Analytical and Numerical Study of Localized Impurity in Bose-Einstein Condensate

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Motivation

Motivated by the recent experimental work of Refs. [1,2], we investigate a localized ^{133}Cs impurity in the center of a trapped ^{87}Rb Bose-Einstein condensate (BEC). Within a zero-temperature mean-field description, we provide a one-dimensional physically intuitive model, which we solve by a time-independent variational approach and numerical calculations. With this, we predict both width and dip in the condensate wave function for increasing strength of the interaction between impurity and BEC.

Impurity in one-dimensional BEC

- One-dimensional dimensionless GPE ($\tau = \omega_z t$, $z = \tilde{z} l_{\rm B}$, and $l_{\rm B} = \sqrt{\hbar/(m_{\rm B}\omega_z)}$),

$$i\frac{\partial}{\partial\tau}\tilde{\Psi}(\tilde{z}) = \left\{-\frac{\partial^2}{\partial\tilde{z}^2} + \frac{1}{2}\tilde{z}^2 + \tilde{G}_{\rm IB}\delta(\tilde{z}) + 2\tilde{G}_{\rm B}|\tilde{\Psi}(\tilde{z})|^2\right\}\tilde{\Psi}(\tilde{z})$$

here $\tilde{G}_{\rm B} = 2\frac{N\omega_r a}{\omega_z l_{\rm B}}$, $\tilde{G}_{\rm IB} = \frac{a_{\rm IB} m_{\rm B}\omega_r}{m_{\rm IB}\omega_z l_{\rm B}}$, $m_{\rm IB} = \frac{m_{\rm I} m_{\rm B}}{m_{\rm I}+m_{\rm B}}$, where $a_{\rm IB}$ is the s-wave scattering length of the impurity-BEC interaction, which is tunable due to a Feshbach resonance.
- Experimental parameters of a 3d experiment:
  - $N = 2.5\times 10^5$ number of ^{87}Rb atoms
  - S-wave scattering length for ^{87}Rb is $a_{\rm B} = 87\ a_0$
  - S-wave scattering length for impurity-BEC coupling $a_{\rm IB} = 650\ a_0$
  - Trap frequencies $\{\omega_x, \omega_y, \omega_z\} = 2\pi\,\{180, 144, 46\}$ Hz,
  - Critical temperature $T_c^0 = \frac{\hbar\tilde{\omega}}{k_{\rm B}} N^{\frac{1}{3}} = 3.204\times 10^{-7}$ K
  - $\frac{T}{T_c^0} = 0.343 \ll 1$ for $T = 110\times 10^{-9}$ K
- Resulting dimensionless parameters in 1d model
  - Impurity-BEC dimensionless coupling constant is $\tilde{G}_{\rm IB} = 0.12$
  - Longitudinal trap frequency $\omega_z = 2\pi\times 46$ Hz
  - Radial trap frequency is $\omega_r = 2\pi\times 160.99$ Hz
  - Dimensionless coupling constant $\tilde{G}_{\rm B} = 5530.11$

Variational approach

- Thomas-Fermi anstaz

$$\tilde{\Psi}(\tilde{z}) = \sqrt{\frac{15\tilde{B}}{4\tilde{A}\left(\tilde{A}^2+5\tilde{B}^2\right)}}\sqrt{\left(1-\frac{\tilde{z}^2}{\tilde{A}^2}\right)\left(1+\frac{\tilde{z}^2}{\tilde{B}^2}\right)}$$

- Total energy of impurity-BEC system

$$E = \frac{15\tilde{B}^2\tilde{G}_{\rm IB}}{4\left(\tilde{A}^3+5\tilde{A}\tilde{B}^2\right)} + \frac{\tilde{G}_{\rm B}}{\left(\tilde{A}^2+5\tilde{B}^2\right)^2}\left[\frac{15\tilde{A}\tilde{B}^2}{7} + \frac{15\tilde{B}^4}{2\tilde{A}} + \frac{5\tilde{A}^3}{14}\right] + \frac{\tilde{A}^2}{2\left(\tilde{A}^2+5\tilde{B}^2\right)}\left[\tilde{B}^2 + \frac{3\tilde{A}^2}{7}\right]$$
$$+ \frac{5}{4\tilde{A}^3\left(\tilde{A}^2+5\tilde{B}^2\right)}\left[\left(2\tilde{A}^3+3\tilde{A}\tilde{B}^2\right) + 3\tilde{B}\left(\tilde{A}^2+\tilde{B}^2\right)\tan^{-1}\left(\frac{\tilde{A}}{\tilde{B}}\right)\right]$$

where last line represents kinetic energy of impurity dip (kinetic energy of envelope neglected)
- Asymptotic behavior of $\tilde{B}$ when $E_{\rm Kin} = 0$, by using the Euler-Lagrange formula, $\frac{\partial E}{\partial B} = 0$

