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# Hydrodynamic Theory of Ultracold Quantum Gases

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June 16, 2011



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## 1 Introduction

As temperature approaches absolute zero, macroscopic quantum phenomena emerge. One of them is Bose-Einstein condensation, a phase transition, which occurs in ultracold atomic or molecular Bose gases, if the thermal de Broglie wavelength becomes as large as the average distance between atoms [1]. In this state, the ground state of the system is macroscopically occupied and all particles can be described by means of one single wave function. However, the condensation depends not only on the temperature, but is also dramatically influenced by the type and strength of the particle interaction. Even at  $T = 0$  K particles will remain out of the condensate, unless there is no particle interaction [2].

Considering contact interaction, it is the rate of collisions between the particles, that plays a key role in the description of condensate dynamics. The collisional relaxation time  $\tau$  describes the period needed to reach local equilibrium. Compared to the frequency of a collective mode  $\omega$ , two distinct cases can be distinguished. On the one hand, there is the collision-less regime, in which the period of the collective mode is much smaller than the relaxation time  $\omega\tau \gg 1$  and the system is not in local equilibrium. On the other hand there is the collision dominated or hydrodynamic regime, in which the period of the collective mode is much longer than the relaxation time  $\omega\tau \ll 1$  and collisions drive the system into a state of local equilibrium [3]. This is necessary in order to apply any kind of hydrodynamic equations. For this region there exist a generic set of equations, the Landau-Khalatnikov equations.

These equations were developed by Landau in 1941 [4] in order to describe the phenomenon of superfluidity in liquid helium II. At  $T = 2.17$  K, referred to as the  $\lambda$ -point [5], liquid helium undergoes a second-order phase transition [6]. In this new, superfluid state several unique properties come into picture. In contrast to liquid helium I, superfluid helium II shows no viscosity, when it flows through a thin capillary. However, rotating the ground of a vessel filled with liquid helium, leads to a rotation within the fluid, that would not be possible, if there was no viscosity. This contradictory behaviour caused Landau to formulate the two-fluid model, which says that the quantum liquid can exhibit two different motions – one viscous and the other ideal – at the same time [7]. Experiments have also shown that in helium-II under certain circumstances, part of the liquid flows from a cooler point to a warmer point in the vessel without any external mechanical forces. This would violate the second law of thermodynamics, unless it is said, that this flow does not carry any heat. Therefore, it was concluded that the superfluid flow is free of entropy. Another important property of the superfluid component is that its flow – according to Landau – is always irrotational. This was a conclusion derived from a microscopic model, which Landau also developed [6].

Although, the Landau-Khalatnikov (LK) two-fluid equations were developed in order to understand superfluid  $^4\text{He}$ , a wider range of applications exist. This is due to the fact, that the LK equations are formulated on the basis of fundamental conservation laws. In addition, like ordinary hydrodynamic equations, the LK equations have to be completed by equations of state. Thus, the specific thermodynamic properties of any hydrodynamic system, which are obtained through microscopic and quantum mechanical considerations, enter and specify the LK equations for the considered matter. As a consequence, the two-fluid model is not restricted to Bose fluids like  $^4\text{He}$  and can also be applied to Fermi liquids e.g. superfluid  $^3\text{He}$ . It is also promising to use the two-fluid model in order to describe the BCS-BEC crossover [8] in a two-component Fermi gas [9].

In this thesis, we will derive the Landau-Khalatnikov two-fluid equations by following and comparing two different approaches. In Section 2, we will use the principle of least action, first

used by Zilsel in 1950 [10] to derive the LK equations. Section 3 will be devoted to the classic derivation of the LK equations mainly based on conservation laws, the Galilean relativity principle, and symmetry considerations performed by Khalatnikov [11]. In Section 4, we will summarize and compare the two treatments. As the various types of sound propagation constitute one of the most famous features of superfluidity, several different wave propagations and their respective velocities will be derived and discussed in Section 5.

## 2 Variational Principle

In this section the Landau-Khalatnikov hydrodynamic equations of motion for a two-fluid system will be derived by using Hamilton's variational principle. This approach goes back to Zilsel in 1950 [10] and was improved further by Jackson [12], Geurst [13], and others going back to an idea by Lin [14] originally applied in normal hydrodynamics. The description given in subsections 2.1 and 2.3 mostly refers to Taylor et al. [15].

To avoid confusion, all thermodynamic variables given per unit volume  $V$ , e.g. the entropy density  $s = S/V$ , will be denoted by small letters. All variables given per unit mass  $m$  will be characterized by a bar, e.g. the chemical potential per unit mass will be labelled with  $\bar{\mu} = \mu/m$ .

### 2.1 Formulation of Action

We will restrict ourselves to a closed system of ideal fluids without viscosity. Therefore not only the total mass, but also the total entropy will be conserved.

The conservation of mass can be formulated in terms of the continuity equation

$$\frac{\partial \rho}{\partial t} + \text{div } \mathbf{j} = 0 , \quad (2.1)$$

where  $\rho = m n$  is the mass density of the total liquid and  $\mathbf{j}$  describes the current density, which decomposes according to

$$\mathbf{j} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s , \quad (2.2)$$

where  $\rho_n$  and  $\rho_s$  are the mass densities of the normal and the superfluid component with the corresponding velocities  $\mathbf{v}_n$  and  $\mathbf{v}_s$ . The total mass density is the sum of the two component densities

$$\rho = \rho_n + \rho_s . \quad (2.3)$$

We assume that the entropy is carried only by the normal fluid component, i.e.

$$s_s = 0 , \quad s_n = s . \quad (2.4)$$

If we denote the entropy density, i.e. entropy per unit volume, with  $s$ , the entropy conservation is described by

$$\frac{\partial s}{\partial t} + \text{div}(s \mathbf{v}_n) = 0 . \quad (2.5)$$

We now consider the Lagrangian density per unit volume  $l$  of the two fluids to be of the form

$$l = \frac{1}{2}(\rho - \rho_n) \mathbf{v}_s^2 + \frac{1}{2} \rho_n \mathbf{v}_n^2 - u(\rho, \rho_n, s) , \quad (2.6)$$

where the internal energy density  $u$  is assumed to be a function of  $\rho_n$ ,  $\rho$ , and  $s$ .

In the formulation of the action  $\mathcal{A}$  both conservation laws (2.1) and (2.5) will be taken into account by using Lagrange multipliers  $\lambda(\mathbf{r}, t)$  and  $\kappa(\mathbf{r}, t)$ , which are yet undetermined auxiliary fields:

$$\mathcal{A} = \int d^3r \int dt \left\{ \frac{1}{2}(\rho - \rho_n)\mathbf{v}_s^2 + \frac{1}{2}\rho_n\mathbf{v}_n^2 - u(\rho, \rho_n, s) + \lambda \left[ \frac{\partial \rho}{\partial t} + \text{div}((\rho - \rho_n)\mathbf{v}_s + \rho_n\mathbf{v}_n) \right] + \kappa \left[ \frac{\partial s}{\partial t} + \text{div}(s\mathbf{v}_n) \right] \right\}. \quad (2.7)$$

In the following we will calculate the variation of the action  $\mathcal{A}$  with respect to  $\rho$ ,  $\rho_n$ ,  $s$ ,  $\mathbf{v}_s$  and  $\mathbf{v}_n$ , which represent its independent variables. To this end we have to consider some useful thermodynamic relations for the internal energy density  $u$ .

## 2.2 Thermodynamic Identities

We start from the fundamental thermodynamic relation of a classic ideal fluid. Usually the internal energy is given for a fluid at rest. In a two-fluid system it is in general not possible to choose a reference frame, in which both the superfluid and the normal component are at rest. Therefore, the internal energy relation has to be extended. This additional energy depends on the relative velocity and the mass, which is still moving even in a reference frame, where one component is at rest. We will choose here the mass that is transported by the normal component, but the superfluid component would work as well. A more detailed discussion of this issue is given in Section 3.3. For now, we only state that the total mass of the normal component  $M_n$  will serve as an extensive variable. Therefore the internal energy  $U$  satisfies the thermodynamic identity for reversible processes

$$dU = TdS - pdV + \mu dN + \frac{\partial U}{\partial M_n} dM_n, \quad (2.8)$$

where  $\mu$  denotes the chemical potential (neither per unit mass nor per unit volume) and  $N$  denotes the total particle number. The total internal energy  $U(\mathbf{r}, t)$  in any given unity volume  $V$ , which is constant by definition, can be written as

$$U = uV. \quad (2.9)$$

Also all other quantities can be written in terms of a corresponding density times volume

$$S = sV, \quad (2.10)$$

$$N = \frac{\rho}{m}V, \quad (2.11)$$

$$M_n = \rho_n V. \quad (2.12)$$

We compare (2.8) in this case of a constant volume with the total differential of (2.9)

$$dU = TdS + \mu dN + \frac{\partial u}{\partial \rho_n} V d\rho_n = V du. \quad (2.13)$$

Dividing (2.13) by the unity volume  $V$  and expressing the chemical potential  $\mu$  as the chemical potential per unit mass  $\bar{\mu} = \mu/m$  we obtain

$$du = Tds + \bar{\mu}d\rho + \frac{\partial u}{\partial \rho_n}d\rho_n . \quad (2.14)$$

Thus, equation (2.14) justifies a posteriori the assumption in equation (2.6) that the internal energy density is a function of  $\rho$ ,  $\rho_n$ , and  $s$ . Furthermore, we read off from equation (2.14) that the partial derivatives of  $u$  with respect to  $\rho$  and  $s$  yield the chemical potential per unit mass

$$\frac{\partial u}{\partial \rho} = \bar{\mu} , \quad (2.15)$$

and the temperature

$$\frac{\partial u}{\partial s} = T , \quad (2.16)$$

respectively. In the next section we will see that the yet unknown partial derivative  $\frac{\partial u}{\partial \rho_n}$  is to be derived from the variation of the action with respect to  $\rho_n$ .

