Stability Analysis for Bose-Einstein Condensates under Parametric Resonance

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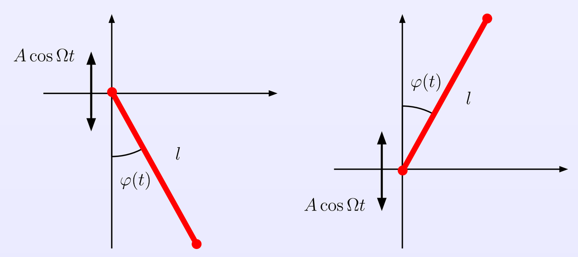
Motivation We conduct a stability analysis for Bose-Einstein condensates (BECs) in a harmonic trap under parametric excitation by periodic modulation of the s-wave scattering length [1, 2]. We are motivated by the classical system of a parametrically driven pendulum, wherein an originally stable equilibrium may be destabilized, and an unstable equilibrium made stable by parametric excitation. Following [3, 4], we obtain equations of motion for the radial and axial widths of the condensate using a Gaussian variational ansatz for the Gross-Pitaevskii condensate wave function. Linearizing about the equilibrium positions, we obtain a system of coupled Mathieu equations, the stability of which has been studied extensively [5-10]. We carry out an analytic stability analysis for the Mathieu equations, and compare with numerical results for the nonlinear equations of motion. We find qualitative agreement between the Mathieu analytics and nonlinear numerics, and conclude that the stability characteristics of two equilibrium radii of a BEC might be inverted by parametric excitation.

Pendulum Physics

- Linearized equation of motion

$$\ddot{\varphi}(t) + \left(\frac{g}{l} + \frac{A\Omega^2}{l}\cos\Omega t\right)\varphi(t) = 0,$$



- Mathieu equation [11, 12]

$$\ddot{x}(t') + \left[c - 2q\cos 2t'\right]x(t') = 0$$

$$c = \pm\frac{4g}{l\Omega^2}, \qquad q = \mp\frac{2A}{l}, \qquad 2t' = \Omega t, \qquad x(t') = \varphi(t)$$

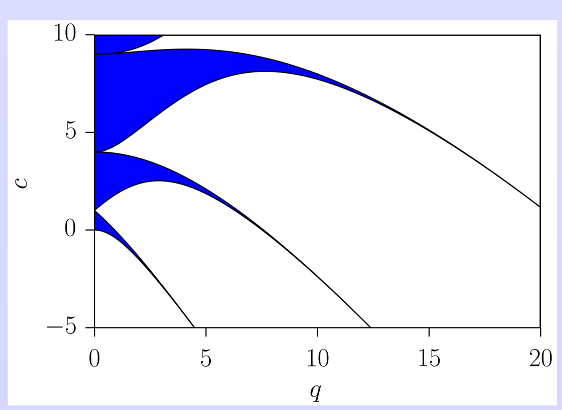
- One method: Fourier series ansatz [5]

$$x(t') = \sum_{n=0}^{\infty} A_n \cos(n\,t') + \sum_{n=1}^{\infty} B_n \sin(n\,t')$$

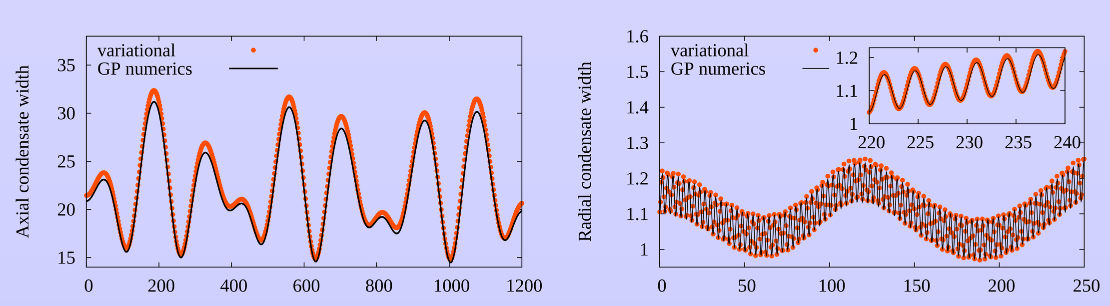
- Obtain decoupled systems of the form

$$\sum_{n=0}^{\infty} A_n\Big[(c-n^2)\cos(n\,t') - q\cos\big((n-1)\,t'\big) - q\cos\big((n+1)\,t'\big)\Big] = 0$$

