a}}, \quad (1)$$

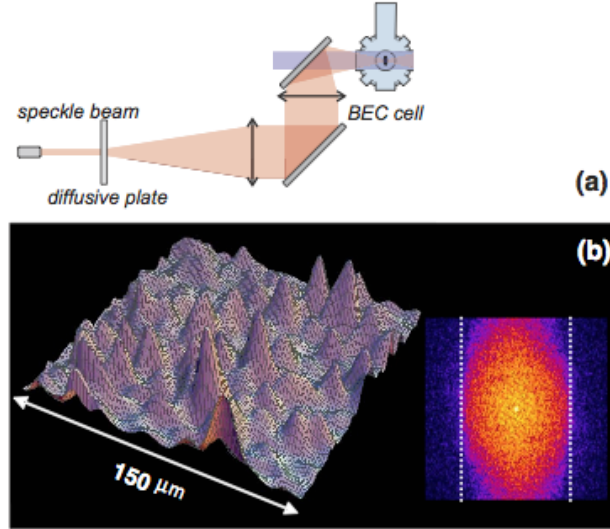


Fig. 1 – (a) Optical setup for the speckle potential for the BEC. The axial direction of the magnetic trap is in the vertical direction of the figure. (b) 3D representation of the speckle potential (left) and its Fourier transform (right). The dotted lines correspond to a length scale of about $10 \mu\text{m}$ in the axial direction (from Ref. [11]).

where a represents the s-wave scattering length and R indicates the disorder strength. This result is due to the fact that the atoms partially occupy the local minima of the disorder potential, so that these fragmented Bose-Einstein condensates represent a loss for the global condensate. The question arises whether, for increasing disorder strength R , a state of a vanishing global condensate density can be established where all atoms accumulate in the minima of the disorder potential. As it turns out, the Huang-Meng result (1) is only reliable for small disorder and, therefore, cannot make a proposition of whether this quantum phase transition occurs or not.

In the following we show in Section 2 that the Huang-Meng result (1) can be reproduced by solving the Gross-Pitaevskii equation perturbatively. Afterwards, Section 3 discusses a non-perturbative approach towards the dirty boson problem which is based on a Gaussian approximation. As a non-trivial result we find a critical disorder strength beyond which a Bose-glass phase emerges, causing a quantum phase transition [18]. In analogy to the Edward-Anderson order parameter of spin gases [19], Section 4 applies a Bose-glass order parameter [20] in order to interpret the non-perturbative result of Section 3. Finally, Section 5 provides a summarizing discussion of the thesis.

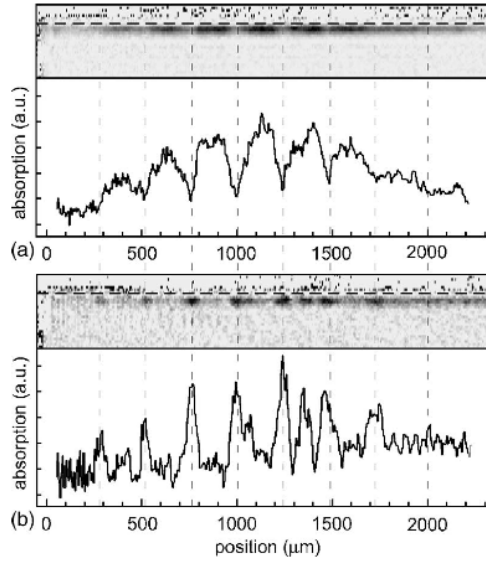


Fig. 2 – Fragmentation of an ultracold cloud of ^{87}Rb atoms in a magnetic waveguide potential of an electroplated conductor. The conductor is indicated by the horizontal dashed line (from Ref. [14]).

2 Perturbation Theory

2.1 Gross-Pitaevskii Equation

A Bose-Einstein-Condensate in a potential $U(\mathbf{x})$ at zero temperature is described by the time-dependent Gross-Pitaevskii equation [21, 22]

$$\left[-\frac{\hbar^2}{2m}\Delta + U(\mathbf{x}) + g|\Psi(\mathbf{x}, t)|^2 \right] \Psi(\mathbf{x}, t) = i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{x}, t). \quad (2)$$

The Gross-Pitaevskii equation is also referred to as nonlinear Schrödinger equation, as it reduces for a vanishing interaction parameter $g = 0$ to the Schrödinger equation. For $g \neq 0$ an additional nonlinear term $g|\Psi(\mathbf{x}, t)|^2 \Psi(\mathbf{x}, t)$ has to be added to account for the 2-particle contact interaction. The second difference is that the Gross-Pitaevskii equation describes a macroscopic quantum state which leads to a different normalization

$$N = \int d^3x |\Psi(\mathbf{x}, t)|^2. \quad (3)$$

The ansatz $\Psi(\mathbf{x}, t) = \Psi(\mathbf{x})e^{-\frac{i}{\hbar}\mu t}$ yields the time-independent Gross-Pitaevskii equation

$$\left[-\frac{\hbar^2}{2m}\Delta + U(\mathbf{x}) + g|\Psi(\mathbf{x})|^2 \right] \Psi(\mathbf{x}) = \mu\Psi(\mathbf{x}), \quad (4)$$

where the chemical potential μ is determined from the normalization (3). As a remark, the wave function $\Psi(\mathbf{x})$ contains the chemical potential μ as a parameter.

2.2 Disorder

Next we will discuss a perturbation approach for solving the stationary Gross-Pitaevskii equation with a random disorder potential $U(\mathbf{x})$. Since $U(\mathbf{x})$ is a random potential, it is characterized by its statistical properties. To this end we introduce $\langle \bullet \rangle$ as the statistical average over many disorder realizations. In the following we will restrict ourselves to a Gaussian process $U(\mathbf{x})$, thus we cut off its cumulant expansion after the second term. This means that its statistical properties are characterized by the first order average $\langle U(\mathbf{x}) \rangle$ and the second order cumulant $\langle U(\mathbf{x})U(\mathbf{x}') \rangle_c$. Taking into account spatial homogeneity, we have

$$\langle U(\mathbf{x}) \rangle = U_0 \quad (5)$$

$$\langle U(\mathbf{x})U(\mathbf{x}') \rangle_c = R(\mathbf{x} - \mathbf{x}'), \quad (6)$$

with some correlation function $R(\mathbf{x} - \mathbf{x}')$. Without loss of generality we can set $U_0 = 0$ by the substitution

$$U'(\mathbf{x}) = U(\mathbf{x}) - U_0, \quad (7)$$

$$\mu' = \mu - U_0, \quad (8)$$

where the disorder average yields $\langle U'(\mathbf{x}) \rangle = 0$. The Gross-Pitaevskii equation is not changed by this substitution as U is simply replaced by U' as well as μ is replaced by μ' . This leads to the following statistical properties of the disorder potential:

$$\langle U(\mathbf{x}) \rangle = 0, \quad (9)$$

$$\langle U(\mathbf{x})U(\mathbf{x}') \rangle = R(\mathbf{x} - \mathbf{x}'). \quad (10)$$

In what follows we will also need the disorder average of the Fourier transformed

$$\langle U(\mathbf{k})U(\mathbf{k}') \rangle = \int d^3x \int d^3x' e^{-i(\mathbf{k}\mathbf{x} + \mathbf{k}'\mathbf{x}')} \langle U(\mathbf{x})U(\mathbf{x}') \rangle. \quad (11)$$

Performing the substitution $\mathbf{x}''(\mathbf{x}) = \mathbf{x} - \mathbf{x}'$ by inserting (10) into (11), we obtain

$$\langle U(\mathbf{k})U(\mathbf{k}') \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') R(\mathbf{k}). \quad (12)$$

2.3 Physical Questioning

The question is what happens to the bosons when we switch on a disorder potential and increase the disorder strength. One would suggest three possible reactions of the bosons:

1. The bosons remain in the global macroscopic condensate, which is determined by the condensate density

2. The bosons condense in small minima of the disorder potential. We refer to them as the fragmented condensate.
3. The bosons reach excited states. This case is not included in the Gross-Pitaevskii equation as it is restricted to zero temperature.

One would assume that with increasing disorder strength more and more bosons are collected in the fragmented condensate and thus lead to a depletion of the global condensate. Now, the question arises whether we can achieve a state where no bosons exist in the macroscopic condensate. If this happens at zero temperature, a quantum phase transition from a superfluid to a Bose-glass will occur [18].

2.4 Perturbative Approach

We will now discuss a perturbative approach where we restrict ourselves to a condensate at rest which can be described without loss of generality by a real wave function $\Psi(\mathbf{x})$. Therefore, we have to solve the stationary Gross-Pitaevskii equation

$$\left[-\frac{\hbar^2}{2m} \Delta + U(\mathbf{x}) - \mu \right] \Psi(\mathbf{x}) + g\Psi(\mathbf{x})^3 = 0 \quad (13)$$

leading to the wave function $\Psi(\mathbf{x})$ as a functional of the disorder potential $U(\mathbf{x})$. We then determine the statistical averages defining both the condensate density

$$\langle \Psi(\mathbf{x}) \rangle^2 = n_0 \quad (14)$$

and the particle density

$$\langle \Psi(\mathbf{x})^2 \rangle = n. \quad (15)$$

2.4.1 Perturbative Expansion

We write the wave function $\Psi(\mathbf{x})$ as

$$\Psi(\mathbf{x}) = \Psi_0 + \Psi_1(\mathbf{x}) + \Psi_2(\mathbf{x}) + \dots, \quad (16)$$

where the respective index n at the contribution $\Psi_n(\mathbf{x})$ indicates the n -th order contribution with respect to the disorder potential. Due to spatial homogeneity the zeroth order contribution is assumed to be a constant. Inserting the ansatz (16) in (13) yields at first

$$(g\Psi_0^2 - \mu)\Psi_0 + \left[\hat{h}\Psi_1(\mathbf{x}) + U(\mathbf{x})\Psi_0 \right] + \left[\hat{h}\Psi_2(\mathbf{x}) + U(\mathbf{x})\Psi_1(\mathbf{x}) + 3g\Psi_0\Psi_1(\mathbf{x})^2 \right] + \dots = 0 \quad (17)$$

with the operator

$$\hat{h} = -\frac{\hbar^2}{2m} \Delta - \mu + 3g\Psi_0^2. \quad (18)$$

Equating the coefficients leads to one equation for each order defining the wave functions $\Psi_n(\mathbf{x})$.

