Geometric Resonances in Bose-Einstein Condensates with Two- and Three-Body Interactions

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Motivation: We study geometric resonances [1] in Bose-Einstein condensates (BECs) for systems with two- and three-body interactions [2] in an axially-symmetric harmonic trap. We use analytical method [3] based on a perturbative expansion and Poincaré-Lindstedt analysis of a Gaussian variational approach [4] and numerical simulations. By changing the anisotropy of the confining potential, we numerically observe and analytically describe strong nonlinear effects: resonances and shifts in the frequencies of collective modes, and coupling of collective modes. We also discuss the stability of a condensate in the presence of an attractive two-body interaction and a repulsive three-body interaction. We show that the small repulsive three-body interaction is able to extend stability region of the condensate.

Variational approach

- At zero temperature, BEC is described by the time-dependent Gross-Pitaevskii equation

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t)=\left[-\frac{\hbar^2}{2M}\Delta+V(\mathbf{r})+g_2N\left|\Psi(\mathbf{r},t)\right|^2+g_3N^2\left|\Psi(\mathbf{r},t)\right|^4\right]\Psi(\mathbf{r},t)$$

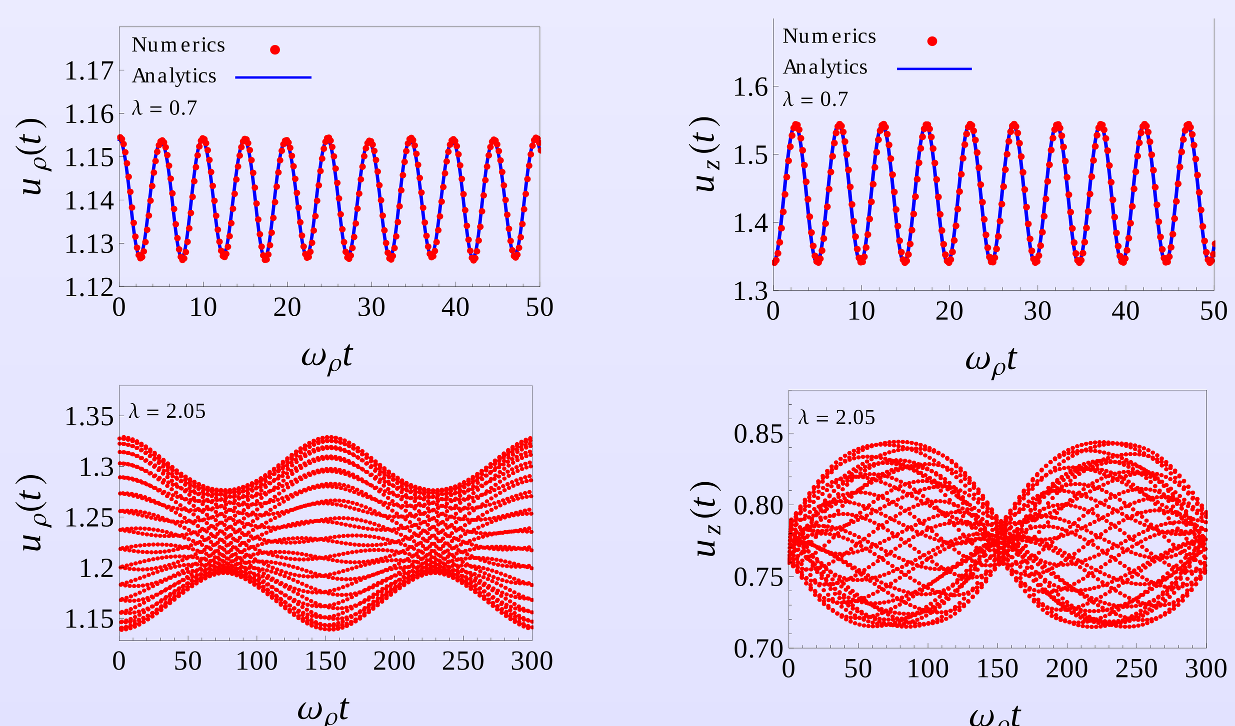
where $V(\mathbf{r})=\frac{1}{2}m\omega_\rho^2\left(\rho^2+\lambda^2z^2\right)$ is harmonic trap with anisotropy λ, and g_2, g_3 are parameters of two- and three-body interactions, respectively.

