Bogoliubov Theory of Dipolar Bose Gas in Weak Random Potential

Freie Universität Berlin

Technische Universität Kaiserslautern

Mahmoud Ghabour[1] and Axel Pelster[2]

[1]Institut für Theoretische Physik, Freie Universität Berlin, Germany
[2]Fachbereich Physik und Forschungszentrum OPTIMAS, Technische Universität Kaiserslautern, Germany

Abstract

We consider a dilute homogeneous Bose gas with both an isotropic short-range contact interaction and an anisotropic long-range dipole-dipole interaction in a weak random potential at low temperature in three dimensions. Within the realm of Bogoliubov theory we analyze how both condensate and superfluid density are depleted due to quantum and thermal fluctuations as well as disorder fluctuations. Afterwards, we calculate with this the resulting velocities of first and second sound within an anisotropic extension of the Landau-Khalatinikov two-fluid model.

Model

- **Grand-canonical Hamiltonian in momentum space**

$$\hat{\mathcal{K}} = \sum_{\mathbf{k}} \left(\frac{\hbar^2\mathbf{k}^2}{2m} - \mu\right) \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{1}{v}\sum_{\mathbf{p},\mathbf{k}} U_{\mathbf{p}-\mathbf{k}}\, \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{k}} + \frac{1}{2v}\sum_{\mathbf{p},\mathbf{k},\mathbf{q}} V_{\mathbf{q}}\, \hat{a}^\dagger_{\mathbf{k}+\mathbf{q}}\hat{a}^\dagger_{\mathbf{p}-\mathbf{q}}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{k}}$$

- **Interaction potential in momentum space** $V_{\mathbf{q}} = g + \frac{C_{\rm dd}}{3}\left(3\cos^2\theta - 1\right)$
- **Random potential**
  - Disorder Ensemble Average $\langle \bullet \rangle = \int \mathcal{D}U \bullet P[U]$, $\int \mathcal{D}U\, P[U] = 1$
  - Assumption $\langle U_{\mathbf{k}} \rangle = 0$, $\frac{1}{v}\langle U_{\mathbf{k}} U_{\mathbf{0}}\rangle = R_{\mathbf{k}}$

Bogoliubov theory

- **Bogoliubov prescription** [1–6] $N - N_0 \ll N$, $\hat{a}_{\mathbf{0}} \approx \hat{a}^\dagger_{\mathbf{0}} \approx \sqrt{N_0}$
- **Simplified Hamiltonian**

$$\hat{\mathcal{K}}' = \left(-\mu + \frac{1}{v}U_0\right) N_0 + \frac{1}{2v}V_0 N_0^2 + \frac{1}{2}\sum'_{\mathbf{k}} \left(\frac{\hbar^2\mathbf{k}^2}{2m} - \mu\right)\left(\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}\right) + \frac{1}{v}\sqrt{N_0}\sum'_{\mathbf{k}} U_{\mathbf{k},\mathbf{0}}\left(\hat{a}^\dagger_{\mathbf{k}} + \hat{a}_{-\mathbf{k}}\right)$$
$$+\frac{1}{2v}N_0\sum'_{\mathbf{k}}(V_0 + V_{\mathbf{k}})\left(\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}\right) + \frac{1}{2v}N_0\sum'_{\mathbf{k}} V_{\mathbf{k}}\left(\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}\right)$$

- **Diagonalization transformation** $\hat{a}_{\mathbf{k}} = u_{\mathbf{k}}\hat{\alpha}_{\mathbf{k}} - v_{\mathbf{k}}\hat{\alpha}^\dagger_{-\mathbf{k}} - z_{\mathbf{k}}$, $\hat{a}^\dagger_{\mathbf{k}} = u_{\mathbf{k}}\hat{\alpha}^\dagger_{\mathbf{k}} - v_{\mathbf{k}}\hat{\alpha}_{-\mathbf{k}} - z^*_{\mathbf{k}}$
- **Grand-canonical potential**

$$\mathcal{F} = v\left(-\mu n_0 + \frac{1}{2}V_0 n_0^2\right) + \frac{1}{2}\sum'_{\mathbf{k}}\left\{E_{\mathbf{k}} - \left[\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + n_0(V_0 + V_{\mathbf{k}})\right]\right\} + \sum'_{\mathbf{k}}\frac{1}{\beta}\ln\left(1 - e^{-\beta E_{\mathbf{k}}}\right)$$
$$-\sum'_{\mathbf{k}}\frac{n_0 R_{\mathbf{k}}}{E^2_{\mathbf{k}}}\left(\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + n_0 V_0\right)$$