The equation for the zeroth order results in

$$\Psi_0 = \sqrt{\frac{\mu}{g}}. \quad (19)$$

The first order term $\Psi_1(\mathbf{x})$ follows from solving

$$\hat{h}\Psi_1(\mathbf{x}) = -U(\mathbf{x})\Psi_0. \quad (20)$$

Performing a Fourier transformation yields an algebraic equation which is solved by

$$\Psi_1(\mathbf{k}) = \frac{-U(\mathbf{k})\Psi_0}{\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + 3g\Psi_0^2} \quad (21)$$

or, correspondingly,

$$\Psi_1(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{-U(\mathbf{k})\Psi_0}{\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + 3g\Psi_0^2} e^{i\mathbf{k}\mathbf{x}}. \quad (22)$$

A similar procedure also applies to the second order $\Psi_2(\mathbf{x})$, which is determined by solving

$$\hat{h}\Psi_2(\mathbf{x}) = -\Psi_1(\mathbf{x}) [U(\mathbf{x}) + 3g\Psi_0\Psi_1(\mathbf{x})]. \quad (23)$$

Here we need the fact that a Fourier transformation of a product of two functions corresponds to a convolution of the Fourier transformed functions

$$\int d^3x \Psi_1(\mathbf{x})U(\mathbf{x})e^{-i\mathbf{k}\mathbf{x}} = \int \frac{d^3k'}{(2\pi)^3} \Psi_1(\mathbf{k} - \mathbf{k}')U(\mathbf{k}') \quad (24)$$

and we obtain in total

$$\Psi_2(\mathbf{k}) = -\frac{\int \frac{d^3k'}{(2\pi)^3} \Psi_1(\mathbf{k} - \mathbf{k}') [U(\mathbf{k}') + 3g\Psi_0\Psi_1(\mathbf{k}')] }{\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + 3g\Psi_0^2}. \quad (25)$$

Now inserting $\Psi_1(\mathbf{k})$ from Eq. (21) yields

$$\Psi_2(\mathbf{k}) = \Psi_0 \int \frac{d^3k'}{(2\pi)^3} \frac{U(\mathbf{k}')U(\mathbf{k} - \mathbf{k}') \left[\frac{\hbar^2\mathbf{k}'^2}{2m} - \mu \right]}{\left(\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + 3g\Psi_0^2 \right) \left[\frac{\hbar^2(\mathbf{k} - \mathbf{k}')^2}{2m} - \mu + 3g\Psi_0^2 \right] \left(\frac{\hbar^2\mathbf{k}'^2}{2m} - \mu + 3g\Psi_0^2 \right)}. \quad (26)$$

2.4.2 Disorder Ensemble Averages

Until now we have calculated the lowest perturbative orders of the wave function $\Psi(\mathbf{x})$ or rather its Fourier transform. In order to determine the densities $n_0(\mu)$ and $n(\mu)$ according to Eqs. (14) and (15) we need to calculate the following disorder averages

$$n_0 = \Psi_0^2 + 2\Psi_0\langle\Psi_1(\mathbf{x})\rangle + 2\Psi_0\langle\Psi_2(\mathbf{x})\rangle + \langle\Psi_1(\mathbf{x})\rangle^2 + \dots, \quad (27)$$

$$n = \Psi_0^2 + 2\Psi_0\langle\Psi_1(\mathbf{x})\rangle + 2\Psi_0\langle\Psi_2(\mathbf{x})\rangle + \langle\Psi_1(\mathbf{x})\rangle^2 + \dots. \quad (28)$$

With (9) and (22) follows immediately

$$\langle\Psi_1(\mathbf{x})\rangle = 0, \quad (29)$$

so the Eqs. (27) and (28) reduce to

$$n_0 = \Psi_0^2 + 2\Psi_0\langle\Psi_2(\mathbf{x})\rangle + \dots, \quad (30)$$

$$n = \Psi_0^2 + 2\Psi_0\langle\Psi_2(\mathbf{x})\rangle + \langle\Psi_1(\mathbf{x})\rangle^2 + \dots. \quad (31)$$

We can now use our solutions for $\Psi_1(\mathbf{k})$ and $\Psi_2(\mathbf{k})$ in Eqs. (21) and (26) in order to determine the two non-trivial terms $\langle\Psi_1(\mathbf{x})\rangle^2$ and $\langle\Psi_2(\mathbf{x})\rangle$. We start with

$$\langle\Psi_1(\mathbf{x})\rangle^2 = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \frac{\langle U(\mathbf{k})U(\mathbf{k}')\rangle\Psi_0^2}{\left(\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + 3g\Psi_0^2\right)\left(\frac{\hbar^2\mathbf{k}'^2}{2m} - \mu + 3g\Psi_0^2\right)} e^{i(\mathbf{k}+\mathbf{k}')\mathbf{x}}, \quad (32)$$

where we make use of (12) and obtain the result

$$\langle\Psi_1(\mathbf{x})\rangle^2 = \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})\Psi_0^2}{\left(\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + 3g\Psi_0^2\right)^2}. \quad (33)$$

The next average we are dealing with is

$$\begin{aligned} \langle\Psi_2(\mathbf{x})\rangle &= \Psi_0 \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \\ &\times \frac{\langle U(\mathbf{k}')U(\mathbf{k}-\mathbf{k}')\rangle\left[\frac{\hbar^2\mathbf{k}'^2}{2m} - \mu\right]}{\left(\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + 3g\Psi_0^2\right)\left[\frac{\hbar^2(\mathbf{k}-\mathbf{k}')^2}{2m} - \mu + 3g\Psi_0^2\right]\left(\frac{\hbar^2\mathbf{k}'^2}{2m} - \mu + 3g\Psi_0^2\right)} e^{i\mathbf{k}\mathbf{x}}. \end{aligned} \quad (34)$$

Again we insert $\langle U(\mathbf{k}')U(\mathbf{k}-\mathbf{k}')\rangle$ from Eq. (12), yielding

$$\langle\Psi_2(\mathbf{x})\rangle = \Psi_0 \int \frac{d^3k'}{(2\pi)^3} R(\mathbf{k}') \frac{\frac{\hbar^2\mathbf{k}'^2}{2m} - \mu}{\left(\frac{\hbar^2\mathbf{k}'^2}{2m} - \mu + 3g\Psi_0^2\right)^2 (3g\Psi_0^2 - \mu)}. \quad (35)$$

With the knowledge of these averages and the explicit expression of Ψ_0 from (19), we are now able to determine both the condensate density (30) and the particle density (31) as a

function of the chemical potential μ :

$$n_0(\mu) = \frac{\mu}{g} + \frac{1}{g} \int \frac{d^3k}{(2\pi)^3} R(\mathbf{k}) \frac{\frac{\hbar^2 \mathbf{k}^2}{2m} - \mu}{\left(\frac{\hbar^2 \mathbf{k}^2}{2m} + 2\mu\right)^2} + \dots, \quad (36)$$

$$n(\mu) = \frac{\mu}{g} + \frac{1}{g} \int \frac{d^3k}{(2\pi)^3} R(\mathbf{k}) \frac{\frac{\hbar^2 \mathbf{k}^2}{2m}}{\left(\frac{\hbar^2 \mathbf{k}^2}{2m} + 2\mu\right)^2} + \dots. \quad (37)$$

In order to obtain a direct relation between condensate density n_0 and particle density n , we have to eliminate the chemical potential μ from Eqs. (36) and (37). At first, we solve (36) for $\mu = \mu(n_0)$ and obtain

$$\mu(n_0) = gn_0 - gn_0 \int \frac{d^3k}{(2\pi)^3} R(\mathbf{k}) \frac{\frac{\hbar^2 \mathbf{k}^2}{2m} - gn_0}{\left(\frac{\hbar^2 \mathbf{k}^2}{2m} + 2gn_0\right)^2} + \dots. \quad (38)$$

Inserting this equation into (37) yields

$$n(n_0) = n_0 + n_0 \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})}{\left(\frac{\hbar^2 \mathbf{k}^2}{2m} + 2gn_0\right)^2} + \dots, \quad (39)$$

which finally leads to

$$n_0(n) = n - n \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})}{\left(\frac{\hbar^2 \mathbf{k}^2}{2m} + 2gn\right)^2} + \dots. \quad (40)$$

This result can be physically interpreted as follows. The disorder correlation yields a depletion of the global condensate due to the occurrence of fragmented condensates in the minima of the disorder potential.

2.4.3 More Efficient Approach

It turns out that one can obtain (39) more easily by using (30) and (31). By inserting (30) into (31), we yield the desired relation

$$n(n_0) = n_0 + \langle \Psi_1(\mathbf{x})^2 \rangle + \dots \quad (41)$$

directly. Since we already know $\langle \Psi_1(\mathbf{x})^2 \rangle$ from (33), taking into account (19) immediately leads to the result (39). This represents a more efficient approach insofar as (39) is obtained without having to determine $\Psi_2(\mathbf{x})$.

2.4.4 Delta Correlation

In this subsection we discuss the depletion of the global condensate for a delta correlation function

$$R(\mathbf{x}) = R\delta(\mathbf{x} - \mathbf{x}'), \quad (42)$$

whose Fourier transform is given by

$$R(\mathbf{k}) = R. \quad (43)$$

In order to evaluate the corresponding integral in (39) we use the relation

$$\frac{\Gamma(x)}{a^x} = \int_0^\infty d\tau \tau^{x-1} e^{-a\tau} \quad (44)$$

with the gamma function $\Gamma(x)$ which is also referred to as the Schwinger trick [23]. With this we obtain

$$n(n_0) = n_0 + \frac{n_0 R}{(2\pi)^3} \int_0^\infty d\tau \tau e^{-2gn_0\tau} \int d^3k e^{-\frac{\hbar^2 \mathbf{k}^2}{2m}\tau} + \dots \quad (45)$$

Now, the \mathbf{k} -integral is Gaussian and can be solved yielding

$$n(n_0) = n_0 + n_0 R \left(\frac{m}{2\pi\hbar^2} \right)^{\frac{3}{2}} \int_0^\infty d\tau \tau^{\frac{1}{2}} e^{-2gn_0\tau} + \dots \quad (46)$$

The τ -integral is solved using again the Schwinger trick (44)

$$n(n_0) = n_0 + R \left(\frac{m}{2\pi\hbar^2} \right)^{\frac{3}{2}} \sqrt{\frac{\pi}{2g}} \sqrt{n_0} + \dots \quad (47)$$

Due to scattering theory the strength g of the contact interaction is related to the s-wave scattering length a according to [24]

$$g = \frac{4\pi\hbar^2}{m} a. \quad (48)$$

Inserting this relation into (47) yields the final result (1) which has originally been derived within a Bogoliubov theory for disordered bosons in Ref. [17] and has then been analyzed further by several groups [25, 26, 27, 28]. As we have performed a perturbative approach, Eq. (1) is only valid for small disorder strengths; thus, it cannot give any information to which degree the global condensate depletion proceeds in case the disorder strength is increased. Nevertheless, we also want to discuss a critical disorder strength R_c where the global condensate $n_0(n, R_c)$ vanishes in order to compare this result to the non-perturbative results of Section 3. In doing so, it is useful to introduce the dimensionless disorder strength

$$r = \sqrt{\frac{\pi}{2g}} R \left(\frac{m}{2\pi\hbar^2} \right)^{\frac{3}{2}}. \quad (49)$$

The result (1) then reads

$$n_0(n, r) = n - r\sqrt{n} \quad (50)$$

and we obtain the critical dimensionless disorder strength within the Huang-Meng theory:

$$r_c^{\text{HM}} = \sqrt{n}. \quad (51)$$

2.4.5 Chemical Potential

In this subsection we calculate the chemical potential $\mu(n)$ for a delta correlation function (42). Inserting (42) into (37) we obtain

$$\mu(n) = gn - R \int \frac{d^3k}{(2\pi)^3} \frac{\frac{\hbar^2 \mathbf{k}^2}{2m}}{\left(\frac{\hbar^2 \mathbf{k}^2}{2m} + 2gn\right)^2}, \quad (52)$$

where the \mathbf{k} -integral is UV-divergent. We evaluate it with dimensional regularization [23] and make use of the Schwinger trick (44), yielding

$$\mu(n) = gn - R \int_0^\infty d\tau \tau e^{-2gn\tau} \int \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 \mathbf{k}^2}{2m} e^{-\frac{\hbar^2 \mathbf{k}^2}{2m} \tau}. \quad (53)$$

