# Thermodynamical Properties of Quantum Gases

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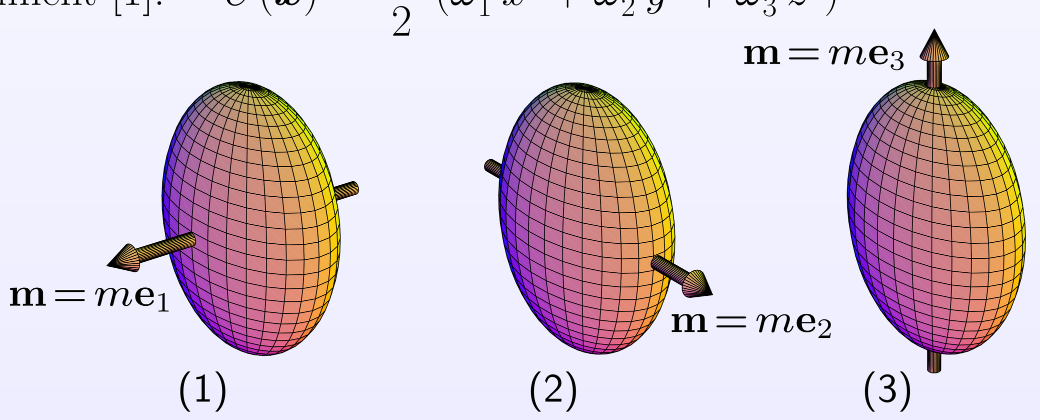
## 1) Chromium Bose-Einstein Condensate

- Trapped interacting Bose gas:

$$\mathcal{A}[\psi^*,\psi] = \int_0^{\hbar\beta} d\tau \int d^3x \left\{ \psi^*(\boldsymbol{x},\tau)\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\boldsymbol{\nabla}^2}{2M} - \mu\right)\psi(\boldsymbol{x},\tau) + U(\boldsymbol{x})\left|\psi(\boldsymbol{x},\tau)\right|^2 + \int d^3x' \left|\psi(\boldsymbol{x}',\tau)\right|^2 V^{(\mathrm{int})}(\boldsymbol{x}-\boldsymbol{x}')\left|\psi(\boldsymbol{x},\tau)\right|^2 \right\}$$

- Stuttgart experiment [1]: $U(\boldsymbol{x}) = \frac{M}{2}\left(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2\right)$

$\omega_1 = 2\pi \cdot 581$ Hz
$\omega_2 = 2\pi \cdot 406$ Hz
$\omega_3 = 2\pi \cdot 138$ Hz




- Interaction potential: $m = 6\, m_B$, $a = 105\, a_B$ [2]

$$V^{(\mathrm{int})}(\boldsymbol{x}) = \frac{4\pi\hbar^2 a}{M}\,\delta(\boldsymbol{x}) + \frac{\mu_0}{4\pi}\left\{\frac{\mathbf{m}^2}{|\boldsymbol{x}|^3} - \frac{3\,[\mathbf{m}\,\boldsymbol{x}]^2}{|\boldsymbol{x}|^5}\right\}$$

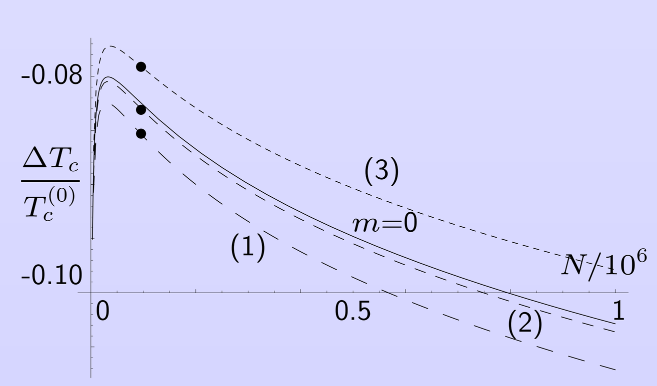
- Critical temperature: sc [3–6] + $\delta$ [7,8] + dd [9,10]