- By using the Gaussian variational ansatz [4], we obtain equations for condensate widths u_ρ and u_z in the dimensionless form:

$$\ddot{u}_\rho(t)+u_\rho(t)-\frac{1}{u_\rho(t)^3}-\frac{p}{u_\rho(t)^3u_z(t)}-\frac{k}{u_\rho(t)^5u_z(t)^2}=0,$$
$$\ddot{u}_z(t)+\lambda^2u_z(t)-\frac{1}{u_z(t)^3}-\frac{p}{u_\rho(t)^2u_z(t)^2}-\frac{k}{u_\rho(t)^4u_z(t)^3}=0\,.$$

Dimensionless parameters are $p=\frac{g_2N}{(2\pi)^{3/2}\hbar\omega_\rho l^3}=\sqrt{\frac{\pi}{2}}\frac{aN}{l}$, $k=\frac{32g_3\hbar\omega_\rho}{9\sqrt{3}g_2^2}p^2$, N is the number of particles, a is the s-wave scattering length, and $l=\sqrt{\hbar/m\omega_\rho}$ is the oscillator length.

- Initial state: $\mathbf{u}(0)=\mathbf{u}_0+\varepsilon\,\mathbf{u}_Q$, $\dot{\mathbf{u}}(0)=\mathbf{0}$
- Real-time dynamics for $p=1$, $k=0.001$, and $\varepsilon=0.1$



- Equilibrium positions:

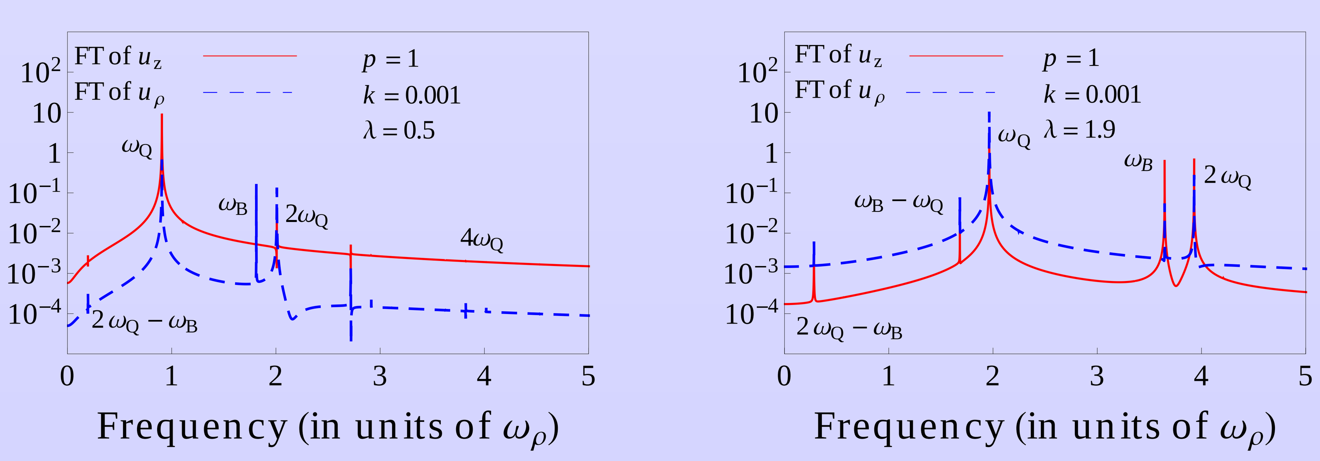
$$u_{\rho0}=\frac{1}{u_{\rho0}^3}+\frac{p}{u_{\rho0}^3u_{z0}}+\frac{k}{u_{\rho0}^5u_{z0}^2},\quad \lambda^2u_{z0}=\frac{1}{u_{z0}^3}+\frac{p}{u_{\rho0}^2u_{z0}^2}+\frac{k}{u_{\rho0}^4u_{z0}^3}$$

- Frequencies of collective modes:

