High-Order Variational Calculation for Periodic Orbits

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1. Motivation

- Model system: Duffing oscillator [1–3]

$$\ddot{x}(t) + \omega_0^2 x(t) + g x^3(t) = 0\,, \qquad x(0) = 1,\quad \dot{x}(0) = 0$$

- Time-periodic solution: $x(t) = x\left(t + \dfrac{2\pi}{\omega}\right)$

$$\omega = \frac{\pi\sqrt{\omega_0^2+g}}{2F\left(\dfrac{\pi}{2}, \sqrt{\dfrac{g}{2(\omega_0^2+g)}}\right)}\,, \qquad F(\varphi,k) = \int_0^\varphi \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}$$

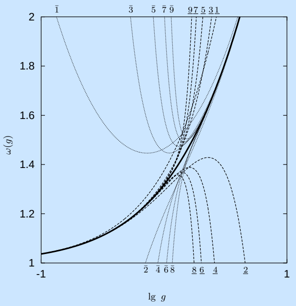
- Frequency:

  Divergent weak-coupling series ($g \to 0$, dashed) :

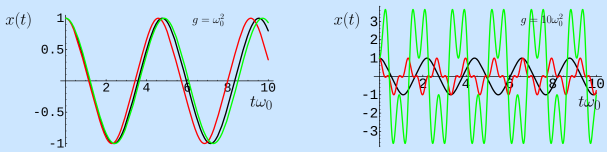
$$\omega = \omega_0 \sum_{n=0}^{\infty} w_n \left(\frac{g}{\omega_0^2}\right)^n$$

  Convergent strong-coupling series ($g \to \infty$, dotted) :

$$\omega = \sqrt{g}\sum_{m=0}^{\infty} b_m \left(\frac{\omega_0^2}{g}\right)^m$$



- Goal: Interpolation between weak- and strong-coupling series
- Time-periodic solution (exact, first and second perturbative order):



- Goal: Reconstruction of $x(t)$ from weak-coupling series for all times

2. Variational Perturbation Theory

- Weak-coupling series: $f(g) = \sum_{n=0}^{\infty} w_n\, g^n \;\; (g \to 0)$

  1) Introduction of variational parameters

  2) Optimization leads to resummation

- Strong-coupling series: $f(g) = g^{p/q}\sum_{m=0}^{\infty} b_m\, g^{-2m/q} \;\; (g\to\infty)$
- Various applications in different fields of theoretical physics [4–15]: quantum mechanics, quantum statistics, critical phenomena, soft matter, Markov processes, Bose-Einstein condensation, etc.

3. Frequency

- Weak-coupling series up to order N: $\omega^{(N)} = \sum_{n=0}^{N} w_n\, \omega_0^{1-2n}\, g^n$
- Introduce variational parameter via square-root substitution:

$$\omega_0 = \Omega\sqrt{1+gr}\,, \qquad r = \frac{1}{g}\left(\frac{\omega_0^2}{\Omega^2}-1\right)$$

- Reexpansion up to order N:

$$\omega^{(N)}(g,\Omega) = \sum_{n=0}^{N} w_n \Omega^{1-2n}\left[\sum_{k=0}^{N-n}\binom{1/2-n}{k}\left(\frac{\omega_0^2}{\Omega^2}-1\right)^k\right] g^n$$

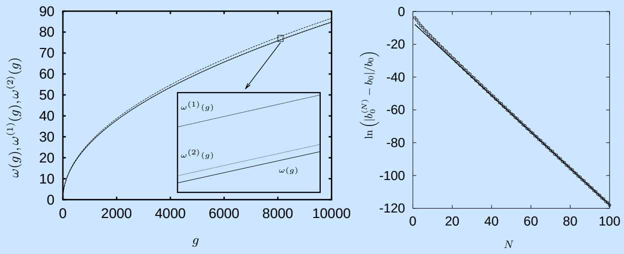
- Optimization according to the principle of minimal sensitivity [16]:

$$\left.\frac{\partial\omega^{(N)}(g,\Omega)}{\partial\Omega}\right|_{\Omega=\Omega^{(N)}(g)} = 0 \quad\text{or}\quad \left.\frac{\partial^2\omega^{(N)}(g,\Omega)}{\partial\Omega^2}\right|_{\Omega=\Omega^{(N)}(g)} = 0$$

- Resummed result: $\omega^{(N)}(g) = \omega^{(N)}(g, \Omega^{(N)}(g))$
- First order:

$$\omega^{(1)} = \omega_0 + \frac{3}{8\omega_0}g \quad\Longrightarrow\quad \omega^{(1)}(g) = \omega_0\sqrt{1+\frac{3g}{4\omega_0^2}}$$

- Exponential convergence [17]: $\ln\dfrac{|b_0^{(N)}-b_0^{(N)}|}{b_0} \approx -6.767 - 1.111\,N$



4. Time-Periodic Solution

- Weak-coupling series up to order N:

$$x^{(N)}(t,\omega^{(N)}) = \sum_{n=0}^{N} x_n(\omega^{(N)}t)\left(\frac{g}{\omega_0^2}\right)^n$$

- Square-root substitution and reexpansion:

$$x^{(N)}(t,\omega^{(N)},\Omega) = \sum_{n=0}^{N} x_n(\omega^{(N)}t)\left(\frac{g}{\Omega^2}\right)^n \times\left[\sum_{k=0}^{N-n}\binom{n+k-1}{k}\left(\frac{\omega_0^2}{\Omega^2}-1\right)^k\right]$$

- Variational approximation:

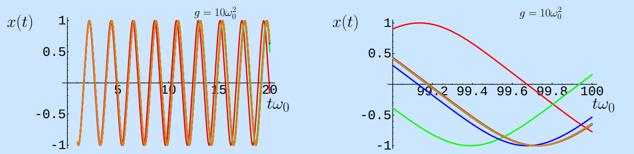
$$x^{(N)}(t,g) = x^{(N)}(t,\omega^{(N)}(g),\Omega^{(N)}(g))$$

- First Order:

$$x^{(1)}(t,\omega^{(1)}) = \cos\omega^{(1)}t + \left[-\cos\omega^{(1)}t + \cos 3\omega^{(1)}t\right]\frac{g}{32\omega_0^2}$$

$$x^{(1)}(t,g) = \cos\omega^{(1)}(g)t + \left[-\cos\omega^{(1)}(g)t + \cos 3\omega^{(1)}(g)t\right]\frac{g}{32\Omega^{(1)}(g)^2}$$

- Variational Result (exact, first, second, third, fourth, fifth order):



- Exponential convergence:

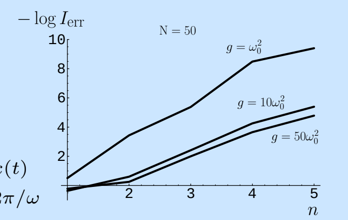
  Error measure:

$$I_{\rm err} = \frac{\int_0^{NT} dt|x_n(t)-x(t)|}{\int_0^{NT} dt|x(t)|}$$

  nth order VPT result: $x_n(t)$

  exact time-periodic solution: $x(t)$

  exact oscillation period: $T = 2\pi/\omega$



5. Outlook

- Goal: Nonlinear dynamical systems with time delay [18–20]
- Goal: Damped Duffing oscillator

  $\Longrightarrow$ Calculate frequency and damping constant
- Goal: Dissipative quantum dynamics [21–25]

  $\Longrightarrow$ Calculate population dynamics and transfer rates

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