### 2.3 Variation of Action

We now calculate the variation of the action with respect to all degrees of freedom and set it to zero. From the variation with respect to the Lagrange multipliers  $\kappa$  and  $\lambda$  we reobtain the conservation laws (2.1) and (2.5). Therefore, those variations are not written down explicitly. From the variation of the action with respect to  $\rho$ ,  $\rho_n$ ,  $s$ ,  $\mathbf{v}_s$  and  $\mathbf{v}_n$ , which represent the independent variables, we obtain five separate equations each following from the variation of the action with respect to one specific variable:

$$\frac{\delta \mathcal{A}}{\delta \rho} : \quad \frac{1}{2} \mathbf{v}_s^2 - \bar{\mu} - \frac{\partial \lambda}{\partial t} - \mathbf{v}_s \cdot \nabla \lambda = 0 , \quad (2.17)$$

$$\frac{\delta \mathcal{A}}{\delta \rho_n} : \quad \frac{1}{2} \mathbf{v}_n^2 - \frac{1}{2} \mathbf{v}_s^2 - \frac{\partial u}{\partial \rho_n} + (\mathbf{v}_s - \mathbf{v}_n) \cdot \nabla \lambda = 0 , \quad (2.18)$$

$$\frac{\delta \mathcal{A}}{\delta s} : \quad -T - \frac{\partial \kappa}{\partial t} - \mathbf{v}_n \cdot \nabla \kappa = 0 , \quad (2.19)$$

$$\frac{\delta \mathcal{A}}{\delta \mathbf{v}_s} : \quad (\rho - \rho_n)(\mathbf{v}_s - \nabla \lambda) = \mathbf{0} , \quad (2.20)$$

$$\frac{\delta \mathcal{A}}{\delta \mathbf{v}_n} : \quad \rho_n(\mathbf{v}_n - \nabla \lambda) - s \nabla \kappa = \mathbf{0} . \quad (2.21)$$

To get these results we have used integration by parts according to the following two examples in order to eliminate spatial or time derivatives of the variables, which are varied

$$\int dt \lambda \frac{\partial \delta \rho}{\partial t} = \underbrace{[\lambda \delta \rho]_{t_1}^{t_2}}_{=0} - \int dt \frac{\partial \lambda}{\partial t} \delta \rho , \quad (2.22)$$

$$\int d^3r \lambda \operatorname{div}(\delta \rho \mathbf{v}_s) = \underbrace{\int_{\partial V} \lambda \delta \rho \mathbf{v}_s \cdot d\mathbf{f}}_{=0} - \int d^3r \nabla \lambda \cdot \mathbf{v}_s \delta \rho . \quad (2.23)$$

Note that in both cases the boundary terms do not contribute as the variations are supposed to vanish there.

In the case of a non-vanishing superfluid component  $\rho - \rho_n = \rho_s \neq 0$  we obtain from the variation with respect to  $\mathbf{v}_s$  in (2.20)

$$\mathbf{v}_s = \nabla\lambda . \quad (2.24)$$

This is a remarkable result as it states that the superfluid flow is irrotational

$$\text{rot } \mathbf{v}_s = \mathbf{0} . \quad (2.25)$$

Inserting (2.24) into (2.18) we can determine the yet missing thermodynamic relation

$$\frac{\partial u}{\partial \rho_n} = \frac{1}{2}(\mathbf{v}_n - \mathbf{v}_s)^2 . \quad (2.26)$$

Taking the gradient of (2.17) and using (2.24) we obtain

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla(\bar{\mu} + \frac{1}{2}\mathbf{v}_s^2) = \mathbf{0} , \quad (2.27)$$

which represents the Euler equation for the superfluid. If we use the vector identity

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{2}\nabla\mathbf{v}^2 - \mathbf{v} \times \text{rot } \mathbf{v} \quad (2.28)$$

and bear in mind that the superfluid flow is irrotational according to (2.25), we can write (2.27) in the well-known form

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = -\nabla\bar{\mu} . \quad (2.29)$$

Equation (2.21), which stems from the variation with respect to  $\mathbf{v}_n$ , reduces to

$$\mathbf{v}_n = \nabla\lambda + \frac{s}{\rho_n}\nabla\kappa . \quad (2.30)$$

Thus, we conclude that  $\mathbf{v}_n$  turns out to be irrotational if  $s/\rho_n$  would be a constant. However, this result is not to be interpreted in a physical manner, but represents a flaw of the theory, which also occurs in the description of normal fluids. There is no reason, why the normal component flow should be irrotational in case  $s/\rho_n$  is a constant. To correct this mistake, a new constraint, mostly referred to as Lin's constraint [14], has to be included. In classical hydrodynamics this constraint represents the conservation of the identity of particles and is closely connected to the conservation of vorticity. However, to apply this constraint correctly in the superfluid two-fluid model is not without difficulties and has been discussed intensively by Jackson [12], Geurst [13], [16] and others [17], [18]. To replicate this discussion here, would go beyond the scope of this thesis. Eventually, a modified version of Lin's constraint, which allows for vorticity also in the special case  $\rho_n/s = \text{const}$ , is given by

$$\frac{\partial s\eta}{\partial t} + \text{div}(s\eta\mathbf{v}_n) = 0 , \quad (2.31)$$

in which the arbitrary function  $\eta$  is a real valued function of  $\mathbf{r}$  and  $t$ . The inclusion of this constraint leads to minor changes in the equations given above, but will not change

the resulting equations of motions. Therefore, the calculations will not be redone including equation (2.31) and we refer for further discussions to the references [15] and [19].

Equation (2.30) can be rewritten using (2.24) yielding the following expression

$$\nabla\kappa = \frac{\rho_n}{s}(\mathbf{v}_n - \mathbf{v}_s), \quad (2.32)$$

which implies the useful identity

$$\text{rot} \left[ \frac{\rho_n}{s}(\mathbf{v}_n - \mathbf{v}_s) \right] = \mathbf{0}. \quad (2.33)$$

We replace  $\nabla\kappa$  in (2.19) using (2.32). Then we take the gradient and solve for  $\nabla \frac{\partial\kappa}{\partial t}$ . Taking the time derivative of (2.32) we obtain the following equation by applying the Schwarz theorem, which says that for functions with continuous 2nd partial derivatives, the partial derivatives with respect to two different variables do commute:

$$\frac{\partial}{\partial t} \left[ \frac{\rho_n}{s}(\mathbf{v}_n - \mathbf{v}_s) \right] + \nabla \left[ \frac{\rho_n}{s} \mathbf{v}_n \cdot (\mathbf{v}_n - \mathbf{v}_s) + T \right] = 0. \quad (2.34)$$

By some laborious algebra this can be transformed into Euler's equation for the normal fluid using the continuity equation (2.1), the conservation of entropy (2.5) and (2.33)

$$\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \cdot \nabla) \mathbf{v}_n = -\frac{s}{\rho_n} \nabla T - \nabla \bar{\mu} - \nabla \frac{1}{2} (\mathbf{v}_n - \mathbf{v}_s)^2 - \frac{\Gamma}{\rho_n} (\mathbf{v}_n - \mathbf{v}_s), \quad (2.35)$$

where the source term  $\Gamma$  is defined by

$$\Gamma = \frac{\partial \rho_n}{\partial t} + \text{div}(\rho_n \mathbf{v}_n). \quad (2.36)$$

Using the continuity equation (2.1) it can be shown that

$$-\Gamma = \frac{\partial \rho_s}{\partial t} + \text{div}(\rho_s \mathbf{v}_s). \quad (2.37)$$

The possibility that  $\Gamma$  is not equal to zero implies that neither  $\rho_s$  nor  $\rho_n$  alone fulfil the continuity equation. Therefore, it can't be excluded within the two-fluid model that particles of the superfluid component loose their superfluid characteristics and thus become a part of the normal fluid component and vice versa.

The continuity equations for mass (2.1) and entropy (2.5) together with the two Euler equations for the superfluid component (2.29) and the normal component (2.35) form a complete set of hydrodynamic equations. However, it is more common to use an equation for the total momentum flux  $\mathbf{j}$ . This can be obtained by combining equation (2.2), (2.29), and (2.35), yielding

$$\begin{aligned} \frac{\partial \mathbf{j}}{\partial t} + \mathbf{v}_s \text{div}(\rho_s \mathbf{v}_s) + (\rho_s \mathbf{v}_s \cdot \nabla) \mathbf{v}_s + \mathbf{v}_n \text{div}(\rho_n \mathbf{v}_n) + (\rho_n \mathbf{v}_n \cdot \nabla) \mathbf{v}_n \\ + \frac{\rho_n}{2} \nabla (\mathbf{v}_n - \mathbf{v}_s)^2 + \rho \nabla \bar{\mu} + s \nabla T = \mathbf{0}. \end{aligned} \quad (2.38)$$

Equation (2.38) can be rewritten in component form by using the Einstein convention, that a summation over double indices is implicitly assumed:

$$\frac{\partial j_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = 0. \quad (2.39)$$



If we use the convenient form of a momentum density tensor  $\Pi$ , we can identify its diagonal entries with the pressure

$$\Pi_{ik} = \rho_s v_{si} v_{sk} + \rho_n v_{ni} v_{nk} + p \delta_{ik} , \quad (2.40)$$

where the gradient of the pressure is given by

$$\nabla p = \rho \nabla \bar{\mu} + s \nabla T + \frac{\rho_n}{2} \nabla (\mathbf{v}_n - \mathbf{v}_s)^2 . \quad (2.41)$$

Using the thermodynamic relation (2.14) and (2.26) we can formulate an explicit expression for the pressure, which is compatible with (2.41):

$$p = Ts + \bar{\mu} \rho + \frac{1}{2} (\mathbf{v}_n - \mathbf{v}_s)^2 \rho_n - u . \quad (2.42)$$

If the continuity equation (2.1) and the conservation of entropy (2.5) are given, the two Euler equations (2.29), (2.35) and the momentum conservation (2.38) form a set of three equations, of which one is redundant. Usually Euler's equation for the superfluid (2.29) and the momentum conservation (2.38) are used to describe the system.

### 3 Derivation of Two-Fluid Equations according to Landau – Khalatnikov

In this section we will describe another derivation of the Landau-Khalatnikov hydrodynamic equations, which was first performed by Khalatnikov [11]. This approach is built upon conservation laws and the Galilean relativity principle. First the conservation laws will be given including some undetermined quantities. Those will then be calculated in another reference frame using Galilean transformations. In this new reference frame it will be possible to determine exact expressions for these quantities, thus completing the system of hydrodynamic equations.

