A Single Vortex in a Bose-Einstein Condensate

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Motivation Within a variational approach we describe the physical properties of a Bose-Einstein condensate in a cylinder-symmetric harmonic trap with a single vortex in the center. At first we analyze the equilibrium configuration and determine the vortex size as well as the Thomas-Fermi radii. Then we calculate the critical rotation frequency for the emergence of the vortex and compare our findings with the literature. Finally, we investigate how the presence of the vortex changes the collective excitation frequencies. All results are obtained analytically in form of an asymptotic series in the limit of strong two-particle interactions.

Vortex BEC

- **Gross-Pitaevskii (GP) Equation**
  - ⋆ Time-independent GP equation [1,2]

$$\mu\Psi(\mathbf{r}) = \left\{-\frac{\hbar^2}{2M}\Delta + V(\mathbf{r}) + gN\,|\Psi(\mathbf{r})|^2\right\}\Psi(\mathbf{r})$$

where $V(\mathbf{r}) = \frac{1}{2}M\omega_\rho^2\left(\rho^2+\lambda^2 z^2\right)$ is a trap with anisotropy parameter λ, and $g = 4N\pi\hbar^2 a/m$ is the interaction strength defined by s-wave scattering length a and number of atoms N.

- **Thomas-Fermi (TF) Approximation**
  - ⋆ Density profile in TF: $n(\mathbf{r}) = |\Psi(\mathbf{r})|^2 = \begin{cases} \frac{\mu - V(\mathbf{r})}{g}, & \text{for } \mu - V(\mathbf{r}) \geq 0 \\ 0, & \text{otherwise.}\end{cases}$
  - ⋆ TF radii: $R_\perp = \sqrt{\frac{2\mu}{M\omega_\rho^2}}$, $R_\parallel = \sqrt{\frac{2\mu}{M(\omega_\rho\lambda)^2}}$
  - ⋆ Chemical potential: $\mu = \left[\frac{15\omega_\rho^3\lambda N g}{8\pi}\left(\frac{M}{2}\right)^{3/2}\right]^{2/5}$
- **Quantization of circulation**
  - ⋆ Madelung transformation [3]: $\Psi(\mathbf{r},t) = \sqrt{n(\mathbf{r},t)}e^{iS(\mathbf{r},t)}$
  - ⋆ Superfluid velocity $\mathbf{v} = \frac{\hbar}{M}\bigtriangledown S(\mathbf{r},t)$
  - ⋆ Velocity field is irrotational $\bigtriangledown\times\mathbf{v} = \mathbf{0}$
  - ⋆ Circulation around a closed contour $k = 2\pi l\frac{\hbar}{M}$, l is an integer number.
- **Structure of Single Vortex**
  - ⋆ Solution ansatz $f(\rho,z)e^{il\phi}$ for GP equation [2,4]:

$$-\frac{\hbar^2}{2M}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right)+\frac{\partial^2 f}{\partial z^2}\right]+\frac{\hbar^2}{2M\rho^2}l^2 f+\frac{M}{2}\omega_\rho^2\left(\rho^2+\lambda^2 z^2\right)f+gf^3=\mu f$$

  - ⋆ Healing length

$$\xi = \sqrt{\frac{\hbar^2}{2Mgf^2}} = \frac{1}{\sqrt{8\pi a f^2}}$$

  - ⋆ For $\rho \ll \xi$ dominant term arises from kinetic energy:

$$-\frac{\hbar^2}{2M}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right)\right]+\frac{\hbar^2}{2M\rho^2}l^2 f = 0$$

Solving for f yields

$$f = c\rho^l,\quad n = c^2\rho^{2l},\quad \rho\ll\xi,\quad \text{where } c \text{ is a suitable constant.}$$

  - ⋆ For $\xi \ll \rho < R_\perp$ kinetic energy can be neglected and we have Thomas-Fermi solution:

$$n_{\rm TF}(\rho,z) = \frac{\mu}{g}\left[1-\left(\frac{\rho}{R_\rho}\right)^2-\left(\frac{z}{R_z}\right)^2\right]$$