$$\tilde{B} = \frac{\tilde{A}\sqrt{16\tilde{A}^3 + 40\tilde{G}_{\rm B} - 105\tilde{G}_{\rm IB}}}{\sqrt{-80\tilde{A}^3 + 120\tilde{G}_{\rm B} + 525\tilde{G}_{\rm IB}}}$$

this result shows good agreement with the numerical values as shown in Fig. 3.
$\Longrightarrow$ kinetic energy of impurity dip negligible
- Asymptotic behavior of $\tilde{A}$ when $\tilde{B}\to 0$

$$\tilde{A} \simeq \sqrt[3]{\frac{105}{16}\tilde{G}_{\rm IB} - \frac{5}{2}\tilde{G}_{\rm B}}$$

Comparing analytical and numerical results

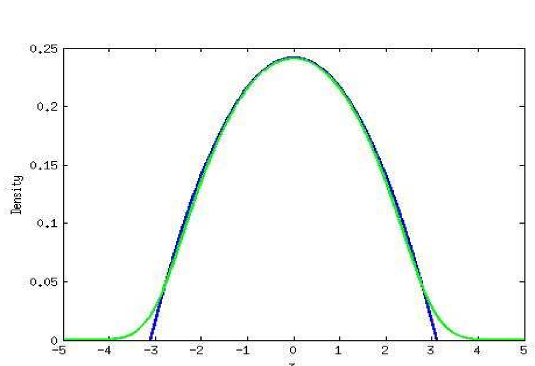

Fig. 1: Blue line represents $\tilde{B}\to\infty$, red line stands for $\tilde{B} = 647.7$, and green line depicts the numerical results. Here $\tilde{G}_{\rm B} = 20$ and $\tilde{G}_{\rm IB} = 0$.

Acknowledgments

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Density profiles

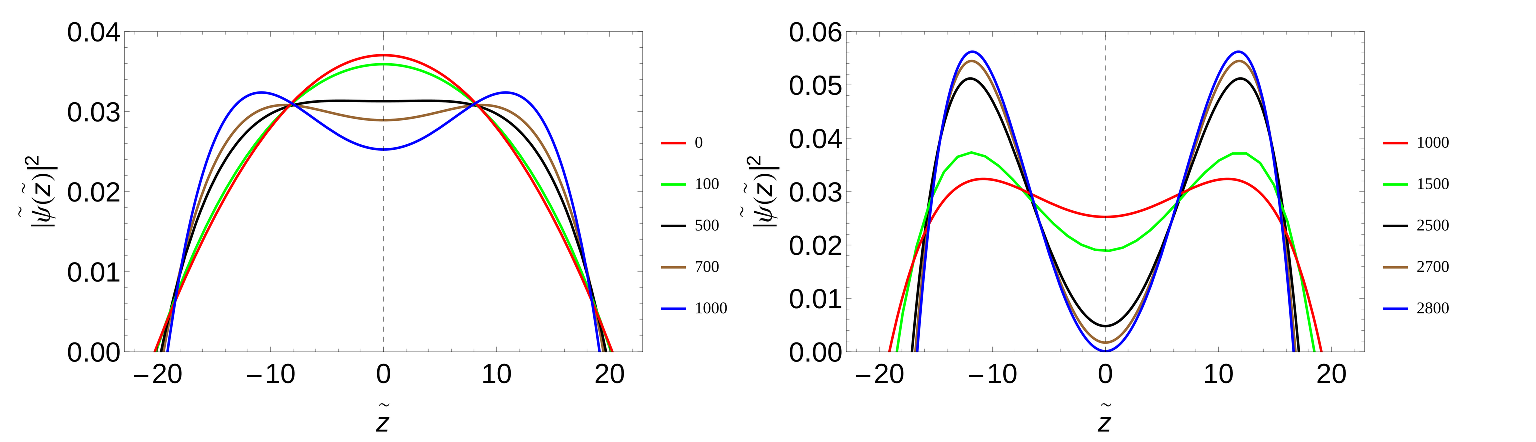

Fig. 2: Condensate density for different values of $\tilde{G}_{\rm IB}$ for experimental BEC coupling constant $\tilde{G}_{\rm B} = 5530.11$.