- Infinite matrix equations: vanishing determinants for nontrivial A_n, B_n
- (q, c) for vanishing determinant gives stability borders (blue: stable)



- Analogous stability behaviour for BEC? [2]



- Modulation of scattering length by Feshbach resonance [1, 13]

$$a = a_{\rm av} + \delta_a\cos\Omega t$$

Variational Approach

- Lagrangian

$$L(t) = \int \mathcal{L}(\mathbf{r},t)\,d\mathbf{r}$$

- Lagrange density

$$\mathcal{L}(\mathbf{r},t) = \frac{i\hbar}{2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) + \frac{\hbar^2}{2m}|\nabla\psi|^2 - V(\mathbf{r})|\psi|^2 - \frac{2\pi\hbar^2 a(t)}{m}|\psi|^4$$

- Harmonic trapping potential $V(\mathbf{r}) = \frac{1}{2}m\omega_\rho^2(\rho^2 + \lambda^2 z^2)$
- Gaussian variational ansatz [3, 4]

$$\psi^{\rm G}(\rho,z,t) = \mathcal{N}(t)\exp\left[-\frac{1}{2}\left(\frac{\rho^2}{\tilde{u}_\rho^2} + \frac{z^2}{\tilde{u}_z^2}\right) + i\left(\rho^2\phi_\rho + z^2\phi_z\right)\right],$$

- Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}} - \frac{\partial L}{\partial q} = 0, \qquad q \in \{\tilde{u}_i, \phi_i\}$$

- Phases

$$\phi_\rho(t) = -\frac{m\dot{\tilde{u}}_\rho}{2\hbar\tilde{u}_\rho}, \qquad \phi_z(t) = -\frac{m\dot{\tilde{u}}_z}{2\hbar\tilde{u}_z}$$

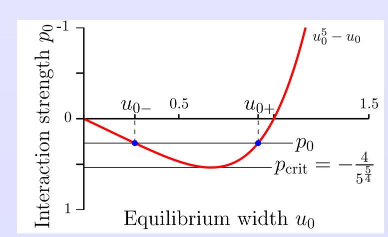
- Equations of motion for dimensionless widths

$$\ddot{u}_\rho + u_\rho = \frac{1}{u_\rho^3} + \frac{p_0 + p_1\cos\left(\frac{\Omega\tau}{\omega_\rho}\right)}{u_\rho^3\,u_z}, \qquad p_0 = \sqrt{\frac{2}{\pi}}\frac{N a_{\rm av}}{a_{\rm ho}},$$

$$\ddot{u}_z + \lambda^2 u_z = \frac{1}{u_z^3} + \frac{p_0 + p_1\cos\left(\frac{\Omega\tau}{\omega_\rho}\right)}{u_\rho^2\,u_z^2}, \qquad p_1 = \sqrt{\frac{2}{\pi}}\frac{N\delta_a}{a_{\rm ho}}$$

- Isotropic condensate: $u_\rho = u_z = u$, $\lambda = 1$ [2,15-17]

$$\ddot{u} + u = \frac{1}{u^3} + \frac{p_0 + p_1\cos\left(\frac{\Omega\tau}{\omega}\right)}{u^4}, \qquad u_0 = \frac{1}{u_0^3} + \frac{p_0}{u_0^4}.$$