#### 3.1 Conservation Laws

The conservation of mass yields the continuity equation

$$\frac{\partial \rho}{\partial t} + \text{div } \mathbf{j} = 0 . \quad (3.1)$$

The mass density  $\rho$  equals the sum of the two component mass densities according to (2.3). In contrast to the previous derivation the explicit expression for the current  $\mathbf{j}$  is not proposed a priori, instead it is only assumed that it transforms under a Galilean boost according to the usual relations given in the next paragraph. Similarly, the conservation of entropy is proposed by using a not yet known entropy flux vector  $\mathbf{f}$ :

$$\frac{\partial s}{\partial t} + \text{div } \mathbf{f} = 0 . \quad (3.2)$$

Note that in this approach, it is not proposed a priori that the superfluid component does not carry any entropy. Instead it will turn out that this will be an outcome of the further

derivations. The conservation of momentum is proposed by using the energy momentum tensor  $\Pi_{ik}$

$$\frac{\partial j_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = 0 . \quad (3.3)$$

In contrast to the derivation based on the variational principle, it is postulated that the flow of the superfluid is irrotational

$$\text{rot } \mathbf{v}_s = \mathbf{0} . \quad (3.4)$$

Therefore, it must be possible to express its motion by the gradient of some yet undetermined scalar function  $\varphi$ . Thus, Euler's equation takes the following form

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s = -\nabla \varphi . \quad (3.5)$$

Using the vector identity (2.28), which is simplified by condition (3.4), we can rewrite (3.5) into the following form, which will be more useful later on

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla \left( \frac{1}{2} \mathbf{v}_s^2 + \varphi \right) = \mathbf{0} . \quad (3.6)$$

In addition, energy conservation yields

$$\frac{\partial e}{\partial t} + \text{div } \mathbf{q} = 0 , \quad (3.7)$$

where  $e$  is the total energy density, i.e. the sum of kinetic and internal energy density and  $\mathbf{q}$  denotes the energy flux vector.

### 3.2 Galilean-Transformation

In order to determine  $\mathbf{j}$ ,  $\Pi_{ik}$ ,  $\mathbf{f}$ ,  $\mathbf{q}$ , and  $\varphi$ , a new reference frame  $K_0$ , which moves along with the superfluid component, will be introduced. For any given element of the fluid, the new reference frame  $K_0$  moves at the velocity  $\mathbf{v}_s$  with respect to the original rest frame  $K$ . The variables in the new reference frame are denoted by an index 0 and are related to the old variables via:

$$\mathbf{v}_n = \mathbf{v}_s + \mathbf{v}_{n0} , \quad (3.8)$$

$$\mathbf{j} = \rho \mathbf{v}_s + \mathbf{j}_0 , \quad (3.9)$$

$$\Pi_{ik} = \Pi_{0ik} + \rho v_{s_i} v_{s_k} + v_{s_i} j_{0k} + j_{0i} v_{s_k} , \quad (3.10)$$

$$e = \frac{1}{2} \rho \mathbf{v}_s^2 + \mathbf{v}_s \cdot \mathbf{j}_0 + e_0 , \quad (3.11)$$

$$\mathbf{q} = \left( \frac{1}{2} \rho \mathbf{v}_s^2 + \mathbf{v}_s \cdot \mathbf{j}_0 + e_0 \right) \mathbf{v}_s + \frac{1}{2} \mathbf{v}_s^2 \mathbf{j}_0 + \Pi_0 \mathbf{v}_s + \mathbf{q}_0 , \quad (3.12)$$

$$\mathbf{f} = s \mathbf{v}_s + \mathbf{f}_0 . \quad (3.13)$$

### 3.3 Thermodynamics

The Galilean-Transformation will not only allow us to determine  $\mathbf{j}$ ,  $\Pi_{ik}$ ,  $\mathbf{f}$  and  $\varphi$ , but it is also very helpful in order to determine the fundamental thermodynamic identity for reversible processes in a superfluid.

Before we go on with the further derivation of the Landau-Khalatnikov two-fluid equations, we formulate the thermodynamic identity for the total energy density  $e_0$ . In the new reference

frame  $K_0$ , the superfluid is at rest, while the normal fluid is moving with a velocity  $\mathbf{v}_{n_0}$  defined by equation (3.8). If we consider this velocity to be an intensive variable of the system, the canonical conjugate extensive variable has to have the dimension of a momentum density  $\mathbf{j}_0$ , i.e. the momentum per unit volume. As only the normal fluid component is moving, the momentum density vector should have the following form

$$\mathbf{j}_0 = \rho_n \mathbf{v}_{n_0} . \quad (3.14)$$

In order to determine the total differential  $de_0$  we remember that the total differential of the internal energy for reversible processes can be expressed as a sum of products formed by an intensive variable times the total differential of its canonical conjugate extensive variable. In the formulation of energy densities instead of total energy, the variables vary slightly from the standard form (see Section 2.2). We choose the following sets of intensive and extensive variables:  $(T, s)$ ,  $(\bar{\mu}, \rho)$ , and  $(\mathbf{v}_{n_0}, \mathbf{j}_0)$  and obtain

$$de_0 = Tds + \bar{\mu}d\rho + \mathbf{v}_{n_0} d\mathbf{j}_0 . \quad (3.15)$$

Using equations (3.8) and (3.14), equation (3.15) can be rewritten:

$$de_0 = Tds + \bar{\mu}d\rho + (\mathbf{v}_n - \mathbf{v}_s) \cdot d[\rho_n(\mathbf{v}_n - \mathbf{v}_s)] . \quad (3.16)$$

By comparing the total differential of the energy  $e$  in the old rest frame  $K$ , which can be written as a composition of internal  $u$  and kinetic energy  $e_{\text{kin}}$  according to

$$e = e_{\text{kin}} + u , \quad (3.17)$$

with the total differential of the energy  $de_0$ , it is possible to find the thermodynamic equation for reversible processes in a superfluid. To this end we formulate  $de$  in terms of kinetic and internal energy densities

$$de = de_{\text{kin}} + du = d\left(\frac{1}{2}\rho_s \mathbf{v}_s^2\right) + d\left(\frac{1}{2}\rho_n \mathbf{v}_n^2\right) + du \quad (3.18)$$

and calculate the total differential of (3.11)

$$de = d\left(\frac{1}{2}\rho \mathbf{v}_s^2\right) + d(\mathbf{j}_0 \cdot \mathbf{v}_s) + de_0 . \quad (3.19)$$

By inserting (3.16) we can rewrite (3.19) yielding

$$de = Tds + \bar{\mu}d\rho + \frac{1}{2}(\mathbf{v}_n - \mathbf{v}_s)^2 d\rho_n + \frac{1}{2}\mathbf{v}_s^2 d\rho_s + \frac{1}{2}\mathbf{v}_n^2 d\rho_n + \rho_s \mathbf{v}_s d\mathbf{v}_s + \rho_n \mathbf{v}_n d\mathbf{v}_n . \quad (3.20)$$

Comparing (3.18) and (3.20) we obtain the total differential of the internal energy

$$du = Tds + \bar{\mu}d\rho + \frac{1}{2}(\mathbf{v}_n - \mathbf{v}_s)^2 d\rho_n . \quad (3.21)$$

As it was postulated in the previous Section 2.2,  $\rho_n$  emerges to be the third extensive quantity of the two-fluid system.

The calculation of the total differential  $du_0$  of the internal energy density in the reference frame  $K_0$ , which can be done by subtracting the total differential of the kinetic energy density in the reference frame  $K_0$  from equation (3.16), yields

$$du_0 = Tds + \bar{\mu}d\rho + (\mathbf{v}_n - \mathbf{v}_s) d[\rho_n(\mathbf{v}_n - \mathbf{v}_s)] - d\left[\frac{1}{2}\rho_n(\mathbf{v}_n - \mathbf{v}_s)^2\right] , \quad (3.22)$$

$$du_0 = Tds + \bar{\mu}d\rho + \frac{1}{2}(\mathbf{v}_n - \mathbf{v}_s)^2 d\rho_n . \quad (3.23)$$

By this direct calculation and comparison with equation (3.21) we obtain that the internal energy in the reference frame  $K_0$  is identical to the internal energy in the reference frame  $K$

$$du_0 = du . \quad (3.24)$$

This is a consequence of the fact, that the internal energy depends only on the relative velocity  $\mathbf{v}_n - \mathbf{v}_s$  and thereby has to be invariant under a Galilean transformation.

### 3.4 Derivation of LK Equations

The first quantity determined by the Galilean-Transformation is  $\mathbf{j}$ . By inserting (3.8) and (3.14) into (3.9) we find

$$\mathbf{j} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n . \quad (3.25)$$

To determine the corresponding remaining expressions for  $\mathbf{q}_0$ ,  $\Pi_0$ , and  $\mathbf{f}_0$  more effort is needed. The goal is to express  $\text{div } \mathbf{q}_0$  as a function of the thermodynamic variables and the velocities including spatial and time derivatives of the latter. However, noticing that in the absence of energy dissipation  $\mathbf{q}_0$  does only depend on the thermodynamic variables and the velocities, but does not depend on their space or time derivatives, it will then be necessary to eliminate those dependencies. Thereby unique expressions for  $\varphi$ ,  $\mathbf{f}$ ,  $\mathbf{q}_0$ , and  $\Pi_0$  will be revealed.

To find the expression for  $\text{div } \mathbf{q}_0$  we will first differentiate  $e$  with respect to time using equation (3.11) and insert  $\frac{\partial e_0}{\partial t}$  defined by (3.16). We use (3.9) to substitute  $\mathbf{j}_0$  with  $\mathbf{j}$  and obtain

$$\frac{\partial e}{\partial t} = \left(\frac{1}{2}\mathbf{v}_s^2 + \bar{\mu} - \mathbf{v}_n \cdot \mathbf{v}_s\right) \frac{\partial \rho}{\partial t} + (\mathbf{j} - \rho \mathbf{v}_n) \cdot \frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_n \cdot \frac{\partial \mathbf{j}}{\partial t} + T \frac{\partial s}{\partial t} . \quad (3.26)$$

We rewrite (3.26) by replacing all time derivatives of the thermodynamic variables or the velocities according to the conservation laws (3.1) to (3.5). The entropy flux  $\mathbf{f}$  is replaced by  $\mathbf{f}_0$  using (3.13)

$$\frac{\partial e}{\partial t} = -\left(\frac{1}{2}\mathbf{v}_s^2 + \bar{\mu} - \mathbf{v}_n \cdot \mathbf{v}_s\right) \text{div } \mathbf{j} - (\mathbf{j} - \rho \mathbf{v}_n) \cdot \nabla \left(\frac{1}{2}\mathbf{v}_s^2 + \varphi\right) - \mathbf{v}_n \cdot \nabla \Pi - T \text{div}(\mathbf{f}_0 + \mathbf{v}_s s) . \quad (3.27)$$