Variational parameters

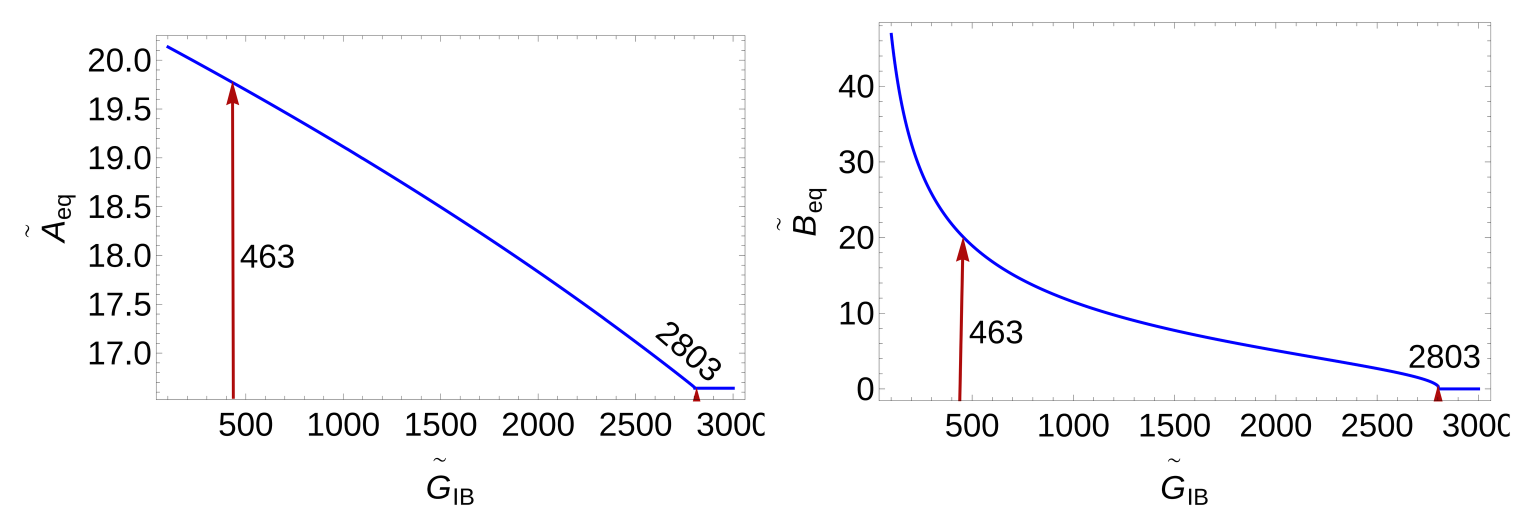

Fig. 3: Variational parameters $\tilde{A}_{\rm eq}$ and $\tilde{B}_{\rm eq}$ versus the coupling constant $\tilde{G}_{\rm IB}$ in the presence of $\tilde{G}_{\rm B} = 5530.11$. We obtain a range for impurity-BEC coupling constant $\tilde{G}_{\rm IB}$ for those values where $\tilde{A}_{\rm eq} > \tilde{B}_{\rm eq}$: $\tilde{G}_{\rm IB} = \{463, 2803\}$.

