Bogoliubov Theory for Dipolar Bose Gas at Low Temperatures

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Bogoliubov theory

- **Hamilton operator:**

$$\hat{H} = \sum_{\mathbf{k}} \left(\frac{\hbar^2 \mathbf{k}^2}{2m} + \hbar \mathbf{wk} - \mu \right) \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} + \frac{1}{2V} \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} V_{\mathbf{q}} \hat{b}^\dagger_{\mathbf{k}-\mathbf{q}} \hat{b}^\dagger_{\mathbf{k}'+\mathbf{q}} \hat{b}_{\mathbf{k}'} \hat{b}_{\mathbf{k}}$$

- **C-number approximation:** $N_0 \gg 1 \quad \hat{b}_0 \approx \hat{b}^\dagger_0 \approx \sqrt{N_0}$

$$\hat{H} = V n_0 \left(\frac{n_0}{2} V_0 - \mu \right) + \frac{1}{2} \sum_{\mathbf{k}\neq 0} \left[\frac{\hbar^2 \mathbf{k}^2}{2m} - \mu + n_0 (V_0 + V_{\mathbf{k}}) \right] \left(\hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} + \hat{b}^\dagger_{-\mathbf{k}} \hat{b}_{-\mathbf{k}} \right)$$
$$+ \frac{1}{2} \sum_{\mathbf{k}\neq 0} \hbar \mathbf{wk} \left(\hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} - \hat{b}^\dagger_{-\mathbf{k}} \hat{b}_{-\mathbf{k}} \right) + \frac{n_0}{2} \sum_{\mathbf{k}\neq 0} V_{\mathbf{k}} \left(\hat{b}_{\mathbf{k}} \hat{b}_{-\mathbf{k}} + \hat{b}^\dagger_{\mathbf{k}} \hat{b}^\dagger_{-\mathbf{k}} \right) + \dots$$

- **Bogoliubov transformation:** $\hat{B}_{\mathbf{k}} = u_{\mathbf{k}} \hat{b}_{\mathbf{k}} + v_{\mathbf{k}} \hat{b}^\dagger_{-\mathbf{k}}, \ \hat{B}^\dagger_{\mathbf{k}} = u^*_{\mathbf{k}} \hat{b}^\dagger_{\mathbf{k}} + v^*_{\mathbf{k}} \hat{b}_{-\mathbf{k}}$

$$u_{\mathbf{k}} = \frac{E_{\mathbf{k}} + \epsilon_{\mathbf{k}}}{2 E_{\mathbf{k}} \epsilon_{\mathbf{k}}}, \quad v_{\mathbf{k}} = \frac{E_{\mathbf{k}} - \epsilon_{\mathbf{k}}}{2 E_{\mathbf{k}} \epsilon_{\mathbf{k}}}, \quad E_{\mathbf{k}} = \sqrt{\epsilon^2_{\mathbf{k}} + 2 n_0 V_{\mathbf{k}} \epsilon_{\mathbf{k}}}, \quad \epsilon_{\mathbf{k}} = \frac{\hbar^2 \mathbf{k}^2}{2m} - \mu + n_0 V_0$$

- **Diagonalized Hamiltonian:**

$$\hat{H} = V n_0 \left(\frac{n_0}{2} V_0 - \mu \right) + \frac{1}{2} \sum_{\mathbf{k}\neq 0} (E_{\mathbf{k}} - \epsilon_{\mathbf{k}} - n_0 V_{\mathbf{k}}) + \sum_{\mathbf{k}\neq 0} (E_{\mathbf{k}} + \hbar \mathbf{wk}) \hat{B}^\dagger_{\mathbf{k}} \hat{B}_{\mathbf{k}} + \dots$$

- **Effective potential:**

$$V_{\text{eff}} = V n_0 \left(\frac{n_0}{2} V_0 - \mu \right) + \frac{1}{2} \sum_{\mathbf{k}\neq 0} (E_{\mathbf{k}} - \epsilon_{\mathbf{k}} - n_0 V_{\mathbf{k}}) + \frac{1}{\beta} \sum_{\mathbf{k}\neq 0} \ln \left(1 - e^{-\beta \tilde{E}_{\mathbf{k}}} \right) + \dots$$