$$\omega_{B,Q}^2=\frac{m_1+m_3\pm\sqrt{(m_1-m_3)^2+8m_2^2}}{2},$$

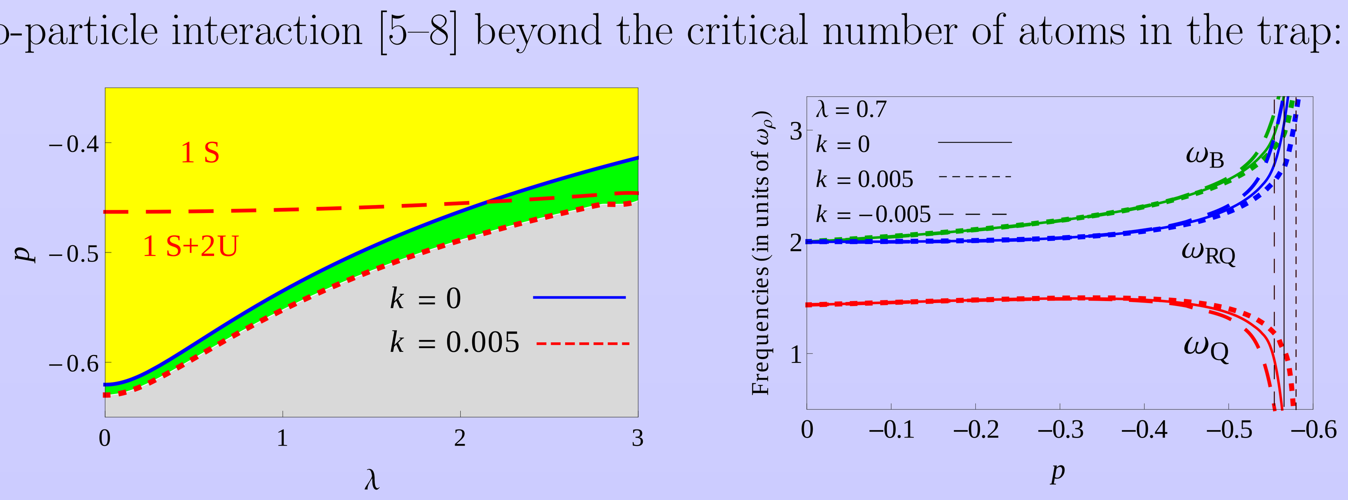
where $m_1=4+\frac{2k}{u_{\rho0}^6u_{z0}^2}$, $m_2=\frac{p}{u_{\rho0}^3u_{z0}^2}+\frac{2k}{u_{\rho0}^5u_{z0}^3}$, $m_3=4\lambda^2-\frac{p}{u_{\rho0}^2u_{z0}^3}$.

Excitation spectra



Stability diagram

- Repulsive three-body interaction extends the stability region of a BEC with attractive two-particle interaction [5–8] beyond the critical number of atoms in the trap:



Frequency shift of collective modes

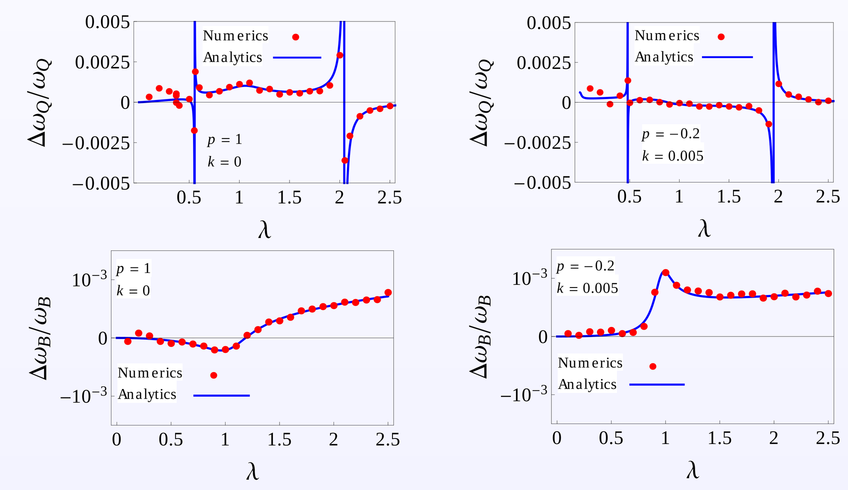
- We apply Poincaré-Lindstedt method [9,10] and perform perturbative expansion in ε:

$$u_\rho(t)=u_{\rho0}+\varepsilon u_{\rho1}(t)+\varepsilon^2u_{\rho2}(t)+\varepsilon^3u_{\rho3}(t)+\ldots..$$
$$u_z(t)=u_{z0}+\varepsilon u_{z1}(t)+\varepsilon^2u_{z2}(t)+\varepsilon^3u_{z3}(t)+\ldots..$$

- Quadrupole mode has a geometric resonance for $\omega_B=2\omega_Q$. This yields $\lambda_1=0.55$ and $\lambda_2=2.056$ for $p=1$, $k=0$ and $\varepsilon=0.1$.