We then calculate  $\text{div } \mathbf{q}$  using (3.12)

$$\text{div } \mathbf{q} = \text{div} \left[ \left(\frac{1}{2}\rho \mathbf{v}_s^2 + \mathbf{v}_s \cdot \mathbf{j}_0 + e_0\right) \mathbf{v}_s \right] + \text{div} \left(\frac{1}{2}\mathbf{v}_s^2 \mathbf{j}_0\right) + \text{div}(\Pi_0 \mathbf{v}_s) + \text{div } \mathbf{q}_0 . \quad (3.28)$$

By using (3.7), we can add (3.27) and (3.28) and solve for  $\text{div } \mathbf{q}_0$

$$\begin{aligned} \text{div } \mathbf{q}_0 = & -\mathbf{j} \cdot \nabla \frac{1}{2}\mathbf{v}_s^2 - \mathbf{v}_s \cdot \mathbf{j}_0 \text{div } \mathbf{v}_s - \mathbf{v}_s \cdot \nabla(\mathbf{v}_s \cdot \mathbf{j}_0) \\ & - \mathbf{v}_s \cdot \nabla e_0 - e_0 \text{div } \mathbf{v}_s - \text{div}(\Pi \mathbf{v}_s) - \mathbf{v}_n \cdot \mathbf{v}_s \text{div } \mathbf{j} \\ & + \bar{\mu} \text{div } \mathbf{j}_0 + \bar{\mu} \text{div}(\rho \mathbf{v}_s) - (\mathbf{j} - \rho \mathbf{v}_n) \cdot \nabla \varphi + (\mathbf{j} - \mathbf{v}_n \rho) \cdot \nabla \frac{1}{2}\mathbf{v}_s^2 \\ & + \mathbf{v}_n \cdot \nabla \Pi + T \text{div } \mathbf{f}_0 + T \text{div}(s \mathbf{v}_s) . \end{aligned} \quad (3.29)$$

Using (3.10) and doing several rearrangements we can rewrite (3.29) once more:

$$\begin{aligned} \text{div } \mathbf{q}_0 = & -(\mathbf{h} \cdot \nabla) \cdot \mathbf{v}_s + (\mathbf{v}_n - \mathbf{v}_s) \cdot (\nabla \cdot \mathbf{h}) + \mathbf{j}_0 \cdot [(\mathbf{v}_n - \mathbf{v}_s) \cdot \nabla] \mathbf{v}_n \\ & - \rho_s (\mathbf{v}_n - \mathbf{v}_s) \cdot \nabla(\varphi - \bar{\mu}) - \nabla T \cdot [\mathbf{f}_0 - s(\mathbf{v}_n - \mathbf{v}_s)] \\ & + \text{div}(\mathbf{f}_0 T + \mathbf{j}_0 \bar{\mu}) . \end{aligned} \quad (3.30)$$

Here we have used the newly introduced tensor  $\mathbf{h}$  to shorten the expression

$$\mathbf{h}_{ik} = \Pi_{0_{ik}} + [e_0 - Ts - \bar{\mu}\rho - (\mathbf{v}_n - \mathbf{v}_s) \cdot \mathbf{j}_0] \delta_{ik} . \quad (3.31)$$

As already mentioned  $\mathbf{q}_0$  and also  $\text{div } \mathbf{q}_0$  must not be a function of spatial derivatives of thermodynamic quantities. Eliminating the dependency on  $\nabla T$  we obtain an expression for  $\mathbf{f}_0$

$$\mathbf{f}_0 = s(\mathbf{v}_n - \mathbf{v}_s) , \quad (3.32)$$

and by using (3.13) also for  $\mathbf{f}$

$$\mathbf{f} = s\mathbf{v}_n . \quad (3.33)$$

Equation (3.33) states that the entropy is only transported by the normal component. Thus, combining (3.2) with (3.33) the complete law of entropy conservation reads

$$\frac{\partial s}{\partial t} + \text{div}(s\mathbf{v}_n) = 0 . \quad (3.34)$$

If neither the superfluid density  $\rho_s$  nor the relative velocity  $(\mathbf{v}_n - \mathbf{v}_s)$  is zero, which is fulfilled except for trivial cases,  $\nabla(\varphi - \bar{\mu})$  has to be zero in order to eliminate the dependency on  $\nabla\bar{\mu}$  in (3.30). Thereby, the unknown scalar function  $\varphi$  can be identified with the chemical potential per unit mass  $\nabla\varphi = \bar{\mu}$ , thus completing Euler's equation for the superfluid component (3.5) yielding

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla \left( \frac{1}{2} \mathbf{v}_s^2 + \bar{\mu} \right) = \mathbf{0} . \quad (3.35)$$

To eliminate the remaining dependency of  $\mathbf{q}_0$  on spatial derivatives of the velocities, the remaining terms on the right-hand side of equation (3.30) have to be written as the divergence of one single expression. In order to find this expression, we rewrite (3.30) in component form

$$\partial_{x_i} q_{0_i} = (v_n - v_s)_i \partial_{x_k} h_{ik} + j_{0_i} (v_n - v_s)_k \partial_{x_k} v_{n_i} - h_{ik} \partial_{x_k} v_{s_i} + \partial_{x_i} (f_{0_i} T + j_{0_i} \bar{\mu}) . \quad (3.36)$$

The last term in (3.36) already fulfils the need to be written as a divergence. All other three terms can be written as a divergence, if  $\mathbf{h}$  takes the following form

$$h_{ik} = j_{0_i} (v_n - v_s)_k . \quad (3.37)$$

Inserting (3.37) into (3.36) we obtain

$$\partial_{x_i} q_{0_i} = \partial_{x_k} [j_{0_i} (v_n - v_s)_k (v_n - v_s)_i] + \partial_{x_i} (f_{0_i} T + j_{0_i} \bar{\mu}) . \quad (3.38)$$

Using (3.14) we can commute  $i$  and  $k$  and finally obtain

$$\partial_{x_i} q_{0_i} = \partial_{x_i} \left[ (\mathbf{v}_n - \mathbf{v}_s)^2 j_{0_i} + f_{0_i} T + j_{0_i} \bar{\mu} \right] . \quad (3.39)$$

Now we have obtained an expression for  $\mathbf{q}_0$ , which does neither depend on spatial or time derivatives of thermodynamic variables nor on spatial or time derivatives of the velocities

$$\mathbf{q}_0 = (\mathbf{v}_n - \mathbf{v}_s)^2 \mathbf{j}_0 + \mathbf{f}_0 T + \mathbf{j}_0 \bar{\mu} . \quad (3.40)$$

Inserting (3.37) into (3.31) we obtain an expression for  $\Pi_0$

$$\Pi_{0_{ik}} = j_{0_i} (v_n - v_s)_k - [e_0 - Ts - \bar{\mu}\rho - (\mathbf{v}_n - \mathbf{v}_s) \cdot \mathbf{j}_0] \delta_{ik} . \quad (3.41)$$

Inserting (3.40) and (3.41) into (3.12) we obtain the energy flux vector in the old rest frame

$$\mathbf{q} = \frac{1}{2}\rho_s \mathbf{v}_s^2 \mathbf{v}_s + \frac{1}{2}\rho_n \mathbf{v}_n^2 \mathbf{v}_n + \frac{1}{2}\rho_n (\mathbf{v}_n - \mathbf{v}_s)^2 \mathbf{v}_n + \mathbf{j}\bar{\mu} + T s \mathbf{v}_n . \quad (3.42)$$

The first two terms represent the transport of the kinetic energy of the two components, each moved along by its respective velocity. The internal energy connected to the chemical potential is moved along with the mass current  $\mathbf{j}$ , the energy connected to entropy is moved by the normal fluid. Note that there is a fifth term, which is connected to the kinetic energy of the relative motion. This energy is also transported by the normal component.

Using (3.10) we can transform (3.41) into the momentum flux tensor in the old reference frame

$$\Pi_{ik} = \rho_n v_{n_i} v_{n_k} + \rho_s v_{s_i} v_{s_k} + p \delta_{ik} . \quad (3.43)$$

All its diagonal entries will be identified with the pressure  $p$

$$p = -e_0 + T s + \bar{\mu} \rho + (\mathbf{v}_n - \mathbf{v}_s) \cdot \mathbf{j}_0 . \quad (3.44)$$

In order to compare this equation with the result (2.42) obtained by using the variational ansatz, we insert (3.9) and (3.11) into (3.44). Thus, (3.44) can be rewritten as

$$p = T s + \bar{\mu} \rho + \frac{1}{2}\rho_s \mathbf{v}_s^2 + \rho_n \left( \frac{1}{2}\mathbf{v}_s^2 - \mathbf{v}_n \cdot \mathbf{v}_s + \mathbf{v}_n^2 \right) - e . \quad (3.45)$$

Expressing the total energy density  $e$  in terms of internal and kinetic energy according to (3.17) we obtain

$$p = T s + \bar{\mu} \rho + \rho_n \left( \frac{1}{2}\mathbf{v}_s^2 - \mathbf{v}_n \cdot \mathbf{v}_s + \frac{1}{2}\mathbf{v}_n^2 \right) - u , \quad (3.46)$$

which is identical to the previous result (2.42).

## 4 Summary and Comparison

In this Section, we will give a short summary of the results obtained in Section 2 and 3. Subsequently, the two approaches will be compared and discussed. At last, the limits of the LK equations will be outlined.