We rewrite this by using the derivative of the Gaussian function and obtain

$$\mu(n) = gn + R \int_0^\infty d\tau \tau e^{-2gn\tau} \int \frac{d^3k}{(2\pi)^3} \frac{d}{d\tau} e^{-\frac{\hbar^2 \mathbf{k}^2}{2m} \tau}. \quad (54)$$

Now we solve the Gaussian integral, yielding

$$\mu(n) = gn + R \left(\frac{m}{2\pi\hbar^2}\right)^{\frac{3}{2}} \int_0^\infty d\tau \tau e^{-2gn\tau} \frac{d}{d\tau} \tau^{-\frac{3}{2}}. \quad (55)$$

After performing the derivative we have

$$\mu(n) = gn - \frac{3}{2} R \left(\frac{m}{2\pi\hbar^2}\right)^{\frac{3}{2}} \int_0^\infty d\tau \tau^{-\frac{3}{2}} e^{-2gn\tau}. \quad (56)$$

Once more we use (44) and obtain with (49)

$$\mu(n) = gn - 3g \sqrt{\frac{n}{\pi}} r \Gamma\left(-\frac{1}{2}\right). \quad (57)$$

The UV-divergence of the \mathbf{k} -integral (52) is now indicated by the negative argument of the gamma function which is evaluated with the help of analytic continuation:

$$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}, \quad (58)$$

so we obtain as the Huang-Meng result for the chemical potential

$$\mu(n, r) = gn + 6g\sqrt{nr}. \quad (59)$$

3 Non-Perturbative Approach

We consider again the stationary Gross-Pitaevskii equation for a real wave function (13). But this time we do not aim at a perturbative calculation, where the condensate wave function $\Psi(\mathbf{x})$ is expressed in lowest orders as a functional of the disorder potential $U(\mathbf{x})$. Instead, we perform a non-perturbative approximative calculation. It is based on the assumption that both, the disorder potential $U(\mathbf{x})$ and the wave function $\Psi(\mathbf{x})$, represent Gaussian processes. In that case, they are completely characterized by all first and second cumulants (see Appendix A). Thus, in addition to the first two cumulants of $U(\mathbf{x})$ in Eqs. (9) and (10) we also have the first two cumulants of the wave function $\Psi(\mathbf{x})$

$$\langle \Psi(\mathbf{x}) \rangle = \sqrt{n_0}, \quad (60)$$

$$\langle \Psi(\mathbf{x})\Psi(\mathbf{x}') \rangle_c = \langle \Psi(\mathbf{x})\Psi(\mathbf{x}') \rangle - \langle \Psi(\mathbf{x}) \rangle \langle \Psi(\mathbf{x}') \rangle \quad (61)$$

and the mixed cumulant

$$\langle U(\mathbf{x})\Psi(\mathbf{x}') \rangle_c = \langle U(\mathbf{x})\Psi(\mathbf{x}') \rangle. \quad (62)$$

Similarly to the argumentation in (9) and (10) spatial homogeneity implies

$$\langle \Psi(\mathbf{x})\Psi(\mathbf{x}') \rangle_c = G_{\Psi\Psi}(\mathbf{x} - \mathbf{x}'), \quad (63)$$

$$\langle U(\mathbf{x})\Psi(\mathbf{x}') \rangle_c = G_{U\Psi}(\mathbf{x} - \mathbf{x}'), \quad (64)$$

where $G_{\Psi\Psi}$ and $G_{U\Psi}$ are the corresponding correlation functions.

3.1 Condensate Density

Averaging the Gross-Pitaevskii equation (13) and inserting Eqs. (60)–(64), we obtain

$$G_{U\Psi}(\mathbf{0}) + g\langle \Psi(\mathbf{x})^3 \rangle = \mu\sqrt{n_0}. \quad (65)$$

As the wave function $\Psi(\mathbf{x})$ is assumed to be Gaussian, the cubic moment $\langle \Psi(\mathbf{x})^3 \rangle$ can be recursively reduced to the first and second cumulant (60), (61) as follows (see Appendix A):

$$\langle \Psi(\mathbf{x})^3 \rangle = \langle \Psi(\mathbf{x}) \rangle \langle \Psi(\mathbf{x})\Psi(\mathbf{x}) \rangle + 2\langle \Psi(\mathbf{x})\Psi(\mathbf{x}) \rangle_c \langle \Psi(\mathbf{x}) \rangle. \quad (66)$$

Using the equation (60) and (61) for $\mathbf{x} = \mathbf{x}'$ together with (63) and (15), i.e.

$$G_{\Psi\Psi}(\mathbf{0}) = n - n_0, \quad (67)$$

this reduces to

$$\langle \Psi(\mathbf{x})^3 \rangle = n_0^{\frac{3}{2}} + 3n_0^{\frac{1}{2}}G_{\Psi\Psi}(\mathbf{0}). \quad (68)$$

Inserting (68) into (65) yields

$$gn_0^{\frac{3}{2}} + [3gG_{\Psi\Psi}(\mathbf{0}) - \mu]n_0^{\frac{1}{2}} + G_{U\Psi}(\mathbf{0}) = 0. \quad (69)$$

Using again (67), Eq. (69) simplifies to

$$(3gn - \mu - 2gn_0)\sqrt{n_0} + G_{U\Psi}(\mathbf{0}) = 0. \quad (70)$$

Thus, in order to determine n_0 , we need to calculate $G_{U\Psi}(\mathbf{0})$.

3.2 Correlation Function $G_{U\Psi}$

At first, we determine the latter quantity by multiplying the Gross-Pitaevskii equation (13) with $U(\mathbf{x}')$ and then taking the disorder average. This leads to the equation

$$-\frac{\hbar^2}{2m}\Delta G_{U\Psi}(\mathbf{x}-\mathbf{x}') + \langle U(\mathbf{x}')U(\mathbf{x})\Psi(\mathbf{x}) \rangle + g\langle U(\mathbf{x}')\Psi(\mathbf{x})^3 \rangle = \mu G_{U\Psi}(\mathbf{x}-\mathbf{x}'). \quad (71)$$

Again we use the Gaussian approximation from Appendix A to reduce both expectation values $\langle U(\mathbf{x}')U(\mathbf{x})\Psi(\mathbf{x}) \rangle$ and $\langle U(\mathbf{x}')\Psi(\mathbf{x})^3 \rangle$ to first and second cumulants. With this we obtain for the first expectation value

$$\langle U(\mathbf{x}')U(\mathbf{x})\Psi(\mathbf{x}) \rangle = \langle U(\mathbf{x}') \rangle \langle U(\mathbf{x})\Psi(\mathbf{x}) \rangle + \langle U(\mathbf{x}')U(\mathbf{x}) \rangle_c \langle \Psi(\mathbf{x}) \rangle, \quad (72)$$

and with (9) and (10) this simplifies to

$$\langle U(\mathbf{x}')U(\mathbf{x})\Psi(\mathbf{x}) \rangle = \sqrt{n_0}R(\mathbf{x}-\mathbf{x}'). \quad (73)$$

Applying the same for the second expectation value, we obtain at first

$$\langle U(\mathbf{x}')\Psi(\mathbf{x})^3 \rangle = \langle U(\mathbf{x}') \rangle \langle \Psi(\mathbf{x})^3 \rangle + 3\langle U(\mathbf{x}')\Psi(\mathbf{x}) \rangle \langle \Psi(\mathbf{x})^2 \rangle. \quad (74)$$

Using (9), (64) and (67) the latter reduces to

$$\langle U(\mathbf{x}')\Psi(\mathbf{x})^3 \rangle = 3[n_0 + G_{\Psi\Psi}(\mathbf{0})]G_{U\Psi}(\mathbf{x}-\mathbf{x}'). \quad (75)$$

Now inserting (73) and (75) into (71) yields

$$\left\{ -\frac{\hbar^2}{2m}\Delta + 3g[n_0 + G_{\Psi\Psi}(\mathbf{0})] - \mu \right\} G_{U\Psi}(\mathbf{x}-\mathbf{x}') = -\sqrt{n_0}R(\mathbf{x}-\mathbf{x}'). \quad (76)$$

By applying a Fourier transformation this differential equation reduces to an algebraic equation with the solution

$$G_{U\Psi}(\mathbf{k}) = \frac{-\sqrt{n_0}R(\mathbf{k})}{\frac{\hbar^2\mathbf{k}^2}{2m} + 3g[n_0 + G_{\Psi\Psi}(\mathbf{0})] - \mu}. \quad (77)$$

By using (67) this simplifies to

$$G_{U\Psi}(\mathbf{k}) = \frac{-\sqrt{n_0}R(\mathbf{k})}{\frac{\hbar^2\mathbf{k}^2}{2m} + 3gn - \mu}. \quad (78)$$

Thus, the correlation function $G_{U\Psi}(\mathbf{0})$ turns out to be

$$G_{U\Psi}(\mathbf{0}) = -\sqrt{n_0} \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})}{\frac{\hbar^2\mathbf{k}^2}{2m} + 3gn - \mu}. \quad (79)$$

Combining (70) and (79) yields for a non-vanishing condensate density n_0 a first self-consistency relation between n , n_0 , and μ :

$$3gn - \mu - 2gn_0 = \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})}{\frac{\hbar^2\mathbf{k}^2}{2m} + 3gn - \mu}. \quad (80)$$

3.3 Correlation Function $G_{\Psi\Psi}$

A similar calculation is performed in order to derive a separate equation for $G_{\Psi\Psi}(\mathbf{0})$. This time we multiply the Gross-Pitaevskii equation (13) with $\Psi(\mathbf{x}')$ and then evaluate the disorder average:

$$-\frac{\hbar^2}{2m}\Delta G_{\Psi\Psi}(\mathbf{x}-\mathbf{x}') + \langle \Psi(\mathbf{x}')U(\mathbf{x})\Psi(\mathbf{x}) \rangle + g\langle \Psi(\mathbf{x}')\Psi(\mathbf{x})^3 \rangle = \mu[G_{\Psi\Psi}(\mathbf{x}-\mathbf{x}') + n_0]. \quad (81)$$

Again we use the Gaussian approximation from Appendix A to reduce the expectation values $\langle \Psi(\mathbf{x}')U(\mathbf{x})\Psi(\mathbf{x}) \rangle$ and $\langle \Psi(\mathbf{x}')\Psi(\mathbf{x})^3 \rangle$ to first and second cumulants. In addition we use our cumulant relations (60)–(64) and (67) and obtain

$$\langle \Psi(\mathbf{x}')U(\mathbf{x})\Psi(\mathbf{x}) \rangle = \sqrt{n_0}[G_{U\Psi}(\mathbf{0}) + G_{U\Psi}(\mathbf{x}-\mathbf{x}')] \quad (82)$$

as well as

$$\langle \Psi(\mathbf{x}')\Psi(\mathbf{x})^3 \rangle = \sqrt{n_0}\langle \Psi(\mathbf{x})^3 \rangle + 3G_{\Psi\Psi}(\mathbf{x}-\mathbf{x}')[G_{\Psi\Psi}(\mathbf{0}) + n_0]. \quad (83)$$