$$\omega_Q=\omega_Q-\varepsilon^2\frac{f_Q(\omega_Q,\omega_B,u_{r0},u_{z0},p,k,\lambda)}{2\omega_Q(\omega_B-2\omega_Q)(\omega_B+2\omega_Q)}$$

$$\omega_B=\omega_B-\varepsilon^2\frac{f_B(\omega_Q,\omega_B,u_{r0},u_{z0},p,k,\lambda)}{2\omega_B(\omega_Q-2\omega_B)(\omega_Q+2\omega_B)}$$



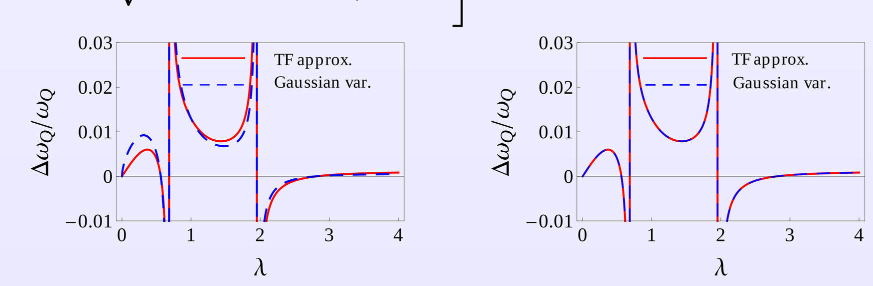
- Good agreement of numerical and analytical results for frequency shifts.

A comparison with Thomas-Fermi approximation

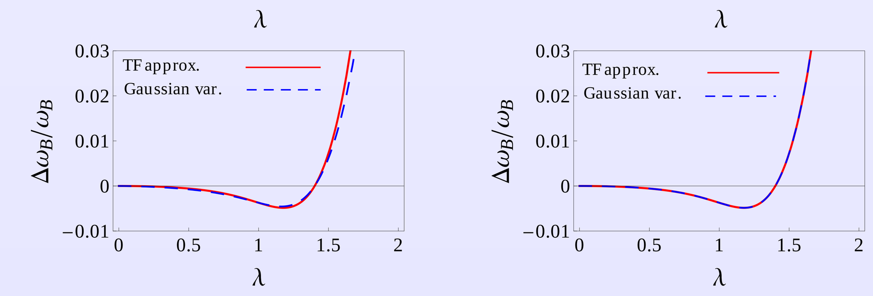
- The limit $p\to\infty$ corresponds to the TF limit [11].
- Frequencies of collective modes become [12]:

$$\omega_B,\omega_Q=\frac{1}{\sqrt{2}}\left[4+3\lambda^2\pm\sqrt{16-16\lambda^2+9\lambda^4}\right]^{1/2}$$

- Geometric resonances for quadrupole mode at $\lambda_{1,2}=(\sqrt{125}\pm\sqrt{29})/\sqrt{72}$, i.e., $\lambda_1\approx0.683$ and $\lambda_2\approx1.952$ [11].



- No resonances for breathing mode.