Variational Approach

- Variational ansatz [5–8]:

$$\Psi(\rho,z,\phi) = C\sqrt{\left(\frac{\rho^2}{\rho^2+R_\rho^2\beta^2}\right)^l\left[1-\left(\frac{\rho}{R_\rho}\right)^2-\left(\frac{z}{R_z}\right)^2\right]}\,e^{il\phi+iB_\rho\rho^2+iB_z z^2}$$

- Variational functions in dimensionless form with $P = Na/l$, $l = \sqrt{\hbar/M\omega_\rho}$

$$\ddot r_z + r_z\left[\lambda^2 + G_0(\beta)\dot\beta^2 + G_1(\beta)\ddot\beta\right] - \frac{G_2(\beta)P}{r_\rho^2 r_z^2} + 2G_1(\beta)\dot r_z\dot\beta = 0$$

$$\ddot r_\rho + r_\rho\left[1 + G_3(\beta)\dot\beta^2 + G_4(\beta)\ddot\beta\right] - \frac{G_5(\beta)}{r_\rho^3} - \frac{G_6(\beta)P}{r_\rho^3 r_z} + 2G_4(\beta)\dot r_\rho\dot\beta = 0$$

$$\left[5G_{14}(\beta)r_\rho^2 - G_{15}(\beta)r_z^2\right]\ddot\beta = -5r_\rho^2 + r_z^2\lambda^2 - 5r_\rho\ddot r_\rho + r_z\ddot r_z + \frac{G_7(\beta)}{r_\rho^2} + \frac{G_8(\beta)P}{r_\rho^2 r_z}$$
$$+ G_9(\beta)\dot r_\rho^2 - G_{10}(\beta)\dot r_z^2 + G_{11}(\beta)r_\rho\dot r_\rho\dot\beta + G_{12}(\beta)\dot\beta + G_{13}(\beta)r_\rho^2\dot\beta^2 + G_{14}(\beta)r_z^2\dot\beta^2 = 0$$

- $G_0(\beta)$–$G_{15}(\beta)$ are functions which depend on relative vortex core size β and have been calculated by using *Mathematica*.

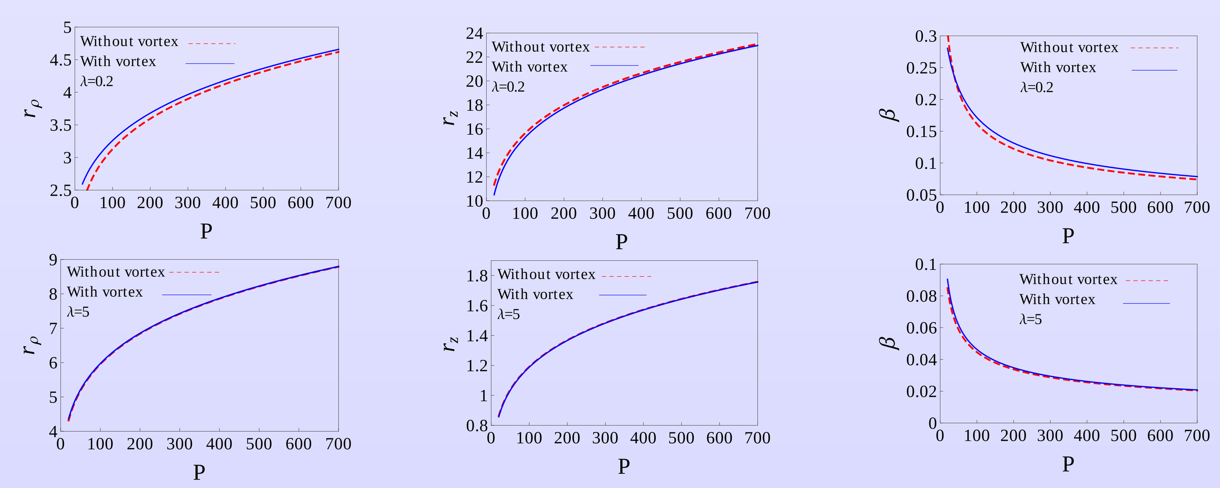
Equilibrium Positions

- Stationary solutions are determined by:

$$\lambda^2 r_{z0} - \frac{G_2 P}{r_{\rho0}^2 r_{z0}^2} = 0,\; r_{\rho0} - \frac{G_6 P}{r_{\rho0}^3 r_{z0}} - \frac{G_5}{r_{\rho0}^3} = 0,\; 5r_{\rho0}^2 - \lambda^2 r_{z0}^2 - \frac{G_7}{r_{\rho0}^2} - \frac{G_8 P}{r_{\rho0}^2 r_{z0}} = 0$$