Now inserting (82) and (83) into (81) yields

$$-\sqrt{n_0}G_{U\Psi}(\mathbf{x}-\mathbf{x}') = \left\{ -\frac{\hbar^2}{2m}\Delta + 3g[G_{\Psi\Psi}(\mathbf{0}) + n_0] - \mu \right\} G_{\Psi\Psi}(\mathbf{x}-\mathbf{x}') + \sqrt{n_0} [G_{U\Psi}(\mathbf{0}) + g\langle \Psi(\mathbf{x})^3 \rangle - \mu\sqrt{n_0}]. \quad (84)$$

Using Eq. (65), which was derived from directly disorder averaging of the Gross-Pitaevskii equation (13), this simplifies to

$$\left\{ -\frac{\hbar^2}{2m}\Delta + 3g[G_{\Psi\Psi}(\mathbf{0}) + n_0] - \mu \right\} G_{\Psi\Psi}(\mathbf{x}-\mathbf{x}') = -\sqrt{n_0}G_{U\Psi}(\mathbf{x}-\mathbf{x}'). \quad (85)$$

After a Fourier transformation we obtain

$$G_{\Psi\Psi}(\mathbf{k}) = \frac{-\sqrt{n_0}G_{U\Psi}(\mathbf{k})}{\frac{\hbar^2\mathbf{k}^2}{2m} + 3g[G_{\Psi\Psi}(\mathbf{0}) + n_0] - \mu}. \quad (86)$$

We insert $G_{U\Psi}(\mathbf{k})$ from (77) and obtain together with (67)

$$G_{\Psi\Psi}(\mathbf{k}) = \frac{n_0R(\mathbf{k})}{\left(\frac{\hbar^2\mathbf{k}^2}{2m} + 3gn - \mu\right)^2}. \quad (87)$$

Transferring this equation into real space, yields

$$G_{\Psi\Psi}(\mathbf{0}) = n_0 \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})}{\left(\frac{\hbar^2\mathbf{k}^2}{2m} + 3gn - \mu\right)^2}. \quad (88)$$

Inserting (88) into (67) we get a second self-consistency relation between n , n_0 and μ :

$$n = n_0 + n_0 \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})}{\left(\frac{\hbar^2\mathbf{k}^2}{2m} + 3gn - \mu\right)^2}. \quad (89)$$

3.4 Perturbative Result

In this section we show that we can reobtain our previous perturbative results by expanding our non-perturbative model in lowest orders of the disorder strength $R(\mathbf{k})$. To this end we interpret both self-consistency relations (80) and (89) as follows. From Eq. (89) we deduce the function

$$n_0(\mu, n) = n \left[1 + \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})}{\left(\frac{\hbar^2 \mathbf{k}^2}{2m} + 3gn - \mu\right)^2} \right]^{-1}, \quad (90)$$

whereupon Eq. (80) allows then to determine the equation of state $n = n(\mu)$. Inserting (90) into (80) yields the following implicit equation for $n = n(\mu)$:

$$gn = \frac{1 + \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})}{\left(\frac{\hbar^2 \mathbf{k}^2}{2m} + 3gn - \mu\right)^2}}{1 + 3 \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})}{\left(\frac{\hbar^2 \mathbf{k}^2}{2m} + 3gn - \mu\right)^2}} \left[\mu + \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})}{\frac{\hbar^2 \mathbf{k}^2}{2m} + 3gn - \mu} \right]. \quad (91)$$

Performing a Taylor expansion in leading order of the disorder strength $R(\mathbf{k})$, we obtain the equation of state

$$n(\mu) = \frac{1}{g} \left[\mu + \int \frac{d^3k}{(2\pi)^3} \frac{\frac{\hbar^2 \mathbf{k}^2}{2m} R(\mathbf{k})}{\left(\frac{\hbar^2 \mathbf{k}^2}{2m} + 2\mu\right)^2} \right] + \dots \quad (92)$$

As one can see, we reproduced (37), which is our result from the perturbative calculation. In the same way we reproduce (36) by inserting (92) into (90)

$$n_0(\mu) = \frac{1}{g} \frac{\mu + \int \frac{d^3k}{(2\pi)^3} \frac{\frac{\hbar^2 \mathbf{k}^2}{2m} R(\mathbf{k})}{\left(\frac{\hbar^2 \mathbf{k}^2}{2m} + 2\mu\right)^2} + \dots}{1 + \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})}{\left(\frac{\hbar^2 \mathbf{k}^2}{2m} + 3gn - \mu\right)^2} + \dots} \quad (93)$$

and then performing a Taylor expansion in leading order of $R(\mathbf{k})$.

3.5 Global Condensate Density

In this section we will consider the disorder induced depletion of the global condensate density $n_0(n, R)$. In the perturbative section we were only able to determine the depletion for small $R(\mathbf{k})$, since the result was only obtained by a perturbation theory for small disorder. We will now consider a delta correlation disorder (42) and obtain a result which is not only reliable for small R . The question is, whether a quantum phase transition to a Bose-glass phase occurs for some critical R_c so that $n_0(n, R_c) = 0$. With the delta correlation disorder we can

specialize the two self-consistency equations (80) and (89). Using the dimensionless disorder strength (49), we obtain

$$3gn - \mu - 2gn_0 = -2\sqrt{2gr}\sqrt{3gn - \mu}, \quad (94)$$

$$n - n_0 = n_0 r \frac{\sqrt{2g}}{\sqrt{3gn - \mu}}. \quad (95)$$

Equation (94) directly yields

$$\sqrt{3gn - \mu} = -\sqrt{2gr} \left(1 \mp \sqrt{1 + \frac{n_0}{r^2}} \right). \quad (96)$$

Inserting (96) into (95), we obtain

$$n = \pm \sqrt{1 + \frac{n_0}{r^2}} (n - n_0). \quad (97)$$

Obviously, only a positive density is physically meaningful. That means we have to consider a positive sign of the right-hand side, leading to the following equation

$$n_0^2 + (r^2 - 2n)n_0 + n^2 - 2nr^2 = 0. \quad (98)$$

This quadratic equation for n_0 is solved by

$$n_0(n, r) = n - \frac{r^2}{2} - \frac{r}{2} \sqrt{r^2 + 4n}. \quad (99)$$

Here the sign of the square root has to be chosen negative, since the global condensate density n_0 as a fraction of the total density n has to stay smaller than the total density $n > n_0$. The result can be physically interpreted as follows. For a given condensate density n one can find a critical disorder strength r_c for which the global condensate vanishes, i.e. a quantum phase transition to a Bose-glass phase would occur at

$$r_c = \sqrt{\frac{n}{2}}. \quad (100)$$

In Figure 3 the function of the global condensate density (99) is plotted together with its first-order approximation which coincides with the Huang-Meng result (50). One can clearly see that the results coincide for small disorder strength R whereas there is a discrepancy for strong disorder.

3.6 Chemical Potential

With equation (96) we also have the possibility of determining the chemical potential $\mu = \mu(n, r)$ as a function of the dimensionless disorder strength r and the total density n . Taking

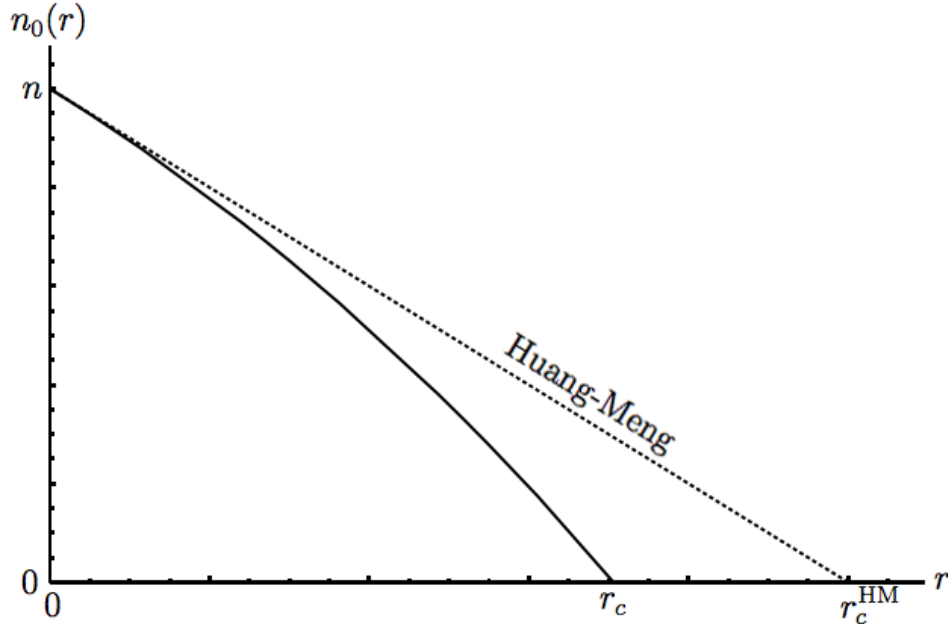


Fig. 3 – The condensate density n_0 from Eq. (99) decreases with the dimensionless disorder strength r from Eq. (49). A quantum phase transition is achieved at r_c from Eq. (100), as then the condensate density vanishes. For $0 \leq r \leq r_c$ a superfluid phase exists, whereas for $r > r_c$ a Bose-glass phase emerges. For comparison also the Huang-Meng result (50) is shown together with its critical dimensionless disorder strength (51).

into account the explanation above, in equation (96) only the negative square root has to be considered. We then find

$$\mu = 3gn - 2gn_0 + 4gr^2 \left(\sqrt{1 + \frac{n_0}{r^2}} - 1 \right). \quad (101)$$

Inserting (99) yields

$$\mu(n, r) = gn - 3gr^2 + gr\sqrt{r^2 + 4n} + 4gr\sqrt{\frac{r^2}{2} + n - \frac{r}{2}\sqrt{r^2 + 4n}} \quad (102)$$

$$(103)$$

which simplifies due to the quadratic completion

$$\sqrt{\frac{r^2}{2} + n - \frac{r}{2}\sqrt{r^2 + 4n}} = \frac{1}{2}\sqrt{r^2 + 4n} - \frac{r}{2} \quad (104)$$

to

$$\mu(n, r) = gn - 5gr^2 + 3gr\sqrt{r^2 + 4n}. \quad (105)$$

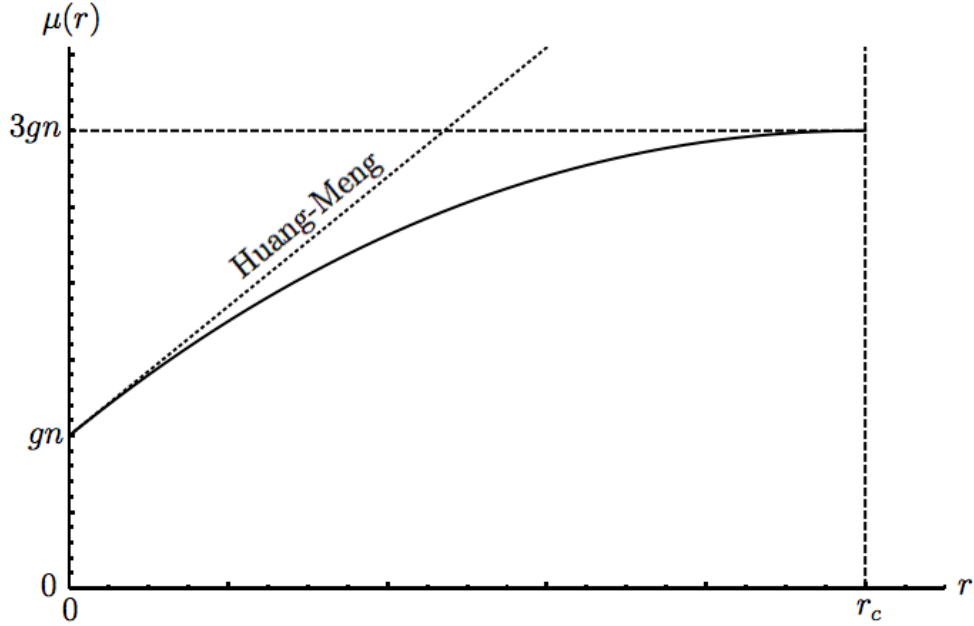


Fig. 4 – The chemical potential μ from (105) increases monotonically with increasing dimensionless disorder strength r from $\mu(n, 0) = gn$ up to $\mu(n, r_c) = 3gn$. For comparison also the Huang-Meng result (59) is shown.