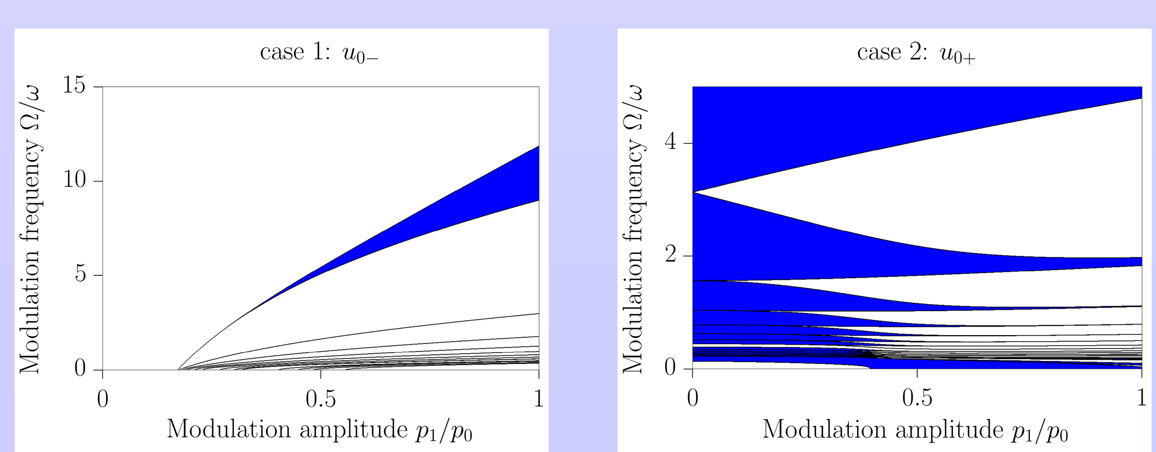
Isotropic Condensate

- Linearize about equilibrium position: $u(\tau) \approx u_0 + \delta u(\tau)$
- Taylor expand nonlinear terms to first order in δu
- Obtain an inhomogeneous Mathieu equation

$$\ddot{x}(t') + \left[c - 2\,q\cos(2\,t')\right]x(t') = -\frac{u_0}{2}\,q\cos(2\,t'),$$

$$q = -\frac{8\,p_1}{u_0^5}\left(\frac{\omega}{\Omega}\right)^2, \qquad 2t' = \frac{\Omega\,\tau}{\omega_\rho},$$

$$c = 4\left(\frac{\omega}{\Omega}\right)^2\left(5 - \frac{1}{u_0^4}\right), \qquad x(t') = \delta u(\tau).$$



Anisotropic Condensate

- Linearize: $u_i = u_{i0} + \delta u_i$, expand nonlinear terms to first order in δu
- Definitions:

$$2t' = \frac{\Omega\tau}{\omega_\rho}, \qquad q = p_1,$$

$$\mathbf{x}(t') = \begin{bmatrix} 2\delta u_r(t') \\ \delta u_z(t') \end{bmatrix}, \qquad \mathbf{A} = 4\left(\frac{\omega_\rho}{\Omega}\right)^2\begin{bmatrix} 8 & \frac{2p_0}{u_{\rho0}^3 u_{z0}^2} \\ \frac{2p_0}{u_{\rho0}^3 u_{z0}^2} & 3\,\lambda^2 + \frac{1}{u_{z0}^4} \end{bmatrix},$$

$$\mathbf{f} = 4\left(\frac{\omega_\rho}{\Omega}\right)^2\begin{bmatrix} \frac{2p_1}{u_{\rho0}^3 u_{z0}} \\ \frac{p_1}{u_{z0}^2 u_{\rho0}^2} \end{bmatrix}, \qquad \mathbf{Q} = -2\left(\frac{\omega_\rho}{\Omega}\right)^2\begin{bmatrix} \frac{6}{u_{\rho0}^4 u_{z0}} & \frac{2}{u_{\rho0}^3 u_{z0}^2} \\ \frac{2}{u_{\rho0}^3 u_{z0}^2} & \frac{2}{u_{z0}^3 u_{\rho0}^2} \end{bmatrix}$$

- Coupled, inhomogeneous Mathieu equations [7]:

$$\ddot{\mathbf{x}}(t') + \left[\mathbf{A} - 2q\,\mathbf{Q}\cos(2t')\right]\mathbf{x}(t') = \mathbf{f}\cos(2t')$$

- Non-homogeneity does not affect stability borders [5, 8]
- Floquet ansatz:

$$\mathbf{x}(t') = \sum_{n=-\infty}^{\infty}\mathbf{u}_{2n}e^{(\beta+2in)t'}$$

- Recursion relation [18]

$$\left[\mathbf{A} + (\beta + 2in)^2\,\mathbf{I}\right]\mathbf{u}_{2n} - q\,\mathbf{Q}(\mathbf{u}_{2n+2} + \mathbf{u}_{2n-2}) = \mathbf{0}$$