- Analytical approximation in limit $P\to\infty$:

$$r_{z0} = \frac{r_{\rho0}}{\lambda},\quad r_{\rho0} = (15P\lambda)^{1/5},\quad \beta_0 = \frac{5^{1/10}}{\sqrt{2}\,(3P\lambda)^{2/5}}$$



Energies and Critical Rotational Frequencies

- Total energy of condensate is given by:

$$E_{\rm total} = \int d^3r\left[\frac{\hbar^2}{2M}\bigtriangledown\Psi(\mathbf{r},t)\bigtriangledown\Psi^*(\mathbf{r},t) + V(\mathbf{r})\,|\Psi(\mathbf{r},t)|^2 + \frac{g}{2}|\Psi(\mathbf{r},t)|^4\right]$$

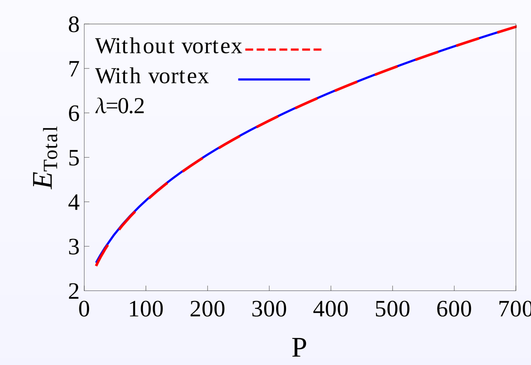
- Energy of condensate with central vortex in TF-limit:

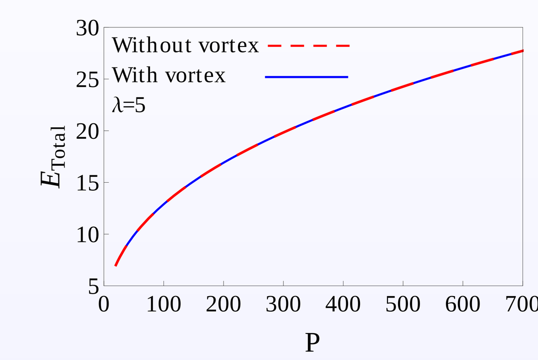
$$E_{\rm total} = \frac{N\hbar\omega_\rho}{560A_1^2}\left[\frac{360P\left(-54+28A_1+63A_1\beta_0^6\right)}{r_{\rho0}^2 r_{z0}} - 8A_1\Big(-30r_{\rho0}^2 + 35r_{\rho0}^2 A_1\beta_0^2 + 6\lambda^2 r_{z0}^2 - 7A_1\lambda^2 r_{z0}^2 - 7A_1\beta_0^2\lambda^2 r_{z0}^2\Big) - \frac{175A_1\left(26+33\beta_0^2\right)}{r_{\rho0}^2} + \frac{3\left(8+16\beta_0^2+11\beta_0^4\right)\text{ArcCoth}\left[\sqrt{1+\beta_0^2}\right]}{r_{\rho0}^2\sqrt{1+\beta_0^2}}\right]$$

- Energy of condensate without vortex in TF-limit:

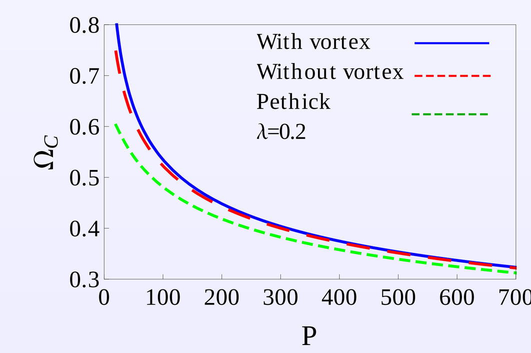
$$E_0 = N\hbar\omega_\rho\frac{5}{14}(15P\lambda)^{2/5}$$