In Figure 4 we plot $\mu(n, r)$ Eq. (105) together with the linear Huang-Meng result (59), again both functions coincide for small disorder.

Inserting our critical disorder strength r_c from (100), we obtain the critical chemical potential

$$\mu_c = 3gn. \quad (106)$$

We can also use equation (105) in order to compute the compressibility as a function of the disorder strength r and the condensate density n . In this case the compressibility is given by [29]

$$\kappa(n, r) = \frac{1}{\frac{\partial \mu(n, r)}{\partial n}} \quad (107)$$

and we obtain

$$\kappa(n, r) = \frac{1}{g + \frac{6gr}{\sqrt{4n+r^2}}}. \quad (108)$$

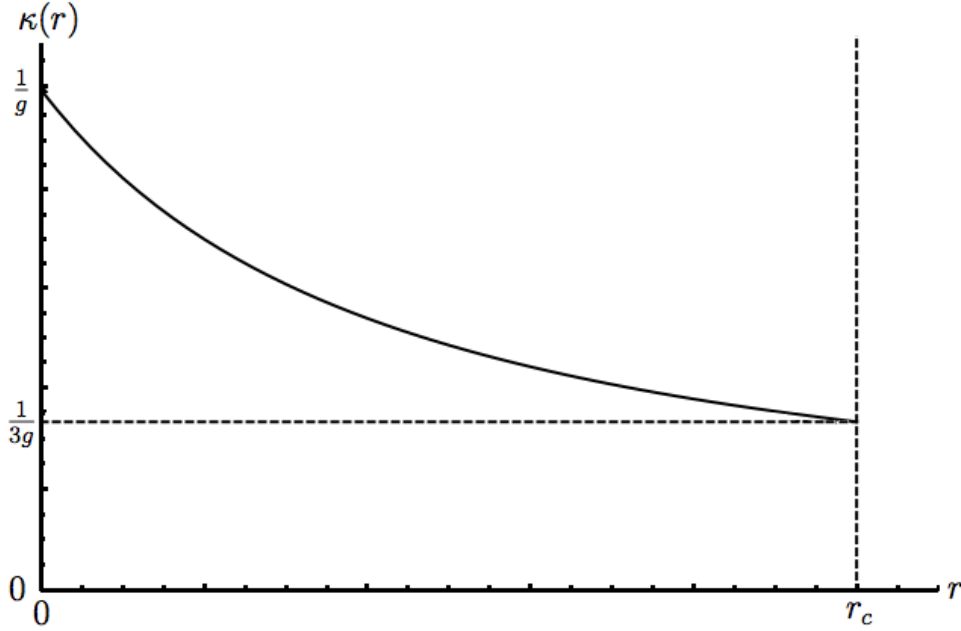


Fig. 5 – The compressibility κ from Eq. (108) decreases monotonically with increasing dimensionless disorder strength r .

We observe in Fig. 5 that the compressibility is decreasing monotonically from the initial value

$$\kappa(n, 0) = \frac{1}{g}, \quad (109)$$

up to the final value

$$\kappa_c(n, r_c) = \frac{1}{3g}. \quad (110)$$

However, the function $\kappa(n, r)$ has no special attributes for the critical dimensionless disorder strength r_c as one would imagine for a phase transition.

4 Superfluid and Bose-Glass Phase

4.1 Order Parameters

In the last section we have seen, that in a non-perturbative approach it is possible to obtain a critical disorder strength r_c , where the global condensate density n_0 vanishes and only the local fragmented condensates remain. Using this model of the coexistence of a fragmented and a global condensate in the superfluid phase, it is reasonable to introduce n_0 as an order

parameter of the global condensate as well as another second order parameter q for the local fragmented condensates. This has recently been done in the work [20], where the global condensate order parameter n_0 is introduced as [30]

$$n_0 = \lim_{|\mathbf{x}-\mathbf{x}'|\rightarrow\infty} \langle \overline{\Psi(\mathbf{x}, \tau)\Psi^*(\mathbf{x}', \tau^+)} \rangle, \quad (111)$$

where $\overline{\cdot\cdot\cdot}$ denotes the quantum average. Furthermore, the Bose-glass order parameter q has been introduced in close analogy to the Edward-Anderson order parameter of spin glasses [19] by both

$$(q + n_0)^2 = \lim_{|\mathbf{x}-\mathbf{x}'|\rightarrow\infty} \langle |\overline{\Psi(\mathbf{x}, \tau)\Psi^*(\mathbf{x}', \tau^+)}|^2 \rangle \quad (112)$$

and

$$q + n_0 = \lim_{|\tau-\tau'|\rightarrow\infty} \langle \overline{\Psi(\mathbf{x}, \tau)\Psi^*(\mathbf{x}, \tau')} \rangle. \quad (113)$$

Here the infinitesimally shifted imaginary time $\tau^+ = \tau + \eta$ with $\eta \downarrow 0$ is necessary to guarantee the normal ordering within the underlying functional integral representation of the 2-point function [20].

In our calculation we restrict ourselves to the special case of a time-independent wave function $\Psi(\mathbf{x})$. With this simplification Eqs. (111)–(113) reduce to

$$n_0 = \lim_{|\mathbf{x}-\mathbf{x}'|\rightarrow\infty} \langle \Psi(\mathbf{x})\Psi^*(\mathbf{x}') \rangle, \quad (114)$$

$$(q + n_0)^2 = \lim_{|\mathbf{x}-\mathbf{x}'|\rightarrow\infty} \langle |\Psi(\mathbf{x})\Psi^*(\mathbf{x}')|^2 \rangle \quad (115)$$

$$q + n_0 = \langle |\Psi(\mathbf{x})|^2 \rangle. \quad (116)$$

4.2 Model

These definitions can be motivated by the following model. We divide the wave function into two parts. One represents the global condensate density n_0 and the other one describes the density of the fragmented condensates q :

$$\Psi(\mathbf{x}) = \sqrt{n_0} + \sqrt{q}e^{i\phi(\mathbf{x})}. \quad (117)$$

The phase of the global condensate can be fixed to zero without loss of generality, whereas the second term for the fragmented condensate is assumed to have a random phase for each space coordinate. Thus, the disorder average corresponds to an average over all possible phases and is then defined by

$$\langle \bullet \rangle = \prod_{\mathbf{y}} \int_0^{2\pi} \frac{d\phi(\mathbf{y})}{2\pi} \bullet. \quad (118)$$

With this we will now determine the average of the wave function

$$\langle \Psi(\mathbf{x}) \rangle = \prod_{\mathbf{y}} \int_0^{2\pi} \frac{d\phi(\mathbf{y})}{2\pi} [\sqrt{n_0} + \sqrt{q}e^{i\phi(\mathbf{x})}]. \quad (119)$$

The average over all realizations cancels out the second term, yielding

$$\langle \Psi(\mathbf{x}) \rangle = \sqrt{n_0}. \quad (120)$$

This is what we hope to find, since our ansatz coincides with the result we already know from (14). In the same way we now consider the average

$$\langle \Psi^*(\mathbf{x}') \Psi(\mathbf{x}) \rangle = \prod_{\mathbf{y}} \int_0^{2\pi} \frac{d\phi(\mathbf{y})}{2\pi} [\sqrt{n_0} + \sqrt{q}e^{-i\phi(\mathbf{x}')}][\sqrt{n_0} + \sqrt{q}e^{i\phi(\mathbf{x})}]. \quad (121)$$

We distinguish between the two cases $\mathbf{x} = \mathbf{x}'$ and $\mathbf{x} \neq \mathbf{x}'$. The first case $\mathbf{x} = \mathbf{x}'$ yields

$$\begin{aligned} \langle |\Psi(\mathbf{x})|^2 \rangle &= \prod_{\mathbf{y}} \int_0^{2\pi} \frac{d\phi(\mathbf{y})}{2\pi} \{n_0 + 2\sqrt{n_0 q} \cos \phi(\mathbf{x}) + q\} \\ &= n_0 + q, \end{aligned} \quad (122)$$

whereas the second case results in

$$\begin{aligned} \lim_{|\mathbf{x}-\mathbf{x}'| \rightarrow \infty} \langle \Psi^*(\mathbf{x}') \Psi(\mathbf{x}) \rangle &= \lim_{|\mathbf{x}-\mathbf{x}'| \rightarrow \infty} \prod_{\mathbf{y}} \int_0^{2\pi} \frac{d\phi(\mathbf{y})}{2\pi} \\ &\quad \times \left\{ n_0 + \sqrt{n_0 q} [e^{i\phi(\mathbf{x})} + e^{-i\phi(\mathbf{x}')}] + qe^{i[\phi(\mathbf{x}) - \phi(\mathbf{x}')]} \right\} \\ &= n_0. \end{aligned} \quad (123)$$

The last average we are considering is

$$\begin{aligned} \lim_{|\mathbf{x}-\mathbf{x}'| \rightarrow \infty} \langle |\Psi(\mathbf{x}) \Psi(\mathbf{x}')|^2 \rangle &= \lim_{|\mathbf{x}-\mathbf{x}'| \rightarrow \infty} \prod_{\mathbf{y}} \int_0^{2\pi} \frac{d\phi(\mathbf{y})}{2\pi} \\ &\quad \times [n_0^2 + q^2 + 2n_0 q + 2\sqrt{n_0 q}(\cos \phi(\mathbf{x}) + \cos \phi(\mathbf{x}'))(n_0 + q) \\ &\quad + 4n_0 q \cos \phi(\mathbf{x}) \cos \phi(\mathbf{x}')] \\ &= (n_0 + q)^2. \end{aligned} \quad (124)$$

Again the random phases cancel out over the different realizations. As we can see Eqs. (114)–(116) are reproduced within our model due to Eqs. (122)–(124).