Extremizing $\frac{\partial \mathcal{F}}{\partial n_0} = 0$ we find condensate density n_0, particle density follows from $n = -\frac{1}{v}\frac{\partial \mathcal{F}}{\partial \mu}$

- **Condensate depletion up to first order in respective fluctuations**

$$n - n_0 = \frac{1}{2v}\sum'_{\mathbf{k}}\left(\frac{\frac{\hbar^2\mathbf{k}^2}{2m} + nV_{\mathbf{k}}}{E_{\mathbf{k}}} - 1\right) + \frac{1}{v}\sum'_{\mathbf{k}}\frac{\frac{\hbar^2\mathbf{k}^2}{2m} + nV_{\mathbf{k}}}{E_{\mathbf{k}}}\frac{1}{e^{\beta E_{\mathbf{k}}} - 1} + \frac{1}{v}\sum'_{\mathbf{k}}\frac{nR_{\mathbf{k}}}{E^4_{\mathbf{k}}}\left(\frac{\hbar^2\mathbf{k}^2}{2m}\right)^2$$

- **Bogoliubov spectrum for collective excitations** $E_{\mathbf{k}} = \sqrt{\left(\frac{\hbar^2\mathbf{k}^2}{2m}\right)^2 + nV_{\mathbf{k}}\frac{\hbar^2\mathbf{k}^2}{m}}$

Superfluidity

- **Grand-canonical Hamiltonian with superfluid and normalfluid velocity** [7]

$$\hat{\mathcal{K}} = \frac{1}{2}\sum_{\mathbf{k}}\left[\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + \frac{1}{2}m\mathbf{v}_s^2 - m\mathbf{u}\mathbf{v}_s\right]\left(\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}\right) + \frac{1}{2}\sum_{\mathbf{k}}\hbar\mathbf{k}(\mathbf{u} - \mathbf{v}_s)\left(\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}\right)$$
$$+\frac{1}{2v}\sum_{\mathbf{p},\mathbf{k}} U_{\mathbf{p}-\mathbf{k}}\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{p}}\right) + \frac{1}{2v}\sum_{\mathbf{p},\mathbf{k},\mathbf{q}} V_{\mathbf{q}}\,\hat{a}^\dagger_{\mathbf{k}+\mathbf{q}}\hat{a}^\dagger_{\mathbf{p}-\mathbf{q}}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{k}}$$

- **Grand-canonical potential**

$$\mathcal{F} = v\left[\frac{1}{2}V_0 n_0^2 - n_0\left(\mu - \frac{1}{2}m\mathbf{v}_s^2 + m\mathbf{u}\mathbf{v}_s\right)\right] + \frac{1}{2}\sum'_{\mathbf{k}}\left\{E_{\mathbf{k}} - \left[\frac{\hbar^2\mathbf{k}^2}{2m} + n_0 V_{\mathbf{k}}\right]\right\} + \sum'_{\mathbf{k}}\frac{1}{\beta}\ln\left(1 - e^{-\beta E_{\mathbf{k}}}\right)$$
$$-\sum'_{\mathbf{k}}\frac{\beta e^{\beta E_{\mathbf{k}}}[\hbar\mathbf{k}(\mathbf{u} - \mathbf{v}_s)]^2}{2(e^{\beta E_{\mathbf{k}}} - 1)^2} - \sum'_{\mathbf{k}}\frac{n_0 R_{\mathbf{k}}}{E^2_{\mathbf{k}}}\frac{\hbar^2\mathbf{k}^2}{2m} - \sum'_{\mathbf{k}}\frac{n_0 R_{\mathbf{k}}}{E^4_{\mathbf{k}}}\frac{\hbar^2\mathbf{k}^2}{2m}\left[\hbar\mathbf{k}(\mathbf{u} - \mathbf{v}_s)\right]^2$$