### 4.1 LK Equations

All unknown quantities could be determined. To summarize the results all equations found are listed below

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} = 0 , \quad (4.1)$$

$$\frac{\partial s}{\partial t} + \operatorname{div}(s \mathbf{v}_n) = 0 , \quad (4.2)$$

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s = -\nabla \bar{\mu} , \quad (4.3)$$

$$\frac{\partial j_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial r_k} = 0 \quad \text{with} \quad \Pi_{ik} = \rho_s v_{s_i} v_{s_k} + \rho_n v_{n_i} v_{n_k} + p \delta_{ik} . \quad (4.4)$$

This is a system of 8 equations, where the current density  $\mathbf{j}$  is given by (3.25). We introduce the thermodynamic potential  $\Omega$  per unit volume, which is given by a Legendre transformation of the internal energy

$$\Omega = u - Ts - \bar{\mu}\rho - \frac{1}{2}\rho_n(\mathbf{v}_n - \mathbf{v}_s)^2. \quad (4.5)$$

Considering (3.46), we notice that it is equivalent to the negative pressure

$$p = -\Omega. \quad (4.6)$$

Using the thermodynamic identity

$$du = Tds + \bar{\mu}d\rho + \frac{1}{2}(\mathbf{v}_n - \mathbf{v}_s)^2 d\rho_n \quad (4.7)$$

we can deduce by considering the total differential of thermodynamic potential per unit volume

$$d\Omega = -sdT - \rho d\bar{\mu} - \rho_n(\mathbf{v}_n - \mathbf{v}_s) \cdot d(\mathbf{v}_n - \mathbf{v}_s), \quad (4.8)$$

that from the two pairs  $(\rho, s)$ ,  $(T, \bar{\mu})$  only one is needed:

$$s(T, \bar{\mu}, \mathbf{v}_n - \mathbf{v}_s) = - \left. \frac{\partial \Omega(T, \bar{\mu}, \mathbf{v}_n - \mathbf{v}_s)}{\partial T} \right|_{\bar{\mu}, \mathbf{v}_n - \mathbf{v}_s}, \quad (4.9)$$

$$\rho(T, \bar{\mu}, \mathbf{v}_n - \mathbf{v}_s) = - \left. \frac{\partial \Omega(T, \bar{\mu}, \mathbf{v}_n - \mathbf{v}_s)}{\partial \bar{\mu}} \right|_{T, \mathbf{v}_n - \mathbf{v}_s}. \quad (4.10)$$

Equations (4.9) and (4.10) can be transformed into equations for  $T$  and  $\bar{\mu}$

$$T = T(s, \rho, \mathbf{v}_n - \mathbf{v}_s), \quad (4.11)$$

$$\bar{\mu} = \bar{\mu}(s, \rho, \mathbf{v}_n - \mathbf{v}_s). \quad (4.12)$$

So we know, that from the 12 variables  $\mathbf{v}_n$ ,  $\mathbf{v}_s$ ,  $\rho_n$ ,  $\rho$ ,  $s$ ,  $T$ ,  $\bar{\mu}$  and  $u$  two are redundant, which reduces the number of degrees of freedom to 10. To complete the eight equations (4.1) to (4.4) two equations of state are needed. To obtain these equations an underlying, microscopic theory has to be considered.

## 4.2 Comparison between Treatments

Comparing the results obtained by the two different approaches, it is apparent that the equations are exactly the same. Also, in both treatments conservation laws are the starting point for all further derivations. However, the approaches differ in the set of initial assumptions. The most striking difference is that in the variational approach, it is assumed that the superfluid carries no entropy

$$s_n = s, \quad s_s = 0, \quad (4.13)$$

while it's potential flow

$$\text{rot } \mathbf{v}_s = 0 \quad (4.14)$$

is not postulated, whereas in the Khalatnikov approach, it is the other way around.

It is therefore tempting to allow  $s_n \neq s$  and  $s_s \neq 0$  in the variational ansatz in the first place and restrict the set of emerging equations in such a way, that only solutions, that fulfil condition (4.14) remain. If this restriction led to condition (4.13) an equivalence between

potential flow and zero superfluid entropy would be proven at least in the formalism of least action.

To this end, equation (2.5) is altered towards

$$\frac{\partial(s_s + s_n)}{\partial t} + \text{div}(s_s \mathbf{v}_s + s_n \mathbf{v}_n) = 0 \quad (4.15)$$

and replaces the entropy conservation implied in the formulation of action (2.7), while the other terms in (2.7) remain the same. In this altered ansatz the variation with respect to  $\mathbf{v}_s$  leads to the following equation

$$\frac{\delta \mathcal{A}}{\delta \mathbf{v}_s} : \quad \mathbf{v}_s = \frac{s_s}{\rho - \rho_n} \nabla \kappa + \nabla \lambda . \quad (4.16)$$

Applying the restriction (4.14) in case the superfluid density  $\rho_s = \rho - \rho_n$  is not constant, there are two possibilities, in which (4.15) is compatible with equation (4.14). The first possibility is, that  $\nabla \kappa$  equals zero. In this case, it can be calculated, that the set of equations reduces to an one-fluid model with  $\mathbf{v}_n = \mathbf{v}_s$  and so on. This is a correct subset of solutions.

The other possibility is, that  $s_s$  equals zero. In this case all equations derived by the altered variational ansatz reduce to the equations obtained by the old ansatz listed in (2.17) to (2.21). However, there is an additional equation left, which emerges from the variation with respect to  $s_s$ . This equation has to be handled with care, as we have seen, that in the subset of solutions compatible with constraint (4.14), the function  $s_s$  has to be zero and therefore the action can no longer depend on it. It is thus necessary to reformulate the action once again and to imply another constraint, which provides, that  $s_s$  equals zero. However, this would be equivalent to formulate the action according to (2.7). We have thus shown, that it is necessary to formulate the entropy conservation according to (4.13) as it is done in (2.5) in order to obtain irrotational superfluid flow (4.14). By the derivation in Section 2.3 it was already shown, that the implication of (2.5), which contains constraint (4.13), is sufficient to provide irrotational flow (4.14). As a conclusion, we have shown that condition (4.13) and (4.14) are equivalent in the variational formalism.

It is not clear, how a similar consideration could be performed in the approach performed by Khalatnikov. If the restriction (4.14) is removed from the set of initial assumptions, Euler's equation can not be obtained immediately. This would be problematic as all further calculations depend on Euler's equation for the superfluid. Adding condition (4.13) would not entirely solve this problem, as only the entropy flux tensor would be known. However, the energy and the momentum flux tensor would still remain undetermined. The approach had to be completely rearranged.

Another difference between the two formalisms is that the so called source term  $\Gamma$  defined by equation (2.36) does not appear explicitly in the derivation of the LK two-fluid equation according to Landau-Khalatnikov. However, it is implicitly included. As already mentioned, equation (2.35), which reveals the source term  $\Gamma$ , can be obtained from rearranging equations (4.1) to (4.4). Therefore, it would be possible to derive the equation for  $\Gamma$  also directly following the Landau-Khalatnikov approach.

Comparing the abilities of the two approaches, it has to be said, that the variational ansatz is not capable of including any dissipative processes. Furthermore, the necessity to include Lin's constraint, which is not accessible to a direct physical interpretation, reduces the elegance and simplicity of this approach. The discussions in relation to Lin's constraint have also shown,



that it can be difficult to include new constraints in order to eliminate certain flaws, without creating new unintended consequences. However, this approach is still very well structured and alterations to the ansatz given in Section 2 can be applied easily by changing the set of independent variables or by applying new constraints with the aid of Lagrange multipliers. For example, there have been successful attempts to extend the variational principle including the gradient of the superfluid density as another degree of freedom. Variations of Lin's principle have also been made in order to allow for superfluid vorticity [16].

The Khalatnikov approach instead, is built upon only a few fundamental arguments and plausible physical reasoning. Although the calculations are lengthy, the main ideas are quite simple and convincing. However, it misses a certain flexibility and it is not obvious how modifications can be reasonably integrated without changing the whole concept.

### 4.3 Limits of LK Equations

The equations given above apply only for small velocities. Above certain critical velocities, the properties of superfluidity are destroyed. Those critical velocities can be derived from the microscopic theory of elementary excitations and yield the so-called Landau criterion. They depend not only on the liquid or gas, but also on the type of motion. E.g. considering liquid helium-II, the critical velocities for flows in a capillary are much higher than in a large volume. These phenomena are related to the formation of quantized vortices, which can only be derived by a microscopic model and would go far beyond the scope of the present thesis [6].

## 5 Propagation of Sound

The different types of wave propagation are one of the most characteristic properties of a superfluid liquid. Due to the comparatively simple experimental examination of these phenomena, they provide a good opportunity to proof theoretical predictions of the two-fluid model. Following the propagation of so called first sound, which is due to pressure and density waves, those different wave phenomena are referred to as second, third and so forth sound. In the first part of this section, we will derive the propagation of first and second sound in a two-fluid liquid by an approach mainly based on Griffin et al. [3]. Subsequently the phenomena of third, fourth and fifth sound, which are related to special geometrical set-ups, will be discussed. At the end of this section there will be a short summary of further wave phenomena [20].

### 5.1 First and Second Sound

In order to analyse the propagation of sound in a superfluid the equations of motion for both fluids (4.1) to (4.4) are linearised. That is all the thermodynamic variables are expanded around their equilibrium values e.g.  $p(\mathbf{r}, t) = p_0 + \delta p(\mathbf{r}, t)$ . In addition, the equilibrium values of the velocities  $\mathbf{v}_{n\text{eq}}$  and  $\mathbf{v}_{s\text{eq}}$  are set to zero. This is equivalent to the assumption, that the velocities are small compared with the velocities of the sound waves.

The linearised system of hydrodynamic equations is

$$\frac{\partial \delta \rho}{\partial t} + \text{div } \delta \mathbf{j} = 0, \quad (5.1)$$

$$\delta \mathbf{j} = \rho_{s_0} \delta \mathbf{v}_s + \rho_{n_0} \delta \mathbf{v}_n, \quad (5.2)$$

$$\frac{\partial \delta s}{\partial t} + s_0 \operatorname{div} \delta \mathbf{v}_n = 0, \quad (5.3)$$

$$\frac{\partial \delta \mathbf{v}_s}{\partial t} = -\nabla \delta \bar{\mu}, \quad (5.4)$$

$$\frac{\partial \delta \mathbf{j}}{\partial t} = -\nabla \delta p. \quad (5.5)$$

For technical reasons we now introduce the entropy per unit mass  $\bar{s} = s/\rho$  instead of the entropy per unit volume  $s$ . Its time derivative

$$\frac{\partial \delta \bar{s}}{\partial t} = \frac{1}{\rho_0} \frac{\partial \delta s}{\partial t} - \frac{s_0}{\rho_0^2} \frac{\partial \delta \rho}{\partial t} \quad (5.6)$$

yields with (5.1), (5.2), and (5.3)

$$\frac{\partial \delta \bar{s}}{\partial t} = \frac{s_0}{\rho_0^2} \rho_{s_0} \operatorname{div}(\delta \mathbf{v}_n - \delta \mathbf{v}_s). \quad (5.7)$$

Differentiating (5.1) with respect to time and inserting (5.5) we obtain

$$\frac{\partial^2 \delta \rho}{\partial t^2} - \Delta \delta p = 0. \quad (5.8)$$