Landau theory

- **Quantum and thermal fluctuations:** smallness parameter η

$$V_{\text{eff}}(n_0) = V \left(-\mu n_0 + \frac{n_0^2}{2} V_0 \right) + \frac{\eta}{2} \sum_{\mathbf{k}\neq 0} \left[E_{\mathbf{k}} - \left(\frac{\hbar^2 \mathbf{k}^2}{2m} - \mu + n_0 (V_0 + V_{\mathbf{k}}) \right) \right]$$
$$+ \frac{\eta}{\beta} \sum_{\mathbf{k}\neq 0} \ln \left(1 - e^{-\beta E_{\mathbf{k}}} \right) + \dots$$

- **Condensate density:** $\dfrac{\partial V_{\text{eff}}(n_0)}{\partial n_0} = 0 \quad \Longrightarrow \quad n_0 = n_0^{(0)} + \eta n_0^{(1)} + \dots$

- **Free energy:** $F(T,V,\mu) = V_{\text{eff}} \left(n_0^{(0)} + \eta n_0^{(1)} + \dots \right)$

$$F(T,V,\mu) = -\frac{V}{2} \frac{\mu^2}{V_0} + \frac{\eta}{2} \sum_{\mathbf{k}\neq 0} \left[\sqrt{ \left(\frac{\hbar^2 \mathbf{k}^2}{2m} \right)^2 + 2\mu \frac{V_{\mathbf{k}}}{V_0} \frac{\hbar^2 \mathbf{k}^2}{2m} } - \frac{\hbar^2 \mathbf{k}^2}{2m} - \mu \frac{V_{\mathbf{k}}}{V_0} \right]$$
$$+ \frac{\eta}{\beta} \sum_{\mathbf{k}\neq 0} \ln \left(1 - e^{-\beta \sqrt{ \left(\frac{\hbar^2 \mathbf{k}^2}{2m} \right)^2 + 2\mu \frac{V_{\mathbf{k}}}{V_0} \frac{\hbar^2 \mathbf{k}^2}{2m} }} \right) + \dots$$

- **Superfluid density:** $(n_s)_{ij} = n \delta_{ij} - (n_n)_{ij}$

$$p_i = -\frac{\partial F}{\partial w_i} = m V (n_n)_{ij} w_j + \dots, \qquad (n_n)_{ij} = \eta \frac{\hbar^2 \beta}{m V} \sum_{\mathbf{k}\neq 0} \frac{e^{\beta E_{\mathbf{k}}}}{(e^{\beta E_{\mathbf{k}}} - 1)^2} k_i k_j + \dots$$

Dipolar Bose gas

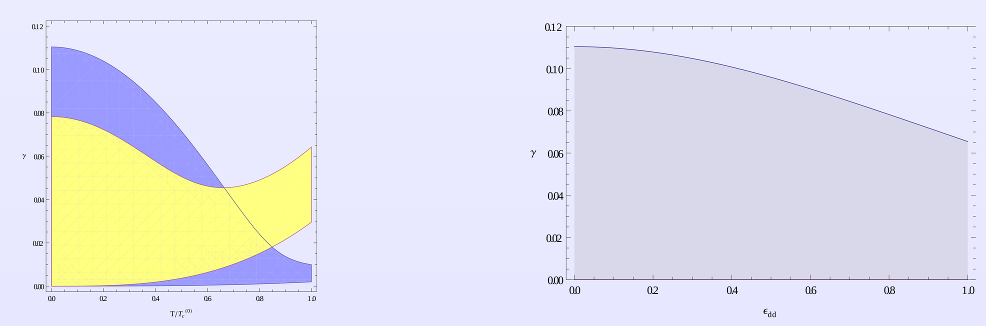
- **Interaction potential:** $C_{dd} = \mu_0 m^2$ (atomic origin) ,$C_{dd} = 4\pi d^2$ (molecular origin)

$$V(\mathbf{x}) = g\delta(\mathbf{x}) + \frac{C_{dd}}{4\pi |\mathbf{x}|^3} \left(1 - 3\cos^2\theta \right)$$
$$V_{\mathbf{k}} = g \left[1 + \epsilon_{dd} \left(3\cos^2\theta - 1 \right) \right], \quad \epsilon_{dd} = \frac{C_{dd}}{3g}$$