- **Momentum in small-velocity limit**

$$\mathbf{P} = \left(-\frac{\partial \mathcal{F}}{\partial \mathbf{u}}\right)_{v,T,\mu} = mv\left(n\mathbf{v}_s + n_{\rm n}\mathbf{v}_{\rm n}\right) + \ldots, \quad n_{\rm n} = n_{\rm R} + n_{\rm th}, \quad \mathbf{v}_{\rm n} = \mathbf{u} - \mathbf{v}_{\rm s}$$

- **Superfluid depletion up to first order in respective fluctuations**

$$(n - n_{\rm s})_{ij} = \frac{1}{v}\sum'_{\mathbf{k}}\frac{2nR_{\mathbf{k}}\hbar^2 k_i k_j}{m\left(\frac{\hbar^2\mathbf{k}^2}{2m}\right)\left(\frac{\hbar^2\mathbf{k}^2}{2m} + 2nV_{\mathbf{k}}\right)^2} + \frac{1}{v}\sum'_{\mathbf{k}}\frac{\beta}{m}\hbar^2 k_i k_j\frac{e^{\beta E_{\mathbf{k}}}}{\left(e^{\beta E_{\mathbf{k}}} - 1\right)^2}$$

Zero-temperature limit for delta-correlated random potential

- **Zero-temperature condensate depletion**

$$n - n_0 = \frac{8}{3\sqrt{\pi}}(na)^{\frac{3}{2}}Q_{\frac{3}{2}}(\epsilon_{\rm dd}) + \frac{m^2 R_0}{8\hbar^4\pi^{\frac{3}{2}}}\sqrt{\frac{n}{a}}Q_{-\frac{1}{2}}(\epsilon_{\rm dd}); \quad \epsilon_{\rm dd} = C_{\rm dd}/3g$$

- **Zero-temperature superfluid depletion**

$$n_{\rm R\parallel} = 4\frac{m^2 R_0}{8\hbar^4\pi^{\frac{3}{2}}}\sqrt{\frac{n}{a}}J_{-\frac{1}{2}}(\epsilon_{\rm dd}), \quad n_{\rm R\perp} = 2\frac{m^2 R_0}{8\hbar^4\pi^{\frac{3}{2}}}\sqrt{\frac{n}{a}}\left[Q_{-\frac{1}{2}}(\epsilon_{\rm dd}) - J_{-\frac{1}{2}}(\epsilon_{\rm dd})\right]$$

- **Functions describing dipolar effect in terms of hypergeometric function**

$$Q_\alpha(\epsilon_{\rm dd}) = (1 - \epsilon_{\rm dd})^\alpha\,{}_2F_1\left(-\alpha, \frac{1}{2}; \frac{3}{2}; \frac{-3\epsilon_{\rm dd}}{1 - \epsilon_{\rm dd}}\right), \quad J_\alpha(\epsilon_{\rm dd}) = \frac{1}{3}(1 - \epsilon_{\rm dd})^\alpha\,{}_2F_1\left(-\alpha, \frac{3}{2}; \frac{5}{2}; \frac{-3\epsilon_{\rm dd}}{1 - \epsilon_{\rm dd}}\right)$$