In order to find a similar expression for  $\bar{s}$  we first need to express the fluctuations of the chemical potential by fluctuations of the pressure  $p$  and the temperature  $T$ . Therefore, we use equation (2.41) to obtain an expression for the gradient of  $\bar{\mu}$

$$\rho \nabla \bar{\mu} = \nabla p - s \nabla T - \frac{\rho_n}{2} \nabla (\mathbf{v}_n - \mathbf{v}_s)^2. \quad (5.9)$$

Linearising (5.9) yields

$$\nabla \delta \bar{\mu} = \frac{\nabla \delta p}{\rho_0} - \bar{s}_0 \nabla \delta T. \quad (5.10)$$

Equation (5.7) is differentiated with respect to time and the velocity  $\delta \mathbf{v}_n$  is expressed in terms of  $\delta \mathbf{j}$  and  $\delta \mathbf{v}_s$  using (5.2), which leads to

$$\frac{\partial^2 \delta \bar{s}}{\partial t^2} = \frac{\bar{s}_0}{\rho_0} \rho_{s_0} \operatorname{div} \left( \frac{\partial \delta \mathbf{j}}{\partial t} - \rho_0 \frac{\partial \delta \mathbf{v}_s}{\partial t} \right). \quad (5.11)$$

For the time derivatives  $\frac{\partial \delta \mathbf{j}}{\partial t}$  and  $\frac{\partial \delta \mathbf{v}_s}{\partial t}$ , equations (5.4) and (5.5) are inserted. Using (5.10) we finally obtain

$$\frac{\partial^2 \delta \bar{s}}{\partial t^2} - \bar{s}_0^2 \frac{\rho_{s_0}}{\rho_{n_0}} \Delta \delta T = 0. \quad (5.12)$$

Equations (5.8) and (5.12) are coupled via the following thermodynamic identities

$$\delta p = \left( \frac{\partial p}{\partial \rho} \right)_{\bar{s}} \delta \rho + \left( \frac{\partial p}{\partial \bar{s}} \right)_{\rho} \delta \bar{s}, \quad (5.13)$$

$$\delta T = \left( \frac{\partial T}{\partial \rho} \right)_{\bar{s}} \delta \rho + \left( \frac{\partial T}{\partial \bar{s}} \right)_{\rho} \delta \bar{s}. \quad (5.14)$$

Equation (5.13) can be obtained by expressing the pressure  $p$  using the thermodynamic potential per unit volume  $\Omega$  according to (4.6) and inserting (4.11) and (4.12) into (4.9) and (4.10). Taking into account that in a linearised version, the dependency on the relative velocity does not contribute as  $p$  only depends on the relative velocity squared, one obtains that the fluctuations of the pressure is a function of  $\rho$  and  $s$  only. Thereby, it is as well a function of  $\rho$  and  $\bar{s} = s/\rho$

$$\delta p = -\Omega(\delta \rho, \delta s) = \delta p(\delta \rho, \delta \bar{s}). \quad (5.15)$$

Equation (5.14) is a direct consequence of linearising (4.11).

Inserting (5.13) and (5.14) into (5.8) and (5.12) we obtain the following two coupled wave equations

$$\frac{\partial^2 \delta \rho}{\partial t^2} - \left[ \left( \frac{\partial p}{\partial \rho} \right)_{\bar{s}} \Delta \delta \rho + \left( \frac{\partial p}{\partial \bar{s}} \right)_{\rho} \Delta \delta \bar{s} \right] = 0, \quad (5.16)$$

$$\frac{\partial^2 \delta \bar{s}}{\partial t^2} - \bar{s}_0^2 \frac{\rho_{s_0}}{\rho_{n_0}} \left[ \left( \frac{\partial T}{\partial \rho} \right)_{\bar{s}} \Delta \delta \rho + \left( \frac{\partial T}{\partial \bar{s}} \right)_{\rho} \Delta \delta \bar{s} \right] = 0. \quad (5.17)$$

With the simplest ansatz of a plane wave propagating through the fluid, that is  $\delta \rho, \delta \bar{s} \propto e^{i(\mathbf{q} \cdot \mathbf{r} - \omega t)}$ , we obtain

$$\frac{\omega^2}{q^2} \delta \rho = \left( \frac{\partial p}{\partial \rho} \right)_{\bar{s}} \delta \rho + \left( \frac{\partial p}{\partial \bar{s}} \right)_{\rho} \delta \bar{s}, \quad (5.18)$$

$$\frac{\omega^2}{q^2} \delta \bar{s} = \left[ \left( \frac{\partial T}{\partial \rho} \right)_{\bar{s}} \delta \rho + \left( \frac{\partial T}{\partial \bar{s}} \right)_{\rho} \delta \bar{s} \right] \bar{s}_0^2 \frac{\rho_{s_0}}{\rho_{n_0}}. \quad (5.19)$$

Solving the system (5.18), (5.19) we obtain the following equation for the velocity of sound  $u = \omega/q$

$$u^4 - u^2 \left[ \left( \frac{\partial p}{\partial \rho} \right)_{\bar{s}} + \frac{\rho_{s_0} \bar{s}_0^2}{\rho_{n_0}} \left( \frac{\partial T}{\partial \bar{s}} \right)_{\rho} \right] + \bar{s}_0^2 \frac{\rho_{s_0}}{\rho_{n_0}} \left[ \left( \frac{\partial T}{\partial \bar{s}} \right)_{\rho} \left( \frac{\partial p}{\partial \rho} \right)_{\bar{s}} - \left( \frac{\partial p}{\partial \bar{s}} \right)_{\rho} \left( \frac{\partial T}{\partial \rho} \right)_{\bar{s}} \right] = 0. \quad (5.20)$$

The lengthy expression

$$\det \frac{\partial(T, p)}{\partial(\bar{s}, \rho)} = \left( \frac{\partial T}{\partial \bar{s}} \right)_{\rho} \left( \frac{\partial p}{\partial \rho} \right)_{\bar{s}} - \left( \frac{\partial p}{\partial \bar{s}} \right)_{\rho} \left( \frac{\partial T}{\partial \rho} \right)_{\bar{s}} \quad (5.21)$$

can be replaced using the cyclic relation for exact differentials also known as triple product rule

$$\left( \frac{\partial \rho}{\partial T} \right)_{\bar{s}} \left( \frac{\partial T}{\partial \bar{s}} \right)_{\rho} \left( \frac{\partial \bar{s}}{\partial \rho} \right)_T = -1. \quad (5.22)$$

For instance, equation (5.22) can be solved for  $\left. \frac{\partial T}{\partial \rho} \right|_{\bar{s}}$  and inserted into (5.21), yielding

$$\begin{aligned} \det \frac{\partial(T, p)}{\partial(\bar{s}, \rho)} &= \left( \frac{\partial T}{\partial \bar{s}} \right)_{\rho} \left( \frac{\partial p}{\partial \rho} \right)_{\bar{s}} + \left( \frac{\partial p}{\partial \bar{s}} \right)_{\rho} \left( \frac{\partial T}{\partial \bar{s}} \right)_{\rho} \left( \frac{\partial \bar{s}}{\partial \rho} \right)_T \\ &= \left( \frac{\partial T}{\partial \bar{s}} \right)_{\rho} \left[ \left( \frac{\partial p}{\partial \rho} \right)_{\bar{s}} + \left( \frac{\partial p}{\partial \bar{s}} \right)_{\rho} \left( \frac{\partial \bar{s}}{\partial \rho} \right)_T \right] \\ &= \left( \frac{\partial T}{\partial \bar{s}} \right)_{\rho} \left( \frac{\partial p}{\partial \rho} \right)_T. \end{aligned} \quad (5.23)$$

In the last step we have used that the pressure  $p$  can be expressed due to equation (4.11) as follows

$$p = p(\rho, T) = p(\rho, \bar{s}(T, \rho)) . \quad (5.24)$$

By using the definition of the equilibrium specific heat per unit mass

$$c_\nu = T \left( \frac{\partial \bar{s}}{\partial T} \right)_\rho \quad (5.25)$$

and inserting (5.23) into (5.20) we finally obtain

$$u^4 - u^2 \left[ \left( \frac{\partial p}{\partial \rho} \right)_{\bar{s}} + \frac{\rho_{s_0}}{\rho_{n_0}} \frac{T \bar{s}_0^2}{c_\nu} \right] + \frac{\rho_{s_0}}{\rho_{n_0}} \left( \frac{T \bar{s}_0^2}{c_\nu} \right) \left( \frac{\partial p}{\partial \rho} \right)_T = 0 . \quad (5.26)$$

This formula is very generic and can be used to describe superfluid liquids or condensed Bose gases in the hydrodynamic limit. However, in the case of any liquid, e.g. He II, equation (5.26) can be considerably simplified, as temperature and pressure fluctuations are essentially uncoupled

$$\left( \frac{\partial p}{\partial T} \right)_\rho \simeq 0 . \quad (5.27)$$

If we express the pressure by using equation (4.11) we find

$$p = p(T, \rho) = p(T(\bar{s}, \rho), \rho) . \quad (5.28)$$

Using (5.28) we can express the partial derivative of  $p$  with respect to  $\bar{s}$  according to

$$\left( \frac{\partial p}{\partial \rho} \right)_{\bar{s}} = \left( \frac{\partial p}{\partial \rho} \right)_T + \left( \frac{\partial p}{\partial T} \right)_\rho \left( \frac{\partial T}{\partial \rho} \right)_{\bar{s}} . \quad (5.29)$$

By inserting (5.27) into (5.29), we obtain

$$\left( \frac{\partial p}{\partial \rho} \right)_{\bar{s}} \simeq \left( \frac{\partial p}{\partial \rho} \right)_T . \quad (5.30)$$

In this case equation (5.26) is considerably simplified and has the two solutions

$$u_1^2 = \left( \frac{\partial p}{\partial \rho} \right)_{\bar{s}} , \quad (5.31)$$

$$u_2^2 = \bar{s}_0^2 \frac{\rho_{s_0}}{\rho_{n_0}} \left( \frac{\partial T}{\partial \bar{s}} \right)_\rho . \quad (5.32)$$

These solutions refer to two distinct physical phenomena referred to as first and second sound.