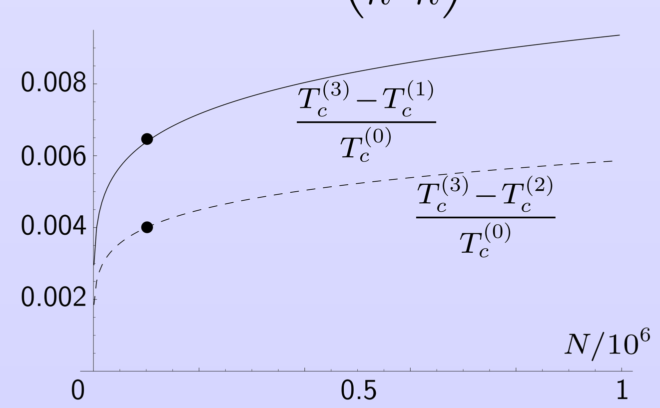
$$\frac{\Delta T_c^{(j)}}{T_c^{(0)}} = -0.73\,\frac{\omega_1+\omega_2+\omega_3}{3\,(\omega_1\omega_2\omega_3)^{1/3}}\,\frac{1}{N^{1/3}} - 3.43\left[\frac{a}{\lambda_c^{(0)}} + \frac{1}{2}f^{(j)}\left(\frac{\omega_1}{\omega_2},\frac{\omega_1}{\omega_3}\right)\frac{\mu_0 m^2 M}{12\pi\hbar^2\lambda_c^{(0)}}\right]$$

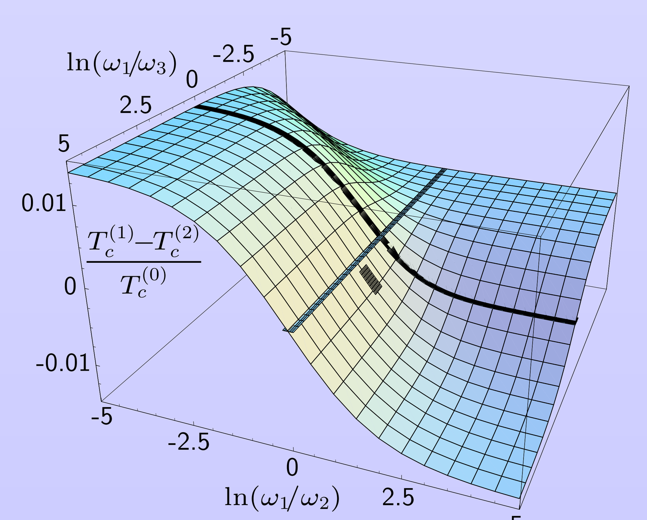
- Geometry factor [11,12]:

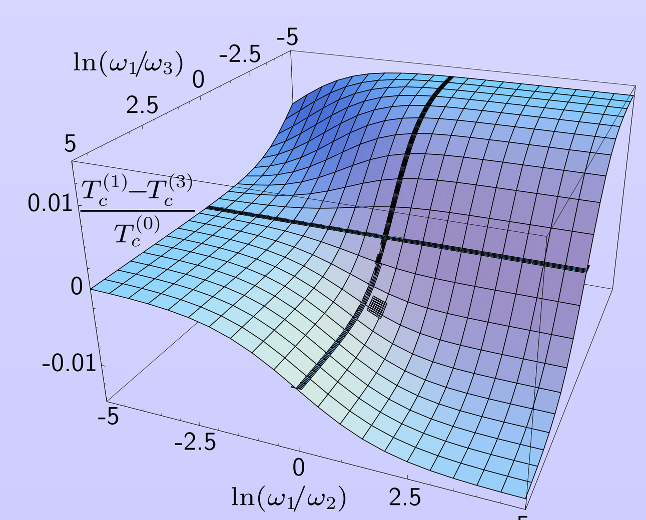
$$f^{(1)}(\eta,\kappa) = 1 + \frac{3\,\kappa\,\eta}{\sqrt{1-\kappa^2}\,(1-\eta^2)}\left\{E\left(\arcsin\sqrt{1-\kappa^2},\sqrt{\frac{1-\eta^2}{1-\kappa^2}}\right) - F\left(\arcsin\sqrt{1-\kappa^2},\sqrt{\frac{1-\eta^2}{1-\kappa^2}}\right)\right\}$$

$$f^{(2)}(\eta,\kappa) = f^{(1)}\left(\frac{\kappa}{\eta},\frac{1}{\eta}\right) \quad , \quad f^{(3)}(\eta,\kappa) = f^{(1)}\left(\frac{1}{\kappa},\frac{\eta}{\kappa}\right)$$




## 2) Canonical Approach to BEC

- $N$-particle partition function [13,14]:

$$Z_N^B(\beta) = \frac{1}{N!}\sum_P \prod_{n=1}^N \left(\int d^3x_n \int_{\boldsymbol{x}_n(0)=\boldsymbol{x}_n}^{\boldsymbol{x}_n(\hbar\beta)=\boldsymbol{x}_{P(n)}} \mathcal{D}^3 x_n\right) e^{-\mathcal{A}[\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N]/\hbar}$$