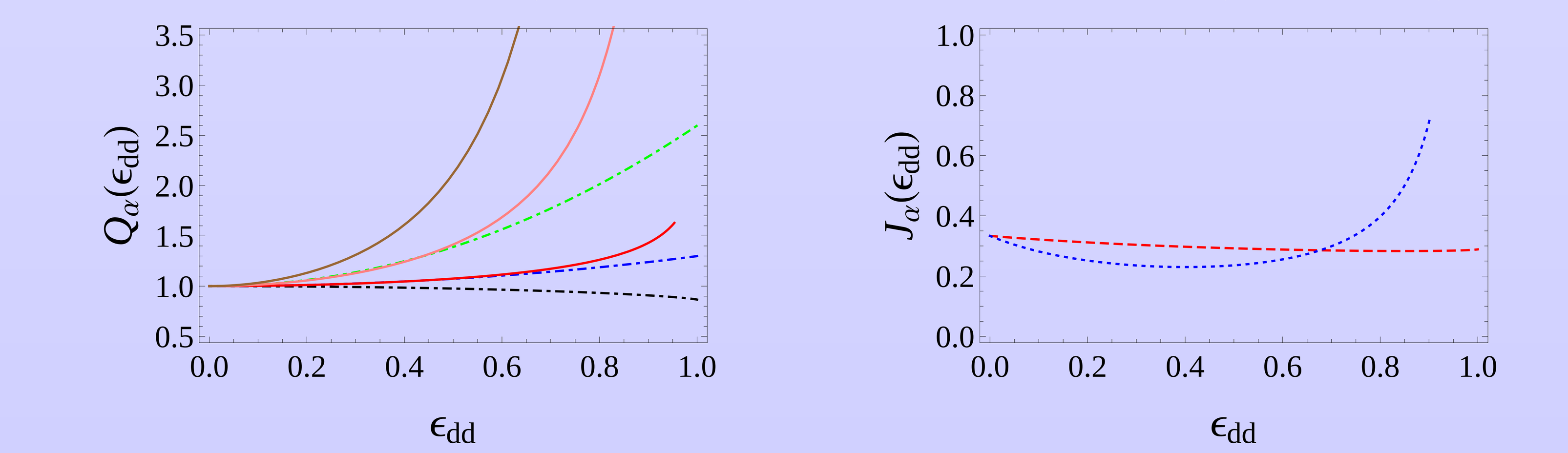

Fig. 1: Dipolar enhancement function $Q_\alpha(\epsilon_{\rm dd})$ (left) versus relative dipolar interaction strength $\epsilon_{\rm dd}$ for different values of α: -5/2 (brown, upper solid), -3/2 (pink, middle solid), -1/2 (red, lower solid), 1/2 (black, lower dotted-dashed), 3/2 (blue, middle dotted-dashed), 5/2 (green, upper dotted-dashed). Function $J_\alpha(\epsilon_{\rm dd})$ (right) versus relative dipolar interaction strength $\epsilon_{\rm dd}$ for different values of α: -1/2 (red, dashed), -5/2 (blue, dotted).

Finite-temperature effects

- **Thermal condensate depletion**

$$\frac{n_{\rm th}}{n} = \frac{\pi^{\frac{3}{2}}\gamma^{-\frac{1}{6}}t^2}{6(\zeta(\frac{3}{2}))^{\frac{4}{3}}}Q_{-\frac{1}{2}}(\epsilon_{\rm dd}) - \frac{\pi^{\frac{7}{2}}\gamma^{-\frac{5}{6}}t^4}{480(\zeta(\frac{3}{2}))^{\frac{8}{3}}}Q_{-\frac{5}{2}}(\epsilon_{\rm dd}) + \ldots; \quad \gamma = na^3, \ t = \frac{T}{T_c^0}, \ T_c^0 = \frac{2\pi\hbar^2 n^{\frac{2}{3}}}{(\zeta(\frac{3}{2}))^{\frac{2}{3}} m k_{\rm B}}$$

- **Thermal superfluid depletion**

$$\frac{n_{\rm th\parallel}}{n} = \frac{\pi^{\frac{7}{2}}\gamma^{-\frac{5}{6}}t^4}{15(\zeta(\frac{3}{2}))^{\frac{8}{3}}}J_{-\frac{5}{2}}(\epsilon_{\rm dd}) + \ldots, \quad \frac{n_{\rm th\perp}}{n} = \frac{\pi^{\frac{7}{2}}\gamma^{-\frac{5}{6}}t^4}{30(\zeta(\frac{3}{2}))^{\frac{8}{3}}}\left[Q_{-\frac{5}{2}}(\epsilon_{\rm dd}) - J_{-\frac{5}{2}}(\epsilon_{\rm dd})\right] + \ldots$$