4.3 Application

Next we will compare this idea of a global condensate and a Bose-glass order parameter with our results from the non-perturbative section, to check whether the introduction of a Bose-glass order parameter is consistent. Comparing (15), (67) and (116) shows that the order parameter q in our non-perturbative result is given by

$$q = G_{\Psi\Psi}(\mathbf{0}). \quad (125)$$

Another relation for $G_{\Psi\Psi}(\mathbf{x} - \mathbf{x}')$ is obtained by taking the long-range limit and inserting (60), (63) and (114) into (61)

$$\lim_{|\mathbf{x}-\mathbf{x}'|\rightarrow\infty} G_{\Psi\Psi}(\mathbf{x} - \mathbf{x}') = 0. \quad (126)$$

In addition to (114) and (116), we also investigate the correlation function $\langle |\Psi(\mathbf{x})\Psi(\mathbf{x}')|^2 \rangle$ in the long-range limit $|\mathbf{x} - \mathbf{x}'| \rightarrow \infty$. We obtain in the Gaussian approximation

$$\langle |\Psi(\mathbf{x})\Psi(\mathbf{x}')|^2 \rangle = n_0^2 + 2n_0 G_{\Psi\Psi}(\mathbf{0}) + 4n_0 G_{\Psi\Psi}(\mathbf{x} - \mathbf{x}') + 2G_{\Psi\Psi}^2(\mathbf{x} - \mathbf{x}') + G_{\Psi\Psi}(\mathbf{0}) \quad (127)$$

by using our cumulant relations from Appendix A. Applying the long-range limit and inserting (126) yields

$$\lim_{|\mathbf{x}-\mathbf{x}'|\rightarrow\infty} \langle |\Psi(\mathbf{x})\Psi(\mathbf{x}')|^2 \rangle = [n_0 + G_{\Psi\Psi}(\mathbf{0})]^2. \quad (128)$$

A comparison of (115) and (128) yields again (125).

We now prove (126) within our non-perturbative approach as follows. Applying a Fourier transformation yields at first

$$G_{\Psi\Psi}(\mathbf{x} - \mathbf{x}') = \int \frac{d^3k}{(2\pi)^3} G_{\Psi\Psi}(\mathbf{k}) e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}. \quad (129)$$

Then we use our non-perturbative result (87) and obtain

$$G_{\Psi\Psi}(\mathbf{x} - \mathbf{x}') = \int \frac{d^3k}{(2\pi)^3} \frac{n_0 R(\mathbf{k})}{\left(\frac{\hbar^2 \mathbf{k}^2}{2m} + 3gn - \mu\right)^2} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}. \quad (130)$$

Inserting the Fourier transform of

$$R(\mathbf{k}) = \int d^3x'' R(\mathbf{x}'') e^{-i\mathbf{k}\mathbf{x}''}, \quad (131)$$

yields

$$G_{\Psi\Psi}(\mathbf{x} - \mathbf{x}') = n_0 \int d^3x'' R(\mathbf{x}'') \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}'-\mathbf{x}'')}}{\left(\frac{\hbar^2 \mathbf{k}^2}{2m} + 3gn - \mu\right)^2}. \quad (132)$$

Applying the Schwinger trick (44) in order to eliminate the denominator, we obtain

$$G_{\Psi\Psi}(\mathbf{x} - \mathbf{x}') = n_0 \int d^3x'' R(\mathbf{x}'') \int_0^\infty d\tau \tau e^{-(3gn-\mu)\tau} \int \frac{d^3k}{(2\pi)^3} e^{-\frac{\hbar^2 \mathbf{k}^2}{2m}\tau + i\mathbf{k}(\mathbf{x}-\mathbf{x}'-\mathbf{x}'')}. \quad (133)$$

Now we solve the \mathbf{k} -integral by using the Gaussian integral

$$\int_{-\infty}^\infty dk e^{-ak^2 - bk} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}, \quad (134)$$

yielding

$$G_{\Psi\Psi}(\mathbf{x} - \mathbf{x}') = n_0 \left(\frac{m}{2\pi\hbar^2}\right)^{\frac{3}{2}} \int d^3x'' R(\mathbf{x}'') \int_0^\infty d\tau \tau^{-\frac{1}{2}} e^{-(3gn-\mu)\tau - \frac{m}{2\hbar^2}(\mathbf{x}-\mathbf{x}'-\mathbf{x}'')^2 \frac{1}{\tau}}. \quad (135)$$

In order to solve this integral we take a look in the literature. It can be found in [31, Eq. 3.471.9]

$$\int_0^\infty dx x^{\nu-1} e^{-\frac{\beta}{x} - \gamma x} = 2 \left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\beta\gamma}), \quad \beta, \gamma > 0 \quad (136)$$

where K_ν is the modified Bessel function of order ν . In view of our integral (135) we set

$$\nu = \frac{1}{2}, \quad \beta = \frac{m}{2\hbar^2}(\mathbf{x} - \mathbf{x}' - \mathbf{x}'')^2, \quad \gamma = 3gn - \mu. \quad (137)$$

Note, that the condition $\beta, \gamma > 0$ is provided, since from the non-perturbative section (106) we know that $3gn > \mu$ holds. With this we obtain

$$G_{\Psi\Psi}(\mathbf{x} - \mathbf{x}') = 2n_0 \left(\frac{m}{2\pi\hbar^2}\right)^{\frac{3}{2}} \int d^3x'' R(\mathbf{x}'') \left[\frac{m}{2\hbar^2} \frac{(\mathbf{x} - \mathbf{x}' - \mathbf{x}'')^2}{3gn - \mu} \right]^{\frac{1}{4}} \times K_{\frac{1}{2}} \left(2\sqrt{\frac{m}{2\hbar^2}(\mathbf{x} - \mathbf{x}' - \mathbf{x}'')^2(3gn - \mu)} \right). \quad (138)$$

We insert the modified Bessel function [31, Eq. 8.469.3]

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \quad (139)$$

and finally obtain

$$G_{\Psi\Psi}(\mathbf{x} - \mathbf{x}') = n_0 \left(\frac{m}{2\pi\hbar^2}\right)^{\frac{3}{2}} \sqrt{\frac{\pi}{3gn - \mu}} \int d^3x'' R(\mathbf{x}'') e^{-\sqrt{\frac{2m}{\hbar^2}(3gn-\mu)|\mathbf{x}-\mathbf{x}'-\mathbf{x}''|}}. \quad (140)$$

Again, note that $3gn - \mu > 0$ is fulfilled so the square root remains a real number. Now we can take the long-range limit and obtain indeed (126) irrespective of the disorder correlation function $R(\mathbf{x} - \mathbf{x}')$.

This result shows that the introduction of the two order parameters coincides with the results obtained in Section 3.

5 Discussion

In this thesis the depletion of a global condensate density due to a random external disorder potential is studied. At first a perturbative approach was successfully applied in order to rederive the result of Huang and Meng of Ref. [17]. In addition, another non-perturbative approach was applied to the same problem in order to derive a depletion which can be applied to strong disorder. The general result were two self-consistency equations which were cross-checked by a perturbative expansion, where again the result of Ref. [17] was reproduced. In the last part of this work a Bose-glass order parameter [20] was motivated, introduced and effectively applied to the non-perturbative result.

The two self-consistency equations from the non-perturbative section could be solved in the special case of a delta disorder correlation function. Furthermore, a critical disorder strength r_c for the phase transition to a Bose-glass phase was found in Eq. (100).

These findings can be compared to other theoretical approaches. For example, in the work of G. M. Falco, T. Nattermann and V. L. Pokrovsky [32] the problem is tackled from a different point of view. In contrast to our approach, they consider the localized state with a fixed disorder strength R and obtain a critical density n_c , where the phase transition to the superfluid occurs at

$$n_c = \frac{1}{3\mathcal{L}^2 a}, \quad (141)$$

where the Larkin length \mathcal{L} is introduced by [32]

$$\mathcal{L} = \frac{\hbar^4}{m^2 R}. \quad (142)$$

Comparing (141) to our result (100) by taking into account the dimensionless disorder strength (49) we find that they do not coincide quantitatively, as the result of G. M. Falco, T. Nattermann and V. L. Pokrovsky leads to

$$r_c^{\text{FNP}} = \sqrt{\frac{3n}{64\pi^3}} \approx 0,04\sqrt{n}, \quad (143)$$

whereas our result is approximately

$$r_c \approx 0,71\sqrt{n}. \quad (144)$$

Besides the different prefactors, however, the qualitative result is the same.

In a paper of P. Navez, A. Pelster and R. Graham [33] also a quantum phase transition was found within a random phase approximation. However, whereas their Fig. 6 indicates a

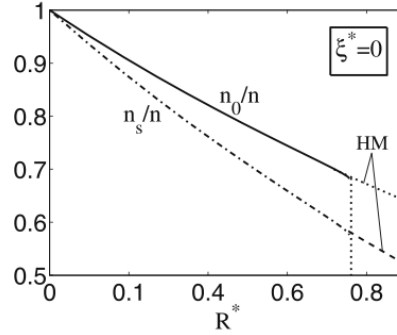


Fig. 6 – Clean part of the condensate fraction n_0 as a function of R^* in the case of uncorrelated disorder in the RPA model (full curve) and in the HM model (dotted curve) and the corresponding superfluid fraction n_s (dot-dashed curve in RPA and dashed curve in the HM model). In our model R^* corresponds to the dimensionless disorder strength r (from Ref. [33]).

first-order phase transition, our result in Fig. 3 suggests a second-order phase transition to the Bose-glass phase. Comparing the position of the critical dimensionless disorder strength

$$r_c^{\text{NPG}} \approx 0,75\sqrt{n} \quad (145)$$

of Ref. [33] to ours, we find that they also agree quantitatively with a discrepancy of about 5%.

All in all the main result of this thesis, a critical disorder strength for the quantum phase transition to a Bose-glass phase (100), agrees at least qualitatively with current approaches of other groups. In addition, also the Huang-Meng result (1) is included in our non-perturbative result, as it is obtained for small disorder.