In the approximation (5.27), that temperature and pressure fluctuations are uncoupled, one can also show, that first sound is a pure density wave and second sound is a pure temperature wave. In case, a wave propagates with the velocity  $u_2$ , we can insert (5.32) into (5.19) and obtain

$$\left( \frac{\partial T}{\partial \rho} \right)_{\bar{s}} \delta \rho = 0 . \quad (5.33)$$

This means, that any temperature fluctuation is directly proportional to an entropy fluctuations and no other quantities interfere. Equation (5.14) reduces to

$$\delta T = \left( \frac{\partial T}{\partial \bar{s}} \right)_\rho \delta \bar{s}. \quad (5.34)$$

If we solve (5.34) for  $\delta \bar{s}$  and insert this into (5.13) we obtain

$$\delta p = \left( \frac{\partial p}{\partial \rho} \right)_{\bar{s}} \delta \rho + \left( \frac{\partial p}{\partial \bar{s}} \right)_\rho \left( \frac{\partial \bar{s}}{\partial T} \right)_\rho \delta T, \quad (5.35)$$

As temperature and pressure fluctuations are uncoupled according to (5.27) this means, that pressure fluctuations are directly proportional to density fluctuations

$$\delta p = \left( \frac{\partial p}{\partial \rho} \right)_{\bar{s}} \delta \rho + \underbrace{\left( \frac{\partial p}{\partial T} \right)_\rho}_{=0} \delta T. \quad (5.36)$$

A consideration similar to the steps above can be done for the propagation of first sound. Thereby, equations (5.34) and (5.35) emerge as well. Thus, it becomes clear, that first and second sound are uncoupled and describe two different physical processes.

**First Sound** is an ordinary phenomenon in liquid, describing the propagation of a density or pressure wave, respectively, that is associated with normal acoustic sound. It is propagated with the velocity  $u_1$  according to equation (5.31). In that case the temperature and entropy fluctuations can be set to zero. Then from equation (5.7) we obtain, that the motion of the two components during the propagation of first sound waves has to be in-phase

$$\delta \mathbf{v}_s = \delta \mathbf{v}_n. \quad (5.37)$$

The velocity of first sound was first successfully measured in 1938 by Findlay et al. [21].

**Second sound** is a phenomenon characteristic for superfluids. It describes the ability to propagate undamped entropy or temperature oscillations. The velocity of second sound  $u_2$  is given by equation (5.32). The density and pressure fluctuations can be set to zero. Therefore, from equation (5.1) it follows, that

$$\delta \mathbf{j} = \rho_{s_0} \delta \mathbf{v}_s + \rho_{n_0} \delta \mathbf{v}_n = \mathbf{0}, \quad (5.38)$$

which means, that the total mass current of the liquid is not oscillating, though each of the components does oscillate. Thus, the two motions are out-of-phase

$$\rho_{s_0} \delta \mathbf{v}_s = -\rho_{n_0} \delta \mathbf{v}_n. \quad (5.39)$$

Second sound was first measured in 1946 by Peshkov in liquid helium II, who used a method proposed by Lifshitz in 1944 [22], [23].

## 5.2 Third Sound

Until now, we have neglected the viscosity  $\eta_n$  of the normal component. However, in order to complete the list of different sound modes in a superfluid, we will discuss a phenomenon referred to as third sound, which only arises, if the normal fluid viscosity is taken into account. Third sound describes the propagation of surface waves along a thin film of superfluid

liquid. Thin means in this context, that an oscillatory motion within a viscous medium is completely damped. This damping principle can be explained easiest for an incompressible liquid. Incompressible means, that the particle density does not change along the trajectory

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0 . \quad (5.40)$$

With equation (5.40) the continuity equation simplifies to

$$\operatorname{div} \mathbf{v} = 0 . \quad (5.41)$$

We consider a plane surface in the  $x - y$ -plane, which does harmonic oscillations with a frequency  $\omega_s$  in the  $x$ -direction. Its velocity is given according to

$$u(t) = u_0 \sin \omega_s t . \quad (5.42)$$

Using symmetry arguments the fluid velocity can only depend on  $z$ , i.e. the distance from the surface. In addition, the following boundary conditions have to be fulfilled:

$$v_x(z = 0) = u_0 \quad v_y(z = 0) = v_z(z = 0) = 0 . \quad (5.43)$$

Thus, using equation (5.41) we obtain

$$v_z = 0 . \quad (5.44)$$

By similar arguments the Navier-Stokes equation for an incompressible liquid [7]

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{v} , \quad (5.45)$$

in which  $\nu$  is the kinematic viscosity defined by  $\nu = \eta/\rho$ , simplifies to

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{v} . \quad (5.46)$$

From the  $z$ -component of equation (5.46), we obtain that the pressure is constant. Thus, we obtain a simplified equation for the velocity in  $x$  direction

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial x^2} . \quad (5.47)$$

With an ansatz of the form

$$v_x = u_0 \exp i(kz - \omega_s t) \quad (5.48)$$

we find that the velocity is given by

$$v_x(z, t) = v_{x_0} \exp(-z/\delta) \sin(z/\delta - \omega_s t) , \quad (5.49)$$

where the damping coefficient of this transverse wave, also called the penetrations depth, equals

$$\delta = \sqrt{\frac{2\eta}{\omega_s \rho}} . \quad (5.50)$$

We can read off from equation (5.49), that the motion parallel to the surface is damped exponentially as we move away from the surface. In return this means, that for a thin film of liquid, which is of a thickness  $d \ll \delta$  the whole liquid sticks to the surface. As a consequence, no surface waves can be excited in this case.

If we consider a superfluid, only the normal component is affected from this clamping effect, but the superfluid component is not due to its zero viscosity. It is therefore possible, that a surface wave propagates even along a very thin film of liquid. These surface waves coincide with temperature waves, as at the peaks of the waves there is an excess of the superfluid component, which carries no heat, whereas at the troughs the temperature is raised due to the accumulation of the normal fluid component. This increase (decrease) of temperature leads to evaporation (condensation).

All equations needed to derive the propagation of third sound are linearised. The temperature  $T$  is given by the equilibrium temperature  $T_{\text{eq}}$  plus a small temperature change  $\delta T$  and the height of the film is given by  $z = z_{\text{eq}} + \delta z$ . The rate of evaporation of the film per unit area of surface is therefore proportional by a factor  $K$  to the temperature change  $\delta T$

$$\frac{dm}{dt} = K\delta T . \quad (5.51)$$

The conservation of mass per unit area yields

$$\frac{\partial \rho z}{\partial t} + \text{div}(\rho_s z \mathbf{v}_s) + K\delta T = 0 . \quad (5.52)$$

Assuming, that the change of the height of the film will be much stronger than the changes in the densities, a linearised version of (5.52) is given by

$$\rho_s z_{\text{eq}} \frac{\partial \delta v_{s_x}}{\partial x} + \rho \frac{\partial \delta z}{\partial t} + K\delta T = 0 . \quad (5.53)$$

Again, due to symmetry arguments the superfluid velocity can be reduced to a one dimensional quantity  $v_{s_x}$ . The heat flow is given by

$$\rho z_{\text{eq}} c \frac{\partial \delta T}{\partial t} = \rho_s z_{\text{eq}} \frac{\partial \delta v_{s_x}}{\partial x} \bar{s} T - LK\delta T , \quad (5.54)$$

where  $c$  is the specific heat capacity,  $\bar{s}$  the entropy per unit mass and  $L$  the heat of vaporization per unit mass. The equilibrium pressure  $p_{\text{eq}}$  is given by the vapour pressure, which is dependent on the temperature and on the force, which ties the liquid to the surface. In shallow water waves, this force would simply be the gravitational force. In the case of superfluid helium II, the dominating forces are van der Waals forces acting between the liquid and the wall. Those forces are proportional to the mass by a constant  $f$ . The deviation of the pressure from equilibrium is therefore given by

$$\delta p = \beta \delta T + f \rho \delta z , \quad (5.55)$$

where  $\beta = \left( \frac{dp}{dT} \right)_{v.p.c.}$  is the slope of the vapour pressure curve. Inserting this specific expression for the pressure, Euler's equation for the superfluid (5.4) becomes

$$\frac{\partial \delta v_{s_x}}{\partial t} = -f \frac{\partial \delta z}{\partial x} + \left( \bar{s} - \frac{\beta}{\rho} \right) \frac{\partial \delta T}{\partial x} . \quad (5.56)$$

With the ansatz of a plane wave for temperature, velocity and height fluctuations

$$\delta z, \delta T, \delta v_{s_x} \propto \exp[i(\omega t - kx)] \quad (5.57)$$

one obtains by using equations (5.53), (5.54), and (5.56)

$$\frac{\omega^2}{k^2} = \frac{\rho_s}{\rho} z_{\text{eq}} f + \frac{\rho_s}{\rho} \bar{s} T \left[ \left( \bar{s} - \frac{\beta}{\rho} \right) - i \frac{Kf}{\rho\omega} \right] / \left[ c - i \frac{KL}{\rho\omega d} \right]. \quad (5.58)$$

Without effects of evaporation, i.e.  $K = 0$ , the velocity of third sound is given by

$$u_{3a}^2 = \frac{\rho_s}{\rho} \left[ z_{\text{eq}} f + \frac{\bar{s} T (\bar{s} - \beta/\rho)}{c} \right]. \quad (5.59)$$

In the case, where the frequencies are small and the evaporation is strong, the sound velocity is given by

$$u_{3b}^2 = \frac{\rho_s}{\rho} z_{\text{eq}} f \left[ 1 + \frac{T\bar{s}}{L} \right]. \quad (5.60)$$

In both cases the similarity to water waves in a gravitational field is apparent. Those waves propagate with a velocity  $c$  according to

$$c^2 = gh, \quad (5.61)$$

where  $g$  is the standard gravity and  $h$  is the height of water. In the case of a two-fluid liquid on a thin film, this term is proportional to the van der Waals forces instead of gravity and is furthermore proportional to the relative amount of the superfluid component. Therefore, at  $T = 0$  K the sound velocity is exactly analogous to the velocity of gravity waves and vanishes at temperatures above the  $\lambda$ -point, where the superfluid density is zero.

For a more detailed view on surface waves and thin films in classic hydrodynamic see [7, p. 83, p. 411]. A more elaborate discussion of third sound is given by Atkins [24], who first discussed this phenomenon in 1953. Experimental verification of the principal constituents of the theory was carried out in 1964 by Everitt et al. [25].