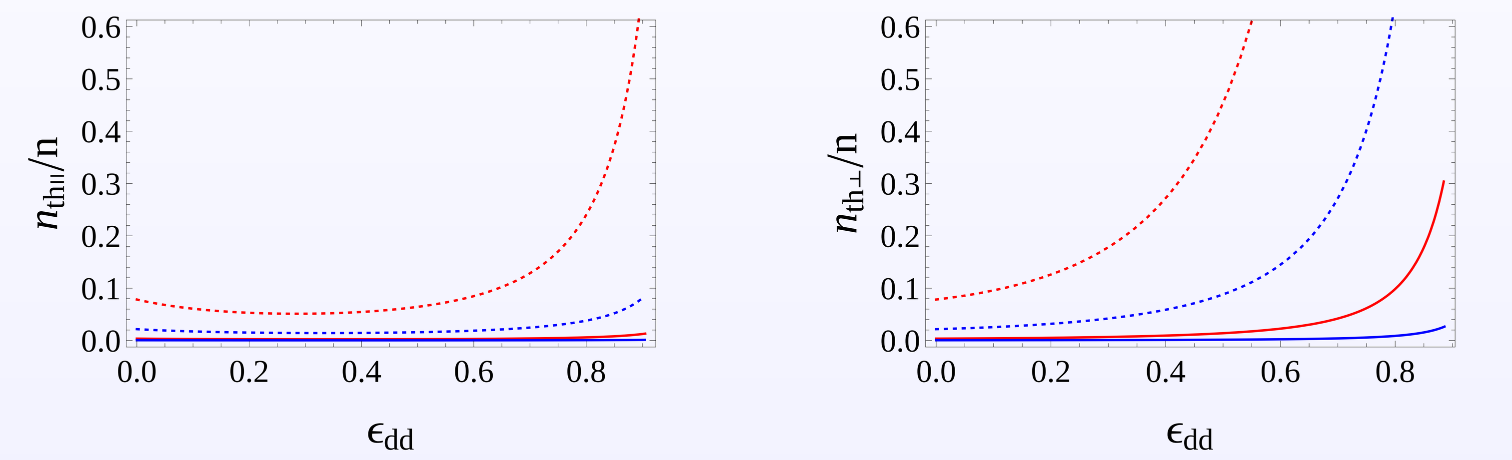

Fig. 2: Thermal superfluid fractional depletions $n_{\rm th\parallel}/n$ and $n_{\rm th\perp}/n$ versus relative dipolar interaction strength $\epsilon_{\rm dd}$ for different values of relative temperature $t = 0.2$ (solid), $t = 0.6$ (dotted) and gas parameter $\gamma = 0.01$ (red), $\gamma = 0.20$ (blue).