- Action:

$$\mathcal{A}[\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N] = \sum_{n=1}^N \int_0^{\hbar\beta} d\tau \left[\frac{M}{2}\dot{\boldsymbol{x}}_n^2 + U(\boldsymbol{x}_n) + \sum_{m=1}^{N}{}' V^{(\mathrm{int})}(\boldsymbol{x}_n-\boldsymbol{x}_m)\right]$$

- Recursion for interaction-free partition function [15–18]:

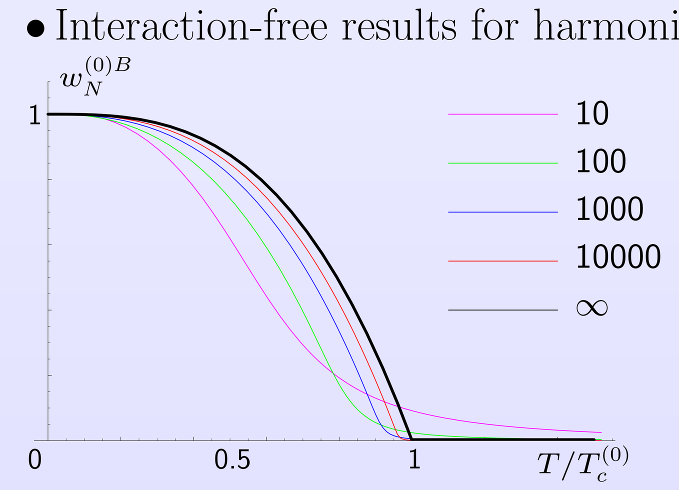
$$Z_N^{(0)B}(\beta) = \frac{1}{N}\sum_{n=1}^N Z_1(n\beta)\, Z_{N-n}^{(0)B}(\beta) \quad \text{with}$$

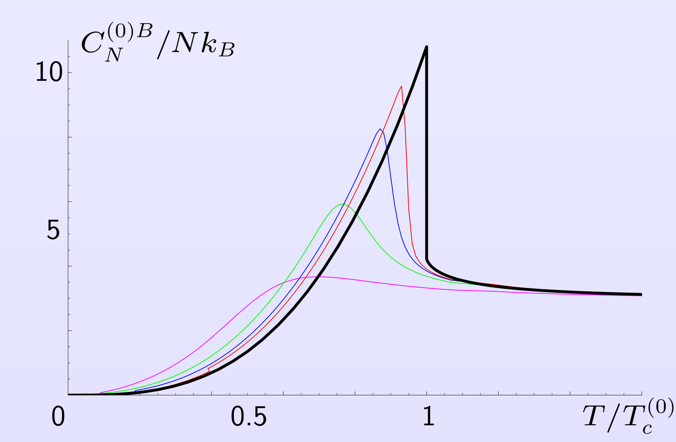
$$Z_1(\beta) = \sum_{\boldsymbol{k}} e^{-\beta E_{\boldsymbol{k}}} \quad \text{and} \quad Z_0^{(0)B}(\beta) = 1$$

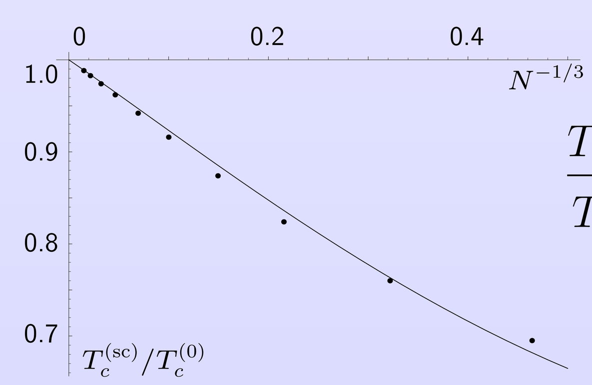
- Ground-state occupancy without interaction [16]:

$$w_N^{(0)B}(\beta) = \frac{1}{N}\sum_{n=1}^N e^{-n\beta E_0}\, Z_{N-n}^{(0)B}(\beta) \,/\, Z_N^{(0)B}(\beta)$$

- Interaction-free results for harmonic case:



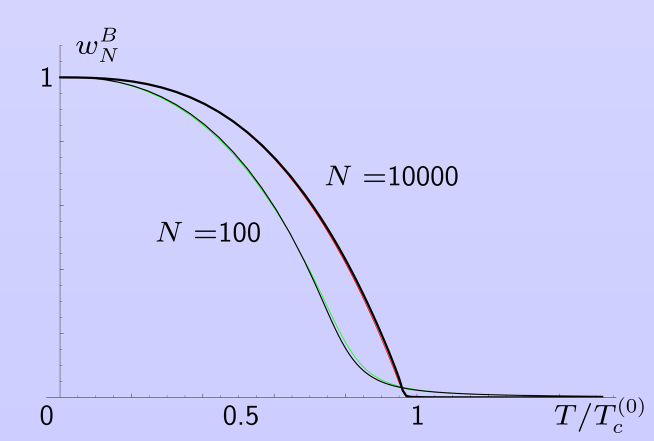



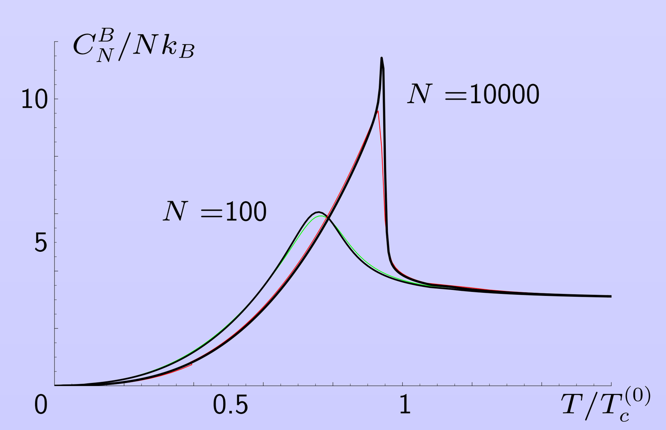




$$\frac{T_c^{(\mathrm{sc})}}{T_c^{(0)}} = 1 - \frac{\zeta(2)}{2\,\zeta^{2/3}(3)}\frac{1}{N^{1/3}} + \frac{\zeta^2(2)}{4\,\zeta^{4/3}(3)}\frac{1}{N^{2/3}} - \frac{\ln\left[8N/\zeta(3)\right]}{9\,\zeta^{1/3}(3)}\frac{1}{N^{2/3}} + \ldots \quad [19]$$

- Interacting Bose gas [20]: $V^{(\mathrm{int})}(\boldsymbol{x}) = g\,\delta(\boldsymbol{x})$

$$Z_N^B(\beta) = \frac{1}{N}\sum_{n=1}^N Z_1(n\beta)\, Z_{N-n}^B(\beta) \times \left\{1 + ng\,\frac{(M\omega/2\pi\hbar)^{3/2}}{\hbar Z_1(n\beta)}\sum_{l=1}^{n-1}\left[Z_1(l\beta)\,Z_1([n-l]\beta)\,Z_1(n\beta)\right]^{1/2} + \ldots\right\}^{-1}$$




## 3) Fermionic Chromium Atoms

- Non-interacting $F = 3/2$ Fermi gas:

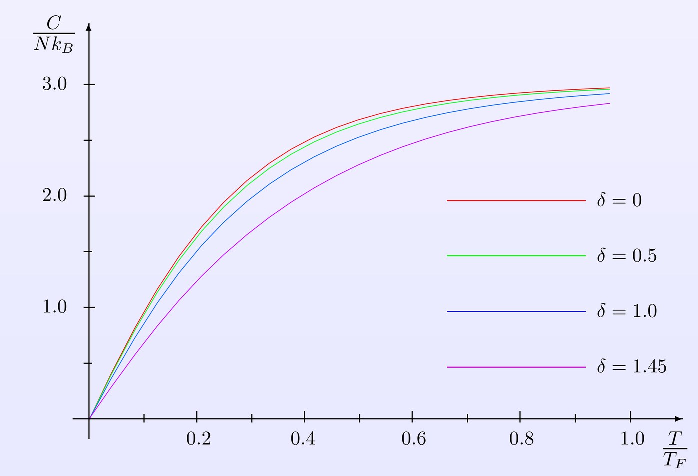
$$U(\boldsymbol{x}) = \frac{M}{2}\,\omega^2\boldsymbol{x}^2\,, \quad T_F = \frac{\hbar\omega}{k_B}\left(\frac{3N}{2}\right)^{1/3}$$

$$\mathcal{Z}^{(0)} = \oint \mathcal{D}\Psi^* \oint \mathcal{D}\Psi\, e^{-\mathcal{A}^{(0)}[\Psi,\Psi^*]/\hbar}$$

$$\mathcal{A}^{(0)}[\Psi,\Psi^*] = \int_0^{\hbar\beta} d\tau \int d^3x\; \psi_i^*(\boldsymbol{x},\tau) H_{ij}\; \psi_j(\boldsymbol{x},\tau)$$

$$H_{ij} = \left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\boldsymbol{\nabla}^2}{2M} + U(\boldsymbol{x}) - \mu\right]\delta_{ij} - \eta F_{ij}^z$$