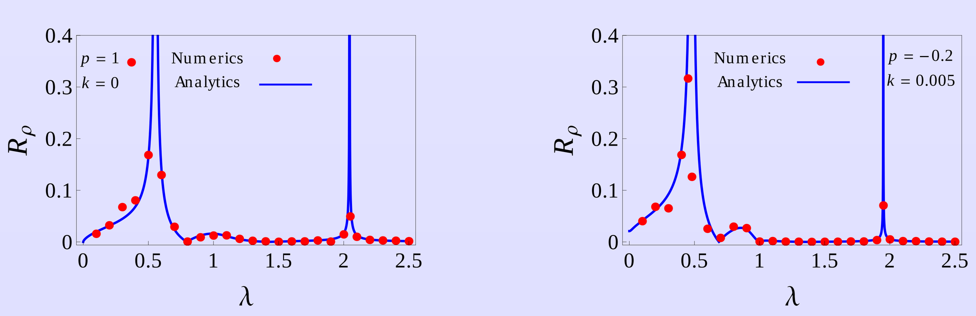


Mode coupling

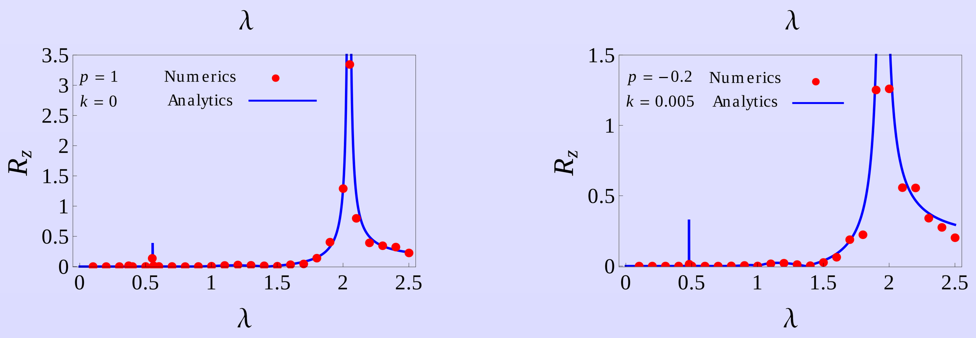
$$\mathbf{u}_2(t)=\begin{pmatrix}A_{\rho Q}\\A_{zQ}\end{pmatrix}\cos\omega_Qt+\begin{pmatrix}A_{\rho B}\\A_{zB}\end{pmatrix}\cos\omega_Bt+\ldots..$$

- Excited quadrupole mode

$$R_\rho=\frac{A_{\rho B}}{A_{\rho Q}}\propto\frac{\omega_B^2-2\omega_Q^2}{\omega_B^2-4\omega_Q^2}$$

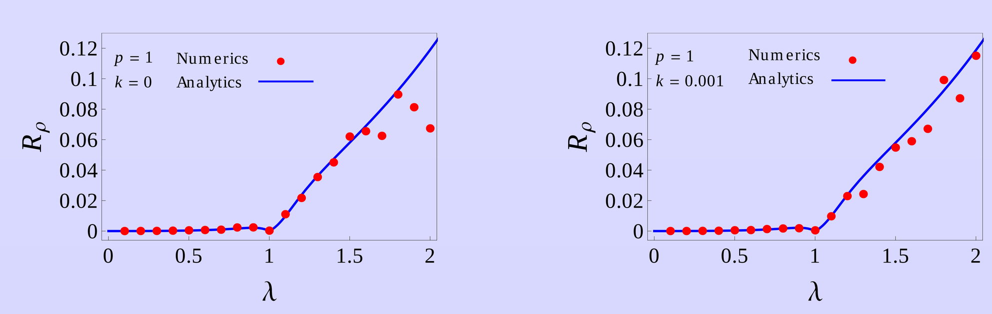


$$R_z=\frac{A_{zB}}{A_{zQ}}\propto\frac{\omega_B^2-2\omega_Q^2}{\omega_B^2-4\omega_Q^2}$$

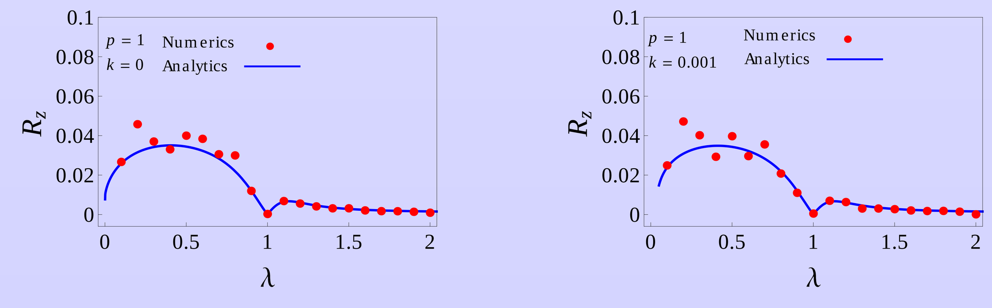


- Excited breathing mode

$$R_\rho=\frac{A_{\rho B}}{A_{\rho Q}}\propto\frac{2\omega_B^2-\omega_Q^2}{4\omega_B^2-\omega_Q^2}$$



$$R_z=\frac{A_{zB}}{A_{zQ}}\propto\frac{2\omega_B^2-\omega_Q^2}{4\omega_B^2-\omega_Q^2}$$



- b_5, b_6, c_1, c_2, $A_{\rho Q}$, and $A_{\rho B}$ are calculated using *Mathematica*.

Summary and outlook

- We have calculated frequency shifts of collective modes of an axially-symmetric BEC with two- and three-body contact interaction for varying trap aspect ratios using numerical Fourier analysis and analytical Poincaré-Lindstedt method.
- We have shown that the influence of a small repulsive three-body interaction extends the stability region of the condensate beyond the critical number of atoms in the trap.
- Due to the nonlinearity of GP equation, collective modes are coupled. Even when we excite only one mode, the others are necessarily excited in the second order of the perturbative expansion and appear in real-time dynamics of the condensate.

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