- **Condensate depletion:**

$$\Delta n_0 = \eta \frac{8}{3\sqrt{\pi}} (na)^{\frac{3}{2}} I\left(\epsilon_{dd}, \frac{3}{2} \right) + \eta \frac{\pi^{\frac{3}{2}}}{6 \lambda^4_{\text{dB}}} n^{-\frac{1}{2}} a^{-\frac{1}{2}} I\left(\epsilon_{dd}, -\frac{1}{2} \right)$$
$$- \eta \frac{\pi^{\frac{7}{2}}}{960 \lambda^8_{\text{dB}}} n^{-\frac{5}{2}} a^{-\frac{5}{2}} I\left(\epsilon_{dd}, -\frac{5}{2} \right) + \dots, \quad \lambda_{\text{dB}} = \sqrt{\frac{2\pi\hbar^2}{m k_B T}}$$
$$I(\epsilon_{dd}, \alpha) = \int_0^1 du \left[1 + \epsilon_{dd} \left(3u^2 - 1 \right) \right]^\alpha$$

- **Validity range:** blue: $\epsilon_{dd} = 0$, yellow: $\epsilon_{dd} = 0.8$



$$0 \leq \frac{\Delta n_0}{n} \leq \frac{1}{2} \quad \underset{\gamma = na^3}{\overset{T=0}{\Longrightarrow}} \quad 0 \leq \gamma(\epsilon_{dd}) \leq \left[\frac{3\sqrt{\pi}}{16 I\left(\epsilon_{dd}, \frac{3}{2} \right)} \right]^2$$

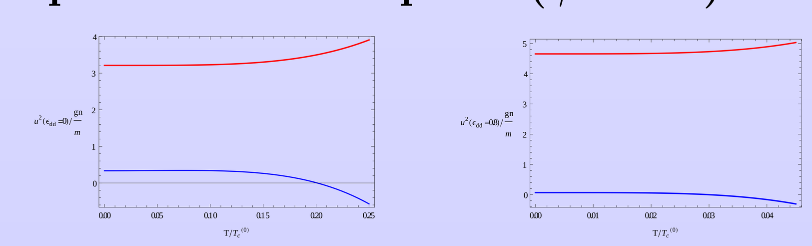
- **Superfluid density:** $(n_n)_{||} \leq (n_n)_\perp$

Sound velocities

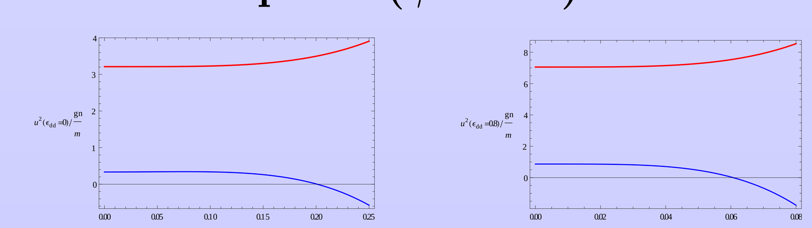
- **Anisotropic Landau-Khalatnikov two-fluid model [1]**
- **Long-wave length dispersion relations:**

$$\omega^2_{1/2} = (k\sin\vartheta \quad k\cos\vartheta) \begin{pmatrix} u^2_\perp & 0 \\ 0 & u^2_{||} \end{pmatrix} \begin{pmatrix} k\sin\vartheta \\ k\cos\vartheta \end{pmatrix}$$

- **Sound velocities perpendicular to dipoles** ($\gamma = 0.06$):



- **Sound velocities parallel to dipoles** ($\gamma = 0.06$):



first sound: pressure and density wave (red) , second sound: entropyand temperature wave (blue)

[1] C. Wille and A. Pelster, to be published