Validity range of Bogoliubov theory for $\frac{n - n_0}{n} \leq \frac{1}{2}$

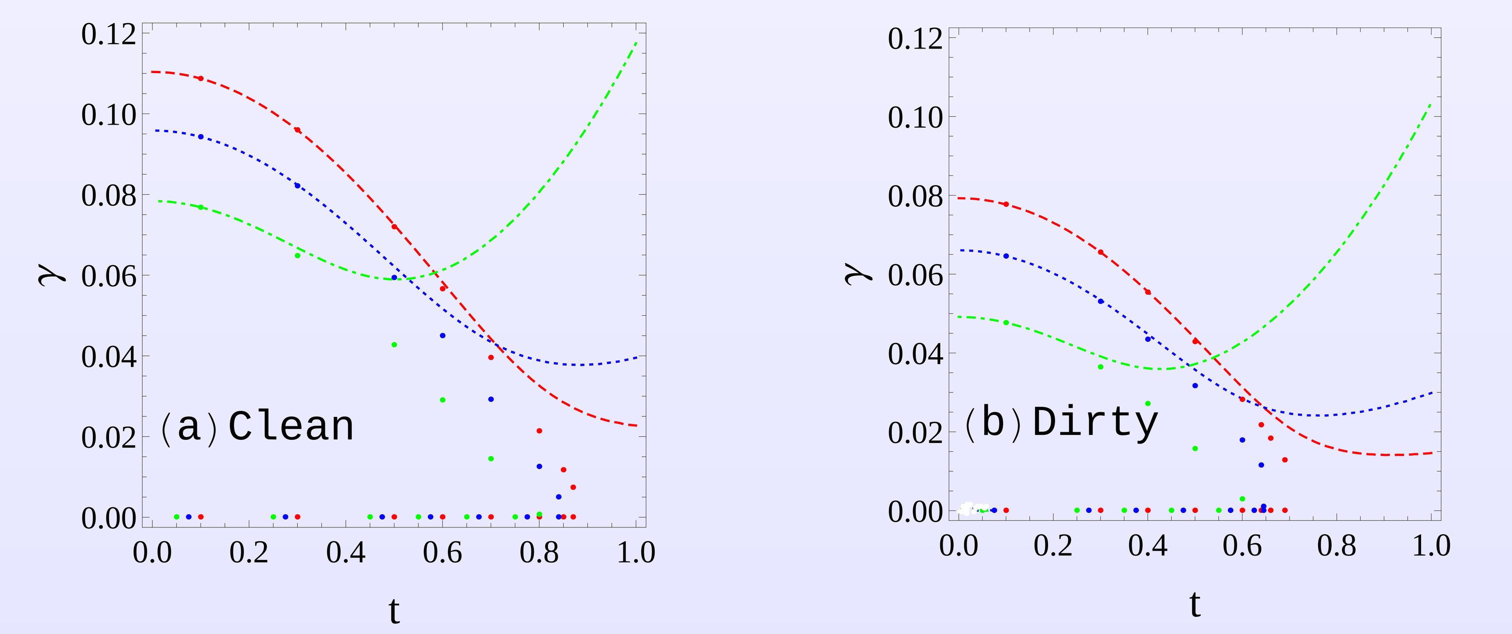

Fig. 3: Validity range of Bogoliubov theory in the $t - \gamma$ plane for (a) clean case and (b) dirty case with $R_0 = \frac{2\hbar^4\pi^{\frac{3}{2}}n^{\frac{1}{3}}}{5m^2}$ for different values of relative dipolar interaction strength $\epsilon_{\rm dd} = 0$ (red, dashed), $\epsilon_{\rm dd} = 0.5$ (blue, dotted), $\epsilon_{\rm dd} = 0.8$ (green, dotted-dashed).

Anisotropic two-fluid model

Sound wave velocities in the anisotropic superfluid Bose gas [8–10]:

- **Sound velocity** $c_\pm$

$$c_\pm^2 = \frac{1}{2}\left(c_1^2 + c_2^2 + c_3^2\right) \pm \left[\frac{1}{4}\left(c_1^2 + c_2^2 + c_3^2\right)^2 - c_1^2 c_2^2\right]^{\frac{1}{2}}$$

Zero-temperature case: $c_3 = 0$, thus $c_+ = c_1$, $c_- = c_2$

- **First sound wave** c_1 $\quad \frac{c_1^2}{c_s^2} = 1 + \epsilon_{\rm dd}(3\cos^2\theta - 1) + 16\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}}Q_{\frac{5}{2}}(\epsilon_{\rm dd}) + \frac{3m^2 R_0}{8\hbar^4\pi^{\frac{3}{2}}n^{\frac{1}{3}}}\gamma^{-\frac{1}{6}}Q_{\frac{1}{2}}(\epsilon_{\rm dd})$
- **Second sound wave** c_2 $\quad \frac{c^2_{2\parallel}}{c_s^2} = \frac{Q_{-\frac{3}{2}}(\epsilon_{\rm dd})}{9J_{-\frac{5}{2}}(\epsilon_{\rm dd})}, \quad \frac{c^2_{2\perp}}{c_s^2} = \frac{2Q_{-\frac{3}{2}}(\epsilon_{\rm dd})}{9\left[Q_{-\frac{5}{2}}(\epsilon_{\rm dd}) - J_{-\frac{5}{2}}(\epsilon_{\rm dd})\right]}$