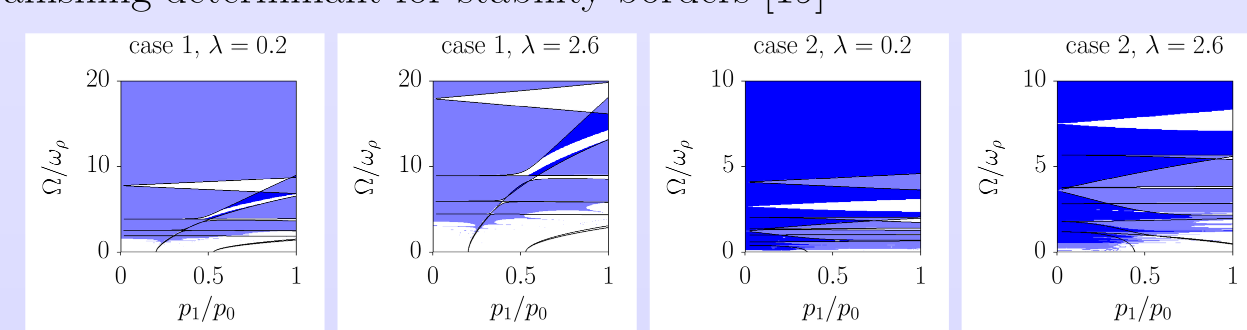
- Ladder operators

$$\mathbf{S}_{2n}^{\pm} = \left[\mathbf{A} + (\beta + 2in)^2\,\mathbf{I} - q\,\mathbf{Q}\,\mathbf{S}_{2n\pm2}^{\pm}\right]^{-1}q\,\mathbf{Q}$$

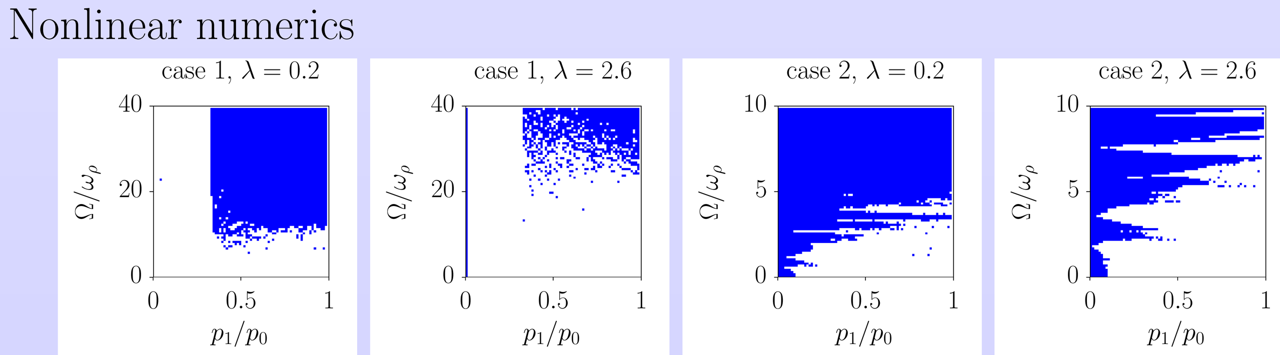
- Continued matrix inversion

$$\Big[\mathbf{A}+\beta^2\mathbf{I}-q^2\mathbf{Q}\Big(\big[\mathbf{A}+(\beta+2i)^2\mathbf{I}-\dots\big]^{-1}+\big[\mathbf{A}+(\beta-2i)^2\mathbf{I}-\dots\big]^{-1}\Big)\mathbf{Q}\Big]\mathbf{u}_0 = \mathbf{0}$$

- Vanishing determinant for stability borders [19]



- Nonlinear numerics



Conclusions and Outlook

- Analogous physics: pendulum and BEC
  - Stabilization of unstable equilibrium by parametric excitation
- Qualitative agreement between analytical Mathieu and numerical nonlinear analysis
- Nonlin. numerics suggest larger stability region for "cigar" trap ($\lambda < 1$)
  - Potential for experiment
- Dipolar BEC [20, 21]

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