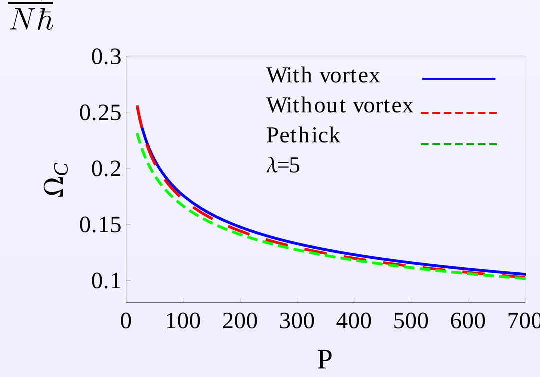
- Creating a vortex costs energy $E_{\rm v} = E_{\rm total} - E_0$





- Critical angular frequency [9,10]: $\Omega_C = \frac{E_{\rm v}}{N\hbar}$





Frequencies of Collective Modes

- Frequencies of collective modes are determined by linearizing equations of motion around equilibrium positions:

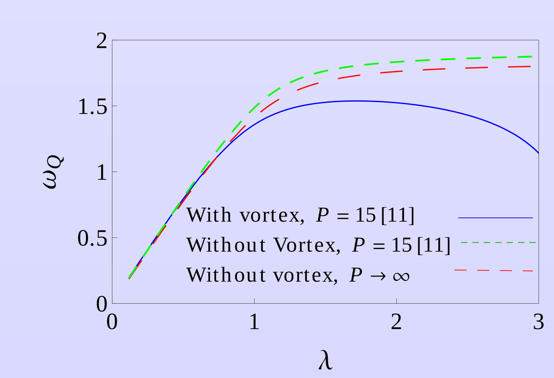
$$\delta\ddot r_{z1}(t) + m_0\delta\ddot\beta_1(t) + m_1\delta r_{\rho1}(t) + m_2\delta u_{z1}(t) + m_3\delta\beta_1(t) = 0,$$
$$\delta\ddot r_{\rho1}(t) + m_4\delta\ddot\beta_1(t) + m_5\delta r_{\rho1}(t) + m_6\delta r_{z1}(t) + m_7\delta\beta_1(t) = 0,$$
$$m_8\delta\ddot\beta_1(t) + 5r_{\rho0}\delta\ddot r_{\rho1}(t) - r_{z0}\delta\ddot z_{z1}(t) + m_9\delta r_{\rho1}(t) + m_{10}\delta r_{z1}(t) + m_{11}\delta\beta_1(t) = 0$$

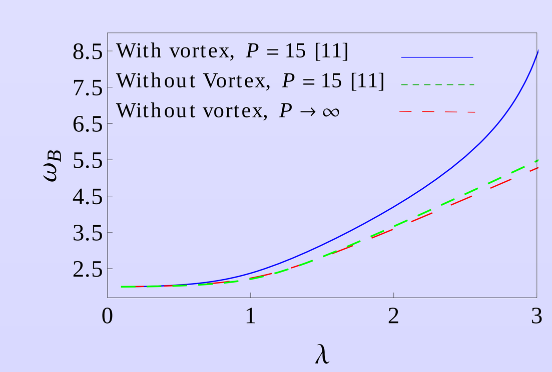
- Frequencies of collective modes are given

$$\omega_{1,2}^2 = \frac{-z_2 \pm \sqrt{z_2^2 - 4z_1 z_3}}{2z_1}$$

- In the limit P going to infinity,

$$\omega_{1,2}^2 = \frac{1}{2}\left[4 + 3\lambda^2 \pm \sqrt{16 - 16\lambda^2 + 9\lambda^4}\right]$$





Summary and Outlook

- ⋆ We studied a Bose-Einstein condensate with a central vortex using a variational approach.
- ⋆ We discussed the equilibrium positions of the condensate, we also found an analytical approximation for the stationary points of the system.
- ⋆ We studied the energy of a condensate with a central vortex as well as the critical frequency of a rotating trap at which the vortex state becomes stable.
- ⋆ We studied frequencies of collective modes.

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