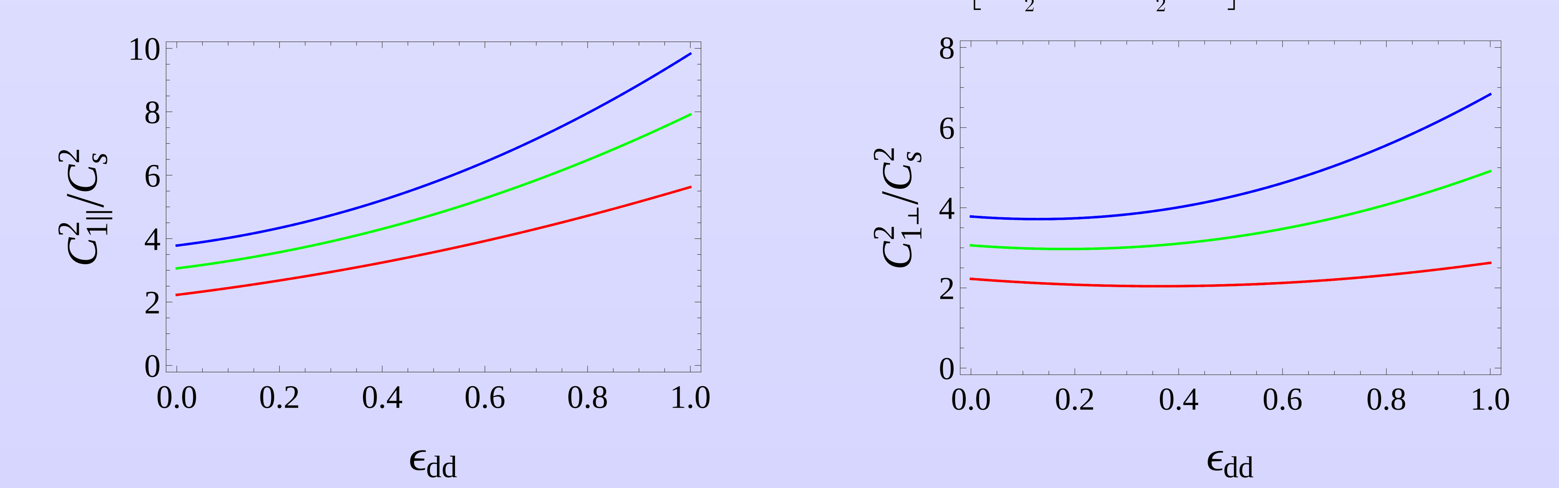

Fig. 4: First sound velocity $c^2_{1\parallel}/c^2_s$ and $c^2_{1\perp}/c^2_s$ versus relative dipolar strength $\epsilon_{\rm dd}$ in the dirty case. Drawn for different values of gas parameter $\gamma = 0.01$ (red), $\gamma = 0.04$ (green), $\gamma = 0.08$ (blue) at zero-temperature and disorder strength $R_0 = \frac{2\hbar^4\pi^{\frac{3}{2}}n^{\frac{1}{3}}}{5m^2}$.