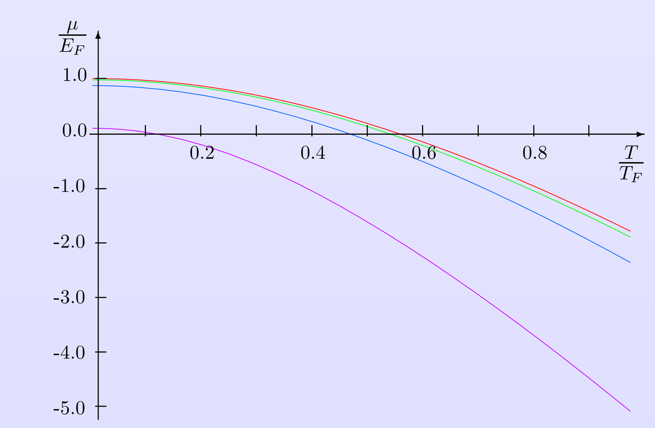
- Heat Capacity:

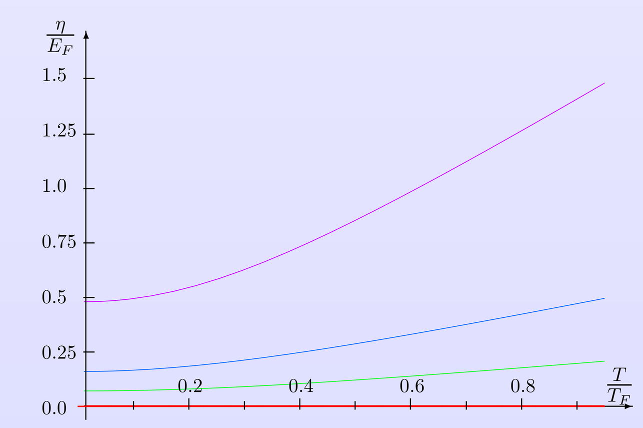



Magnetization per particle:

$$\delta \equiv \frac{\partial \ln \mathcal{Z}^{(0)} / \partial\eta}{\partial \ln \mathcal{Z}^{(0)} / \partial\mu}$$

- Chemical and magneto-chemical motential:




- Fermionic chromium: Paris experiment [21]

1) Nuclear spin $I = 3/2^+$
2) Large magnetic dipole moment, $m = 6\, m_B$

- **Goal**: Interacting dipolar gas

$$\mathcal{A}^{\mathrm{int}}[\Psi,\Psi^*] = \frac{1}{2}\int_0^{\hbar\beta} d\tau \int d^3x \int d^3x'\, V_{iji'j'}^{(\mathrm{int})}(\boldsymbol{x}-\boldsymbol{x}')\,\psi_i^*(\boldsymbol{x},\tau)\psi_j(\boldsymbol{x},\tau)\psi_{i'}^*(\boldsymbol{x}',\tau)\psi_{j'}(\boldsymbol{x}',\tau)$$

$$V_{iji'j'}^{(\mathrm{int})}(\boldsymbol{x}-\boldsymbol{x}') = V_{iji'j'}^{(\mathrm{contact})}(\boldsymbol{x}-\boldsymbol{x}') + V_{iji'j'}^{(\mathrm{dd})}(\boldsymbol{x}-\boldsymbol{x}')$$

1) Contact interaction [22,23]:

$$V_{iji'j'}^{(\mathrm{contact})}(\boldsymbol{x}) = \delta(\boldsymbol{x})\left(c_1\,\delta_{ij}\,\delta_{i'j'} + c_3\,\mathbf{F}_{ij}\,\mathbf{F}_{i'j'}\right)$$

2) Dipole-dipole interaction [24]:

$$V_{iji'j'}^{(\mathrm{dd})}(\boldsymbol{x}) = \frac{\mu_0\, m^2}{4\pi}\left\{\frac{\mathbf{F}_{ij}\,\mathbf{F}_{i'j'}}{|\boldsymbol{x}|^3} - 3\,\frac{[\mathbf{F}_{ij}\,\boldsymbol{x}]\,[\mathbf{F}_{i'j'}\,\boldsymbol{x}]}{|\boldsymbol{x}|^5}\right\}$$

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