A Appendix - Moments of a Gaussian Distribution

A.1 Moments and Cumulants

We consider a stochastic variable x with a probability density $P(x)$. The average of a function $f(x)$ is then defined by

$$\langle f(x) \rangle = \int dx f(x) P(x). \quad (146)$$

As the probability density $P(x)$ is normalized, we have

$$\langle 1 \rangle = 1. \quad (147)$$

In statistics the generating function $C(k)$ of a random variable x is defined by

$$C(k) = \langle e^{ikx} \rangle. \quad (148)$$

Applying the exponential series, we obtain

$$C(k) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} M_n, \quad (149)$$

where $M_n = \langle x^n \rangle$ are the so called moments with

$$M_0 = 1 \quad (150)$$

due to (147). We conclude, that knowing all moments corresponds to knowing all statistical properties, i.e. the generating function. It is useful to introduce in addition so called cumulants K_n via

$$\ln C(k) = \left[\sum_{n=1}^{\infty} \frac{(ik)^n}{n!} K_n \right]. \quad (151)$$

It turns out that one can express all cumulants K_n in terms of lower order moments by comparing the respective coefficients. To do so we insert (149) in the left-hand side of Eq. (151)

$$\ln C(k) = \ln \left[\sum_{n=0}^{\infty} \frac{(ik)^n}{n!} M_n \right] \quad (152)$$

and Taylor expand the logarithm

$$\ln C(k) = \lim_{N \rightarrow \infty} \sum_{m=1}^N \frac{(-1)^{m-1}}{m} \left[\sum_{n=1}^N \frac{(ik)^n}{n!} M_n \right]^m. \quad (153)$$

Applying the multinomial formula [34, page 106]

$$\lim_{N \rightarrow \infty} \left(\sum_{n=0}^N x_n \right)^m = \lim_{N \rightarrow \infty} \sum_{k_1, k_2, \dots, k_N} \sum_{\sum k_i = m} \binom{m}{k_1, k_2, \dots, k_N} x_1^{k_1} x_2^{k_2} \cdots x_N^{k_N} \quad (154)$$

and (151), we obtain

$$\begin{aligned} \lim_{N \rightarrow \infty} \left[\sum_{n=0}^N \frac{(ik)^n}{n!} K_n \right] &= \lim_{N \rightarrow \infty} \sum_{m=1}^N \sum_{m_1, \dots, m_N} \sum_{\sum m_i = m} \frac{(-1)^{m-1}}{m} \binom{m}{m_1, \dots, m_N} \\ &\quad \times \left[\frac{(ik)^1}{1!} M_1 \right]^{m_1} \cdots \left[\frac{(ik)^m}{m!} M_m \right]^{m_m}. \end{aligned} \quad (155)$$

A careful comparison of the coefficients for the a -th order terms yields [35]

$$K_a = a! \sum_{\sum_{n=1}^a n m_n = a} \sum_{\sum m_i = m} \frac{(-1)^{m-1}}{m} \binom{m}{m_1, \dots, m_a} \prod_{n=1}^a \left(\frac{M_n}{n!} \right)^{m_n}. \quad (156)$$

Applying (156) for the first four cumulants we find the following relations:

$$K_1 = M_1 \quad (157)$$

$$K_2 = M_2 - M_1^2 \quad (158)$$

$$K_3 = M_3 - 3M_1 M_2 + 2M_1^3 \quad (159)$$

$$K_4 = M_4 - 3M_2^2 + 12M_2 M_1^2 - 4M_1 M_3 - 6M_1^4. \quad (160)$$

A.2 Gaussian Distribution

In the Gaussian case all cumulants with an order higher than two are zero

$$K_3 = K_4 = \dots = 0., \quad (161)$$

so we are able to express in Eqs. (157)–(160) every moment in terms of K_1 and K_2 :

$$M_1 = K_1 \quad (162)$$

$$M_2 = K_2 + K_1^2 \quad (163)$$

$$M_3 = 3K_1 K_2 + K_1^3 \quad (164)$$

$$M_4 = 3K_2^2 + 6K_2 K_1^2 + K_1^4. \quad (165)$$

Next we aim at a more convenient expression between moments and cumulants. To this end we mention that the generating function (152) reduces due to (161) to

$$C(k) = e^{iK_1 k - \frac{1}{2} K_2 k^2}. \quad (166)$$

With the definition of the delta function and (146) we obtain for an expectation value

$$\langle f(x) \rangle = \int dx' f(x') \langle \delta(x' - x) \rangle. \quad (167)$$

Applying the Fourier representation of the delta function yields

$$\langle f(x) \rangle = \int dx' f(x') \int \frac{dk}{2\pi} e^{-ikx'} \int dx e^{ikx} P(x). \quad (168)$$

We insert (166) and obtain

$$\langle f(x) \rangle = \int dx' f(x') \int \frac{dk}{2\pi} e^{-\frac{1}{2}K_2 k^2 + i(K_1 - x')k}. \quad (169)$$

Performing the Gaussian integral with (134) yields

$$\langle f(x) \rangle = \int dx' f(x') \frac{1}{\sqrt{2\pi K_2}} e^{-\frac{(x' - K_1)^2}{2K_2}}. \quad (170)$$

In particular we obtain for $f(x) = x^n$

$$\langle x^n \rangle = \int dx x^n \frac{1}{\sqrt{2\pi K_2}} e^{-\frac{(x - K_1)^2}{2K_2}}. \quad (171)$$

We rearrange (171) according to

$$\langle x^n \rangle = \int dx x^{n-1} K_2 \left(\frac{x - K_1}{K_2} + \frac{K_1}{K_2} \right) \frac{e^{-\frac{(x - K_1)^2}{2K_2}}}{\sqrt{2\pi K_2}} \quad (172)$$

and obtain at first

$$\langle x^n \rangle = K_1 \langle x^{n-1} \rangle + K_2 \int dx x^{n-1} \left[-\frac{d}{dx} \frac{e^{-\frac{(x - K_1)^2}{2K_2}}}{\sqrt{2\pi K_2}} \right] \quad (173)$$

Now an integration by parts yields

$$\langle x^n \rangle = K_1 \langle x^{n-1} \rangle + (n-1)K_2 \int dx x^{n-2} \frac{e^{-\frac{(x - K_1)^2}{2K_2}}}{\sqrt{2\pi K_2}} \quad (174)$$

which corresponds to

$$\langle x^n \rangle = K_1 \langle x^{n-1} \rangle + (n-1)K_2 \langle x^{n-2} \rangle, \quad (175)$$

or in terms of moments

$$M_n = K_1 M_{n-1} + (n-1)K_2 M_{n-2}. \quad (176)$$

This recursive relation is often used in our calculations when dealing with the Gaussian approximation. In the next two subsections we will generalize it, in order to use it for many variables and also for a continuum of variables. In doing so, however, the spirit of the recursive relation essentially stays the same.

With this recursive relation (176) we determine the first four moments in terms of the cumulants K_1 and K_2

$$M_1 = K_1 \quad (177)$$

$$M_2 = K_1 M_1 + K_2 \quad (178)$$

$$M_3 = K_1 M_2 + 2K_2 M_1 \quad (179)$$

$$M_4 = K_1 M_3 + 3K_2 M_2 \quad (180)$$

$$= K_1^2 M_2 + 2K_1 K_2 M_1 + 3K_2 M_2, \quad (181)$$

using the natural initial condition (150) which corresponds to the normalization of the probability distribution (147). Note that the findings (177)–(181) are identical to (162)–(165).

As a remark we note that the same recursion relation (176) can also be rederived with determinant methods. To this end we use a general expression between moments and cumulants [36, page 18]

$$M_n = \begin{vmatrix} K_1 & -1 & 0 & 0 & 0 & \dots \\ K_2 & K_1 & -1 & 0 & 0 & \dots \\ K_3 & \binom{2}{1} K_2 & K_1 & -1 & 0 & \dots \\ K_4 & \binom{3}{1} K_3 & \binom{3}{2} K_2 & K_1 & -1 & \dots \\ K_5 & \binom{4}{1} K_4 & \binom{4}{2} K_3 & \binom{4}{3} K_2 & K_1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}_n. \quad (182)$$

In the Gaussian approximation (161) this simplifies to

$$M_n = \begin{vmatrix} K_1 & -1 & 0 & 0 & 0 & \dots \\ K_2 & K_1 & -1 & 0 & 0 & \dots \\ 0 & 2K_2 & K_1 & -1 & 0 & \dots \\ 0 & 0 & 3K_2 & K_1 & -1 & \dots \\ 0 & 0 & 0 & 4K_2 & K_1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}_n \quad (183)$$

Using the Laplace expansion of the last column, we reobtain (176).

A.3 Moments and Cumulants - Many Variables

In this section we extend the recursion formula between moments and cumulants (176) for many random variables

$$\mathbf{x} = (x_1, x_2, \dots, x_N). \quad (184)$$

The general expression for the average of a function $f(\mathbf{x})$ now reads

$$\langle f(\mathbf{x}) \rangle = \int d^N x' f(\mathbf{x}') \langle \delta(\mathbf{x} - \mathbf{x}') \rangle. \quad (185)$$

Again we apply the Fourier representation of the delta function and obtain

$$\langle f(\mathbf{x}) \rangle = \int d^N x' f(\mathbf{x}') \int \frac{d^N k}{(2\pi)^N} e^{-i\mathbf{k}\mathbf{x}'} \int d^N x e^{i\mathbf{k}\mathbf{x}} P(\mathbf{x}). \quad (186)$$

At this point we restrict ourselves to a Gaussian process where the generating function for many random variables is given by

$$C(\mathbf{k}) = e^{i\mathbf{K}_1 \mathbf{k} - \frac{1}{2} \mathbf{k}^T K_2 \mathbf{k}}, \quad (187)$$

where the first cumulant \mathbf{K}_1 is described by a vector and the second cumulant K_2 represents a matrix which has without loss of generality the property

$$K_2 = K_2^T. \quad (188)$$

Inserting (187) and into (186) yields

$$\langle f(\mathbf{x}) \rangle = \int d^N x' f(\mathbf{x}') \int \frac{d^N k}{(2\pi)^N} e^{i\mathbf{k}(\mathbf{K}_1 - \mathbf{x}') - \frac{1}{2} \mathbf{k}^T K_2 \mathbf{k}}. \quad (189)$$

We solve the Gaussian integral and obtain

$$\langle f(\mathbf{x}) \rangle = \int d^N x f(\mathbf{x}) \frac{1}{\sqrt{(2\pi)^N \det K_2}} e^{-\frac{1}{2}(\mathbf{x} - \mathbf{K}_1) K_2^{-1} (\mathbf{x} - \mathbf{K}_1)}. \quad (190)$$

Now we specialize the general function $f(\mathbf{x})$ to the polynomial

$$f(\mathbf{x}) = x_{i_1} x_{i_2} \cdots x_{i_n} \quad (191)$$

and obtain

$$\langle x_{i_1} x_{i_2} \cdots x_{i_n} \rangle = \frac{1}{\sqrt{(2\pi)^N \det K_2}} \int d^N x x_{i_2} \cdots x_{i_n} x_{i_1} e^{-\frac{1}{2} \sum_{i,j} (x_i - K_{1i}) K_{2ij}^{-1} (x_j - K_{1j})}. \quad (192)$$

In order to rewrite this and use the integration by parts similarly to the section before we take a look at the derivation

$$-\sum_{k=1}^N K_{2_{i_1 k}} \frac{d}{dx_k} e^{-\frac{1}{2} \sum_{i,j}^N (x_i - K_{1i}) K_{2ij}^{-1} (x_j - K_{1j})} = \sum_{k=1}^N \sum_{j=1}^N K_{2_{i_1 k}} K_{2_{kj}}^{-1} (x_j - K_{1j}) \times e^{-\frac{1}{2} \sum_{i,j}^N (x_i - K_{1i}) K_{2ij}^{-1} (x_j - K_{1j})}, \quad (193)$$

where we can use

$$\sum_{k=1}^N K_{2_{i_1 k}} K_{2_{kj}}^{-1} = \delta_{i_1 j}. \quad (194)$$

With this Eq. (193) reduces to

$$-\sum_{k=1}^N K_{2_{i_1 k}} \frac{d}{dx_k} e^{-\frac{1}{2} \sum_{i,j}^N (x_i - K_{1i}) K_{2ij}^{-1} (x_j - K_{1j})} = (x_{i_1} - K_{1_{i_1}}) e^{-\frac{1}{2} \sum_{i,j}^N (x_i - K_{1i}) K_{2ij}^{-1} (x_j - K_{1j})}. \quad (195)$$