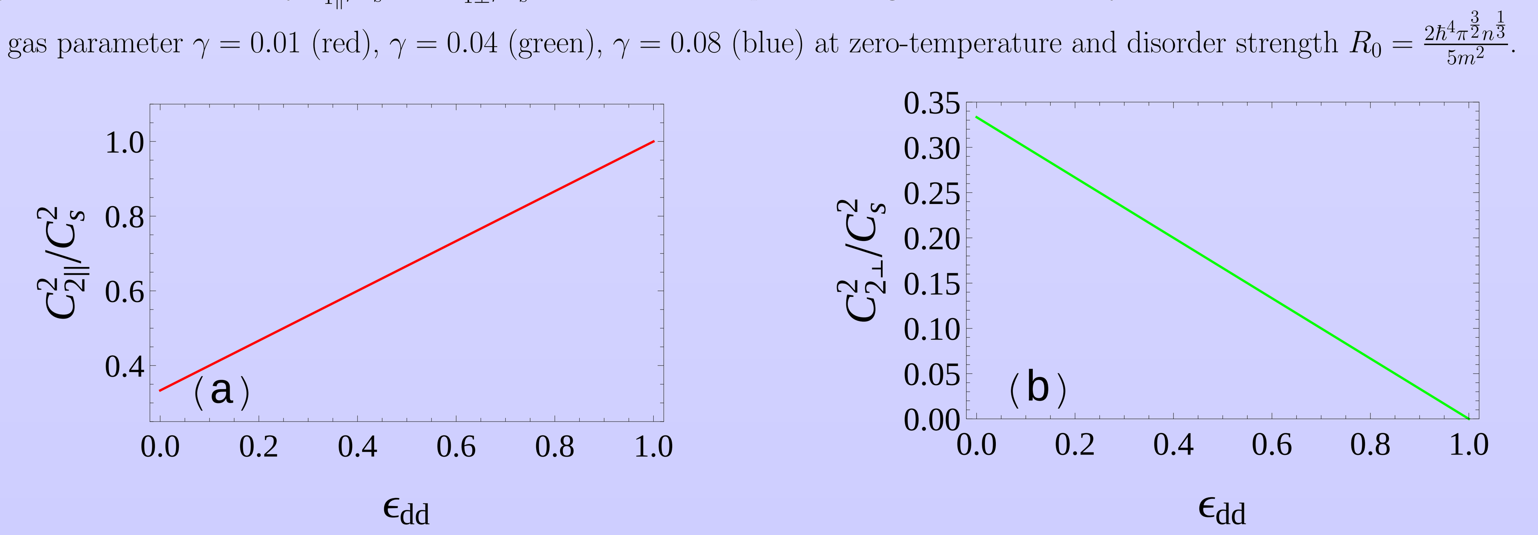

Fig. 5: Second sound velocity (a) $c^2_{2\parallel}/c^2_s$ and (b) $c^2_{2\perp}/c^2_s$ versus relative dipolar strength $\epsilon_{\rm dd}$ at zero-temperature limit.

Bogoliubov Theory of Dipolar Bose Gas in Weak Random Potential

Mahmoud Ghabour[1] and Axel Pelster[2]

[1]Institut für Theoretische Physik, Freie Universität Berlin, Germany
[2]Fachbereich Physik und Forschungszentrum OPTIMAS, Technische Universität Kaiserslautern, Germany

References

[1] N. N. Bogoliubov. J. Phys.(USSR) 11, **4** (1947).

[2] K. Huang and H. F. Meng, *Hard-sphere Bose gas in random external potentials*, Phys. Rev. Lett. **69**, 644 (1992).

[3] G. M. Falco, A. Pelster, and R. Graham, *Thermodynamics of a Bose-Einstein condensate with weak disorder*, Phys. Rev. A **75**, 063619 (2007).

[4] C. Krumnow and A. Pelster, *Dipolar Bose-Einstein condensates with weak disorder*, Phys. Rev. A **84**, 021608(R) (2011)

[5] B. Nikolic, A. Balaz, and A. Pelster, *Dipolar Bose-Einstein condensates in weak anisotropic disorder*, Phys. Rev. A **88**, 013624 (2013)

[6] M. Ghabour and A. Pelster, *Bogoliubov Theory of Dipolar Bose Gas in Weak Random Potential*, arXiv:1410.3070

[7] M. Ueda, *Fundamentals and New Frontiers of Bose-Einstein Condensation* (World Scientific, Singapore, 2010).

[8] L. Landau. Phys. Rev. **60**, 356 (1941).

[9] I. Khalatnikov, *An Introduction To The Theory Of Superfluidity* (Perseus, New York, 2000)

[10] D. Vollhardt and P. Wölfle, *The Superfluid Phases of Helium 3* (Dover, New York, 2013).

M. Ghabour thanks his family for financial support during this work. A. Pelster thanks for the support from the German Research Foundation (DFG) via the Collaborative Research Center SFB /TR49 Condensed Matter Systems with Variable Many-Body Interactions.