Asymptotic approximation

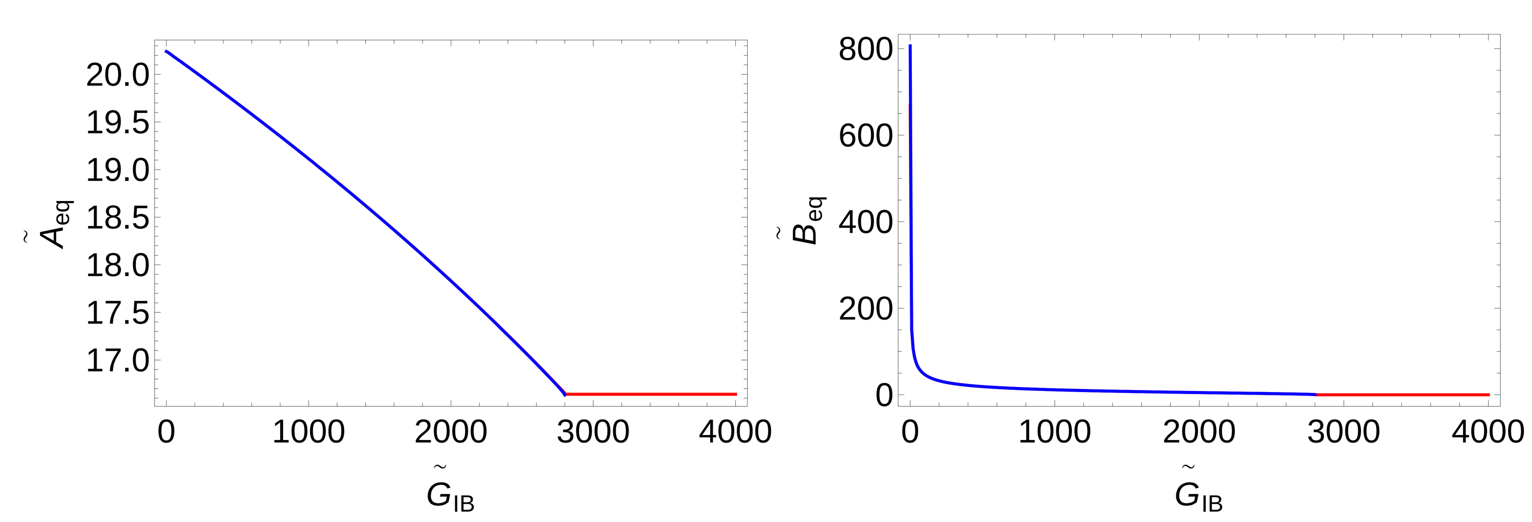

Fig. 3: Variational parameters $\tilde{A}_{\rm eq}$ (left) and $\tilde{B}_{\rm eq}$ (right) versus the coupling constant $\tilde{G}_{\rm IB}$ in the presence of $\tilde{G}_{\rm B} = 5530.11$. Here blue line represents the asymptotic solution and red line stands for numerical solution of the system by minimizing the energy. The asymptotic solution is not valid anymore after the critical value of the impurity-BEC coupling constant $\tilde{G}_{\rm IB} = 2803$.

Impurity dip and width

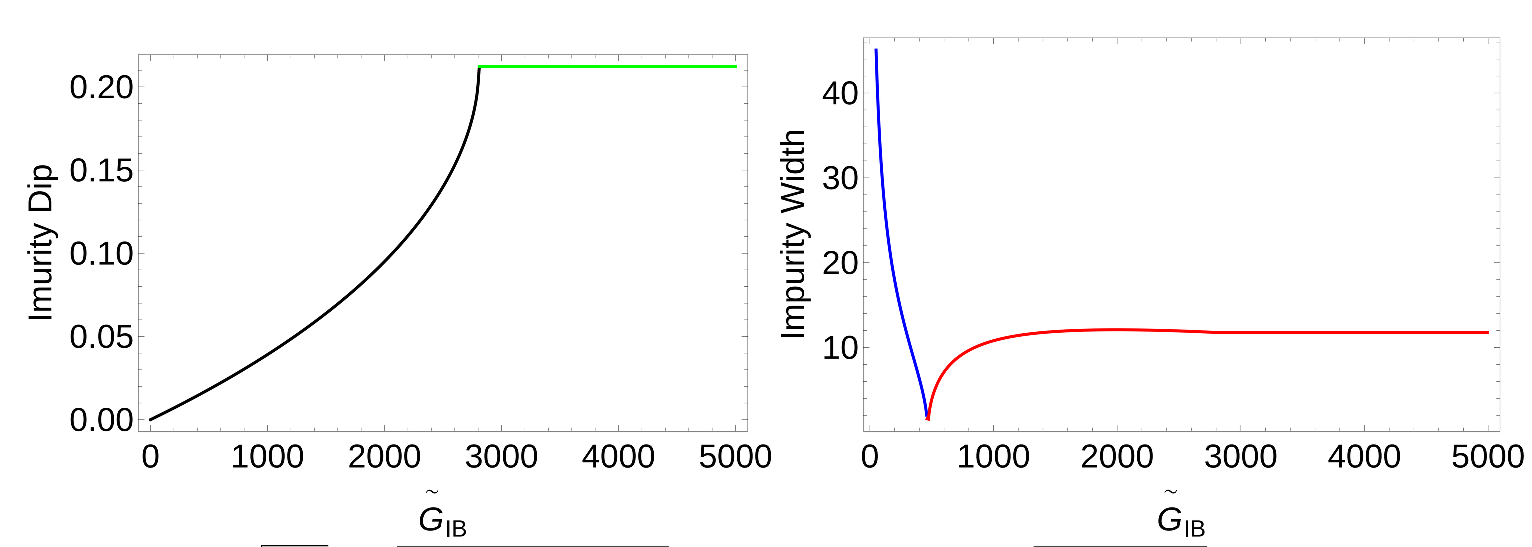

Fig. 4: Impurity dip $\sqrt{3/4\tilde{A}} - \sqrt{15\tilde{B}^2/4\tilde{A}\left(\tilde{A}^2+5\tilde{B}^2\right)}$ (left) and impurity width $\sqrt{\left|\frac{1}{2}\left(\tilde{A}^2-\tilde{B}^2\right)\right|}$ (right) versus impurity-BEC coupling constant $\tilde{G}_{\rm IB}$. At the value $\tilde{G}_{\rm IB} = 2803$, the impurity dip becomes constant as shown in left graph. Here the blue line depicts the negative value of the width of the impurity which is unphysical, and the red line represents the positive width of the impurity in a BEC cloud. At the value $\tilde{G}_{\rm IB} = 463$, the impurity width becomes positive constant as shown in right graph.

References

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