### 5.3 Fourth Sound

Fourth Sound describes the wave propagation in a thin capillary, whose diameter is much smaller than the penetration depth  $\delta$  defined by (5.50). Analogous to third sound in a thin film, the normal component of the liquid sticks to the wall and its motion can be neglected, i.e.  $\delta v_n = 0$ . As in a capillary the liquid can not expand, oscillatory motions produce density changes in first order. The heat flow into the walls of the capillary are ignored. For symmetry reasons, we can assume that all quantities only depend on the direction of the tube  $x$  and that the velocities in radial direction are equal to zero. Therefore, the linearised hydrodynamic equations (5.1) to (5.5) read in this case

$$\frac{\partial \delta \rho}{\partial t} + \rho_{s0} \frac{\partial v_s}{\partial x} = 0, \quad (5.62)$$

$$\frac{\partial v_s}{\partial t} = -\frac{1}{\rho_0} \frac{\partial \delta p}{\partial x} + \bar{s}_0 \frac{\partial \delta T}{\partial x}, \quad (5.63)$$



$$\frac{\partial s}{\partial t} = \rho_0 \frac{\partial \delta \bar{s}}{\partial t} + \bar{s}_0 \frac{\partial \delta \rho}{\partial t} = 0. \quad (5.64)$$

Again, we assume, that temperature and pressure fluctuations are uncoupled according to (5.27), which simplifies the relations (5.13) and (5.14) yielding

$$\delta p = \left( \frac{\partial p}{\partial \rho} \right)_{\bar{s}} \delta \rho, \quad (5.65)$$

$$\delta T = \left( \frac{\partial T}{\partial \bar{s}} \right)_{\rho} \delta \bar{s}. \quad (5.66)$$

Taking the partial derivative of equation (5.64) with respect to  $x$ , we obtain

$$\frac{\partial^2 \delta \bar{s}}{\partial t \partial x} = -\frac{\bar{s}_0}{\rho_0} \frac{\partial^2 \delta \rho}{\partial t \partial x}. \quad (5.67)$$

Inserting (5.65) and (5.66) into (5.63) and performing the partial derivative with respect to time, we obtain

$$\frac{\partial^2 v_s}{\partial t^2} = -\frac{1}{\rho_0} \left( \frac{\partial p}{\partial \rho} \right)_{\bar{s}} \frac{\partial^2 \delta \rho}{\partial x \partial t} - \bar{s}_0 \left( \frac{\partial T}{\partial \bar{s}} \right)_{\rho} \frac{\partial^2 \delta \bar{s}}{\partial x \partial t}. \quad (5.68)$$

Inserting (5.67) into (5.68) the entropy per unit mass  $\bar{s}$  can be eliminated. Taking the partial derivative of equation (5.62) with respect to  $x$ , we can also eliminate the density and obtain a wave equation for  $v_s$ :

$$\frac{\partial^2 v_s}{\partial t^2} = \rho_{s_0} \left[ \frac{1}{\rho_0} \left( \frac{\partial p}{\partial \rho} \right)_{\bar{s}} + \frac{\bar{s}_0^2}{\rho_0} \left( \frac{\partial T}{\partial \bar{s}} \right)_{\rho} \right] \frac{\partial^2 v_s}{\partial x^2}. \quad (5.69)$$

With the ansatz of a plane wave  $v_s \propto \exp i(kx - \omega t)$  this yields

$$\frac{\omega^2}{k^2} = \frac{\rho_{s_0}}{\rho_0} \left( \frac{\partial p}{\partial \rho} \right)_{\bar{s}} + \frac{\rho_{s_0}}{\rho_0} \bar{s}_0^2 \left( \frac{\partial T}{\partial \bar{s}} \right)_{\rho}. \quad (5.70)$$

By equations (5.31) and (5.32), we can write the velocity of fourth sound as a combination of the velocities of first and second sound

$$u_4^2 = \frac{\rho_{s_0}}{\rho_0} u_1^2 + \frac{\rho_{n_0}}{\rho_0} u_2^2. \quad (5.71)$$

From this equation, we can read off, that at temperatures close to zero temperature, when the normal component of the liquid vanishes, the wave propagates like a normal density wave with the velocity of first sound. On the other hand, for high temperatures it seems like the second term would dominate, but the velocity of third sound itself depends on the density of the superfluid component and vanishes at high temperatures. Therefore, the decomposition of  $u_3^2$  in equation (5.71) is slightly deceptive.

Like the phenomenon of third sound, fourth sound was first studied in 1958 by Atkins [24] and validated in 1965 by Shapiro et al. [26].

## 5.4 Fifth Sound

Fifth Sound occurs in geometrical set-ups similar to those used for the detection of third sound. The essential conditions are again boundary conditions, that force the normal component to stay at rest. In fact, fifth sound represents a special case of third sound, in which the vaporization does not contribute to the wave propagation. The equations (5.53) to (5.58) can be used with the following alterations: The vapor pressure does not come into effect, and thus the coefficient of the vapor pressure slope  $\beta$  can be set to zero. Also, the evaporation coefficient  $K$  is set to zero. The continuity equation (5.53) and the heat flow (5.54) then reduce to

$$\rho_s z_{\text{eq}} \frac{\partial \delta v_{sx}}{\partial x} + \rho \frac{\partial \delta z}{\partial t} = 0, \quad (5.72)$$

$$\rho z_{\text{eq}} c \frac{\partial \delta T}{\partial t} = \rho_s z_{\text{eq}} \frac{\partial \delta v_{sx}}{\partial x} \bar{s} T. \quad (5.73)$$

In case, the van der Waals forces are small compared to the temperature fluctuations, Euler's equation (5.56) is simplified yielding

$$\frac{\partial \delta v_{sx}}{\partial t} = \bar{s}_0 \frac{\partial \delta T}{\partial x}. \quad (5.74)$$

The velocity of a wave propagating under these circumstances is then given by equation (5.58) with  $K = \beta = f = 0$

$$u_5^2 = \frac{\rho_{s0}}{\rho_0} \bar{s}_0^2 \frac{T}{c} = \frac{\rho_{n0}}{\rho_0} u_2^2. \quad (5.75)$$

By using the expression for the velocity of second sound (5.32), it becomes apparent, that fifth sound is related to pure temperature waves like second sound under special geometrical conditions.

Fifth sound was proposed theoretically in 1979 by Rudnick et al. [27] and validated experimentally in 1979 by Williams et al. [28].

## 5.5 Further Sound Modes

There are several further wave phenomena or sound modes, that can occur in a hydrodynamic system, which can be described by the LK two-fluid equations. However, they can not all be discussed here. Instead in the following they are listed and references for further reading are given.

The **fifth wave mode** describes the heavily damped propagation of second sound in a capillary. Theoretical prediction in 1971 by Wiechert et al. [29], experimental verification in 1982 by Wiechert et al. [30].

**Two-phase sound** refers to the coupling of second sound between the vaporized phase and the liquid phase. Theoretical prediction and experimental verification in 1979 by Putterman et al. [31].

**Surface tension sound** is a wave mode, that occurs only on curved surfaces on thin films. The velocity of propagation increases with a decrease of the curvature radius. Theoretical prediction in 1975 by Saam et al. [32], experimental verification in 1979 by Rosenbaum et al. [33].

**Second surface sound** describes density waves in the normal component on surfaces of films, which are thicker than the penetration depth. Theoretical prediction in 1972 by Andreev et al., experimental verification in 1974 by Eckardt et al. [20].

**Zeroth Sound** describes sound propagation in a Fermi liquid at very low temperatures. Theoretical prediction in 1957 by Landau [20], experimental verification in 1963 by Keen et al. [34].

## 6 Conclusion and Outlook

In this thesis we derived the Landau-Khalatnikov equations first by the principle of least action and second by an approach based on the Galilean relativity principle and symmetry arguments according to Khalatnikov. Both treatments were based on the fundamental role of conservation laws and led to the same results. However, they started with different sets of initial assumptions. Using the variational principle, the equivalence of the two sets could be derived. The strengths and weaknesses of the two approaches were pointed out emphasizing the potential of the variational ansatz to be extended by inclusions of new degrees of freedom.

The obtained LK equations were linearised in order to discuss the phenomena of sound propagation. First and second sound were derived from these equations and discussed in the case of uncoupled temperature and pressure fluctuations. The phenomena of third, fourth and fifth sound, which appear under special geometrical conditions, were also derived and discussed. Subsequently, a summary of further wave phenomena in superfluids was given.

In order to unfold full applicability the LK equation need to be completed by equations of state, which can only be obtained from a microscopic theory. As superfluidity is a macroscopic quantum phenomenon, it would be convincing to use the Bogoliubov theory for a weakly interacting gas of Bosons with contact interaction at low temperatures and calculate the partition function in order to obtain the equations of state from the corresponding free energy [2].

Another interesting step would be to extend the two-fluid model, which is by now only capable of describing a system with isotropic contact interaction, to a model, which includes the effects of anisotropic, e.g. dipole-dipole interaction. At  $T = 0$  K it is proposed, that in a Bose gas with dipole-dipole interaction and weak disorder, the superfluid density becomes anisotropic [35]. This is also proposed to happen at finite temperatures in systems without disorder [36]. In such a case the scalar functions  $\rho_n$  and  $\rho_s$  would have to be replaced by a diagonal  $3 \times 3$  matrices  $\rho_{n_{ij}} = \rho_{n_i} \delta_{ij}$ ,  $\rho_{s_{ij}} = \rho_{s_i} \delta_{ij}$  with the following property

$$\rho_{n_x} = \rho_{n_y} \neq \rho_{n_z}, \quad \rho_{s_x} = \rho_{s_y} \neq \rho_{s_z}, \quad (6.1)$$

where  $z$  points along the direction of the dipoles and  $\rho \delta_{ij} = \rho_{n_{ij}} + \rho_{s_{ij}}$ . The two fluid equations would then have to be rewritten accordingly. For example the kinetic energy density would be

$$e_{\text{kin}} = \frac{1}{2} v_{s_i} (\rho \delta_{ij} - \rho_{n_{ij}}) v_{s_j} + \frac{1}{2} v_{n_i} \rho_{n_{ij}} v_{n_j}. \quad (6.2)$$

A third aim would be to include disorder in the hydrodynamic description of a superfluid. It has been proposed theoretically by Huang and Meng [37], that a disorder potential leads to the formation of local condensates in the potential minima and that this causes a reduction of the superfluid density even at zero temperature, whereas without disorder the complete fluid is in a superfluid state [38]. This begs the question, whether disorder leads to a third

component of the fluid. In a system at finite temperature with disorder the superfluid density, the normal density, and presumably a third component, resulting from the formation of local condensates, would then have to be described in a unified three-fluid model.

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## **Declaration**

I hereby declare, that I have written this thesis on my own and that I have listed all used sources of information in the references.

Carolin Wille, Berlin, 16th June, 2011