We insert (195) into (192) and obtain with (192)

$$\begin{aligned} \langle x_{i_1} x_{i_2} \cdots x_{i_n} \rangle &= K_{1_{i_1}} \langle x_{i_2} \cdots x_{i_n} \rangle \\ &\quad - \sum_{k=1}^N K_{2_{i_1 k}} \int d^N x x_{i_2} \cdots x_{i_n} \frac{d}{dx_k} \frac{e^{-\frac{1}{2} \sum_{i,j}^N (x_i - K_{1i}) K_{2ij}^{-1} (x_j - K_{1j})}}{\sqrt{(2\pi)^N \det K_2}} \end{aligned} \quad (196)$$

Now an integration by parts yields

$$\begin{aligned} \langle x_{i_1} x_{i_2} \cdots x_{i_n} \rangle &= K_{1_{i_1}} \langle x_{i_2} \cdots x_{i_n} \rangle \\ &\quad + \sum_{k=1}^N K_{2_{i_1 k}} \int d^N x \frac{d}{dx_k} (x_{i_2} \cdots x_{i_n}) \frac{e^{-\frac{1}{2} \sum_{i,j}^N (x_i - K_{1i}) K_{2ij}^{-1} (x_j - K_{1j})}}{\sqrt{(2\pi)^N \det K_2}}, \end{aligned} \quad (197)$$

which reduces with (192) to the recursive relation

$$\begin{aligned} \langle x_{i_1} x_{i_2} \cdots x_{i_n} \rangle &= K_{1_{i_1}} \langle x_{i_2} \cdots x_{i_n} \rangle \\ &\quad + \sum_{l=2}^n K_{2_{i_1 l}} \langle x_{i_2} \cdots x_{i_{l-1}} x_{i_{l+1}} \cdots x_{i_n} \rangle. \end{aligned} \quad (198)$$

In the special case that all random variables coincide, i.e.

$$i_1 = i_2 = \dots = i_n, \quad (199)$$

the recursion relation (198) reduces to the previous one (176).

A.4 Random Fields

In this subsection we extend our model from many random variables x_1, x_2, \dots, x_N to many random fields $\Psi_1(\mathbf{x}_1), \Psi_2(\mathbf{x}_2), \dots, \Psi_N(\mathbf{x}_N)$. In straight analogy to Eq. (190) the average of a general functional $f[\Psi_1, \dots, \Psi_N]$ is given by

$$\langle f[\Psi_1, \dots, \Psi_N] \rangle = \prod_{k=1}^N \int \mathfrak{D}\Psi_k f[\Psi_1, \dots, \Psi_N] \quad (200)$$

$$\times e^{-\frac{1}{2} \sum_{i,j} \int d^3x \int d^3x' [\Psi_i(\mathbf{x}) - K_{1_i}(\mathbf{x})] K_{2_{ij}}^{-1}(\mathbf{x}, \mathbf{x}') [\Psi_j(\mathbf{x}') - K_{1_j}(\mathbf{x}')]}, \quad (201)$$

where each functional integral amounts to integrate at each space point \mathbf{x} over all possible values $\Psi_k(\mathbf{x})$:

$$\int \mathfrak{D}\Psi_k = \prod_{\mathbf{x}} \int d\Psi_k(\mathbf{x}). \quad (202)$$

We specialize our general functional to

$$f[\Psi_1, \dots, \Psi_N] = \Psi_{i_1}(\mathbf{x}_{i_1}) \cdots \Psi_{i_n}(\mathbf{x}_{i_n}) \quad (203)$$

and obtain

$$\langle \Psi_{i_1}(\mathbf{x}_{i_1}) \cdots \Psi_{i_n}(\mathbf{x}_{i_n}) \rangle = \prod_{k=1}^N \int \mathfrak{D}\Psi_k \Psi_{i_1}(\mathbf{x}_{i_1}) \cdots \Psi_{i_n}(\mathbf{x}_{i_n}) \times e^{-\frac{1}{2} \sum_{i,j} \int d^3x \int d^3x' [\Psi_i(\mathbf{x}) - K_{1_i}(\mathbf{x})] K_{2_{ij}}^{-1}(\mathbf{x}, \mathbf{x}') [\Psi_j(\mathbf{x}') - K_{1_j}(\mathbf{x}')]} \quad (204)$$

Similarly to the calculation in the previous subsection, we now consider the functional derivative

$$\begin{aligned} & - \sum_{k=1}^N \int d^3x_k K_{2_{i_1 k}}(\mathbf{x}_{i_1}, \mathbf{x}_k) \frac{\delta}{\delta \Psi_k(\mathbf{x}_k)} \\ & \times \exp \left\{ -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \int d^3x \int d^3x' [\Psi_i(\mathbf{x}) - K_{1_i}(\mathbf{x})] K_{2_{ij}}^{-1}(\mathbf{x}, \mathbf{x}') [\Psi_j(\mathbf{x}') - K_{1_j}(\mathbf{x}')] \right\} \\ & = \sum_{k=1}^N \int d^3x_k K_{2_{i_1 k}}(\mathbf{x}_{i_1}, \mathbf{x}_k) \sum_{j=1}^N \int d^3x' K_{2_{kj}}^{-1}(\mathbf{x}_k, \mathbf{x}') [\Psi_j(\mathbf{x}') - K_{1_j}(\mathbf{x}')] \\ & \times \exp \left\{ -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \int d^3x \int d^3x' [\Psi_i(\mathbf{x}) - K_{1_i}(\mathbf{x})] K_{2_{ij}}^{-1}(\mathbf{x}, \mathbf{x}') [\Psi_j(\mathbf{x}') - K_{1_j}(\mathbf{x}')] \right\}. \end{aligned} \quad (205)$$

Here we use that K_2 and K_2^{-1} are inverse to each other which means

$$\sum_{k=1}^N \int d^3x_k K_{2_{i_1 k}}(\mathbf{x}_{i_1}, \mathbf{x}_k) K_{2_{kj}}^{-1}(\mathbf{x}_k, \mathbf{x}') = \delta_{i_1 j} \delta(\mathbf{x}_{i_1} - \mathbf{x}'). \quad (206)$$

With this Eq. (205) reduces to

$$\begin{aligned}
& - \sum_{k=1}^N \int d^3 x_k K_{2_{i_1 k}}(\mathbf{x}_{i_1}, \mathbf{x}_k) \frac{\delta}{\delta \Psi_k(\mathbf{x}_k)} \\
& \times \exp \left\{ -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \int d^3 x \int d^3 x' [\Psi_i(\mathbf{x}) - K_{1_i}(\mathbf{x})] K_{2_{ij}}^{-1}(\mathbf{x}, \mathbf{x}') [\Psi_j(\mathbf{x}') - K_{1_j}(\mathbf{x}')] \right\} \\
& = [\Psi_{i_1}(\mathbf{x}_{i_1}) - K_{1_{i_1}}(\mathbf{x}_{i_1})] \\
& \times \exp \left\{ -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \int d^3 x \int d^3 x' [\Psi_i(\mathbf{x}) - K_{1_i}(\mathbf{x})] K_{2_{ij}}^{-1}(\mathbf{x}, \mathbf{x}') [\Psi_j(\mathbf{x}') - K_{1_j}(\mathbf{x}')] \right\}
\end{aligned} \tag{207}$$

in analogy to (195). Inserting (207) into (204) then yields with (204)

$$\begin{aligned}
\langle \Psi_{i_1}(\mathbf{x}_{i_1}) \cdots \Psi_{i_n}(\mathbf{x}_{i_n}) \rangle & = K_{1_{i_1}}(\mathbf{x}_{i_1}) \langle \Psi_2(\mathbf{x}_{i_2}) \cdots \Psi_{i_n}(\mathbf{x}_{i_n}) \rangle \\
& - \sum_{k=1}^N \int d^3 x_k K_{2_{i_1 k}}(\mathbf{x}_{i_1}, \mathbf{x}_k) \prod_{k'=1}^N \int \mathfrak{D} \Psi_{k'} \Psi_{i_2}(\mathbf{x}_{i_2}) \cdots \Psi_{i_n}(\mathbf{x}_{i_n}) \\
& \times \frac{\delta}{\delta \Psi_k(\mathbf{x}_k)} e^{-\frac{1}{2} \sum_{i,j}^N \int d^3 x \int d^3 x' [\Psi_i(\mathbf{x}) - K_{1_i}(\mathbf{x})] K_{2_{ij}}^{-1}(\mathbf{x}, \mathbf{x}') [\Psi_j(\mathbf{x}') - K_{1_j}(\mathbf{x}')]}.
\end{aligned} \tag{208}$$

Performing a functional integration by parts yields

$$\begin{aligned}
\langle \Psi_{i_1}(\mathbf{x}_{i_1}) \cdots \Psi_{i_n}(\mathbf{x}_{i_n}) \rangle & = K_{1_{i_1}}(\mathbf{x}_{i_1}) \langle \Psi_2(\mathbf{x}_{i_2}) \cdots \Psi_{i_n}(\mathbf{x}_{i_n}) \rangle \\
& + \sum_{k=1}^N \int d^3 x_k K_{2_{i_1 k}}(\mathbf{x}_{i_1}, \mathbf{x}_k) \prod_{k'=1}^N \int \mathfrak{D} \Psi_{k'} \\
& \times \frac{\delta}{\delta \Psi_k(\mathbf{x}_k)} [\Psi_{i_2}(\mathbf{x}_{i_2}) \cdots \Psi_{i_n}(\mathbf{x}_{i_n})] \\
& \times e^{-\frac{1}{2} \sum_{i,j}^N \int d^3 x \int d^3 x' [\Psi_i(\mathbf{x}) - K_{1_i}(\mathbf{x})] K_{2_{ij}}^{-1}(\mathbf{x}, \mathbf{x}') [\Psi_j(\mathbf{x}') - K_{1_j}(\mathbf{x}')]},
\end{aligned} \tag{209}$$

which reduces to the recursion formula

$$\begin{aligned}
\langle \Psi_{i_1}(\mathbf{x}_{i_1}) \cdots \Psi_{i_n}(\mathbf{x}_{i_n}) \rangle & = K_{1_{i_1}}(\mathbf{x}_{i_1}) \langle \Psi_{i_2}(\mathbf{x}_{i_2}) \cdots \Psi_{i_n}(\mathbf{x}_{i_n}) \rangle \\
& + \sum_{l=2}^n K_{2_{i_1 i_l}}(\mathbf{x}_{i_1}, \mathbf{x}_{i_l}) \langle \Psi_{i_2}(\mathbf{x}_{i_2}) \cdots \Psi_{i_{l-1}}(\mathbf{x}_{i_{l-1}}) \Psi_{i_{l+1}}(\mathbf{x}_{i_{l+1}}) \cdots \Psi_{i_n}(\mathbf{x}_{i_n}) \rangle
\end{aligned} \tag{210}$$

in analogy to (198) from the previous subsection.

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Commitment

This is to certify that I wrote this work on my own and that the references include all the sources of information I have utilised.

Berlin, June 28th 2010

Moritz von Hase