Dynamics of Matter Waves in Optical Lattices

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1. Ginzburg-Landau Theory [1,2]

Bose-Hubbard Hamiltonian [3–5] coupled to local sources $j_i(\tau), j_i^*(\tau)$:

$$\hat{H}(\tau) = -\sum_{ij} J_{ij}\hat{a}_i^\dagger \hat{a}_j + \sum_i \left\{ \frac{U}{2}\hat{n}_i(\hat{n}_i - 1) - \mu\hat{n}_i + j_i^*(\tau)\hat{a}_i + j_i(\tau)\hat{a}_i^\dagger \right\}$$

Grand-canonical free energy ($\hbar = 1$): $\mathcal{F} = -\frac{1}{\beta}\ln \mathrm{Tr}\left(\hat{T}e^{-\int_0^\beta \hat{H}(\tau)\mathrm{d}\tau}\right)$

Matsubara space with $\omega_m = 2\pi m/\beta$:

$$\mathcal{F}[j, j^*] = F_0 - \frac{1}{\beta}\Bigg\{ \sum_{ij}\sum_{m_1 m_2} G_{ij}^{(2)}(\omega_{m_1}|\omega_{m_2}) j_i(\omega_{m_1}) j_j^*(\omega_{m_2})$$
$$+ \sum_{\substack{ij\\kl}} \sum_{\substack{m_1 m_2\\m_3 m_4}} G_{ijkl}^{(4)}(\omega_{m_1}, \omega_{m_2}|\omega_{m_3}, \omega_{m_4}) j_i(\omega_{m_1}) j_j(\omega_{m_2}) j_k^*(\omega_{m_3}) j_l^*(\omega_{m_4}) + \cdots \Bigg\}$$

Cumulant decomposition and hopping expansion [6], e.g.:

$$G_{ij}^{(2)}(\omega_{m_1}|\omega_{m_2}) = \delta_{m_1,m_2}\left\{\delta_{ij}C_i^{(2)}(\omega_{m_1}) + J_{ij}C_i^{(2)}(\omega_{m_1})C_j^{(2)}(\omega_{m_2}) + \cdots\right\}$$

Order parameter field:

$$\psi_i(\omega_m) = \langle \hat{a}_i(\omega_m)\rangle = \beta\frac{\partial \mathcal{F}}{\partial j_i^*(\omega_m)} \quad , \quad \psi_i^*(\omega_m) = \langle \hat{a}_i^\dagger(\omega_m)\rangle = \beta\frac{\partial \mathcal{F}}{\partial j_i(\omega_m)}$$

Legendre transformation yields effective action:

$$\Gamma\left[\psi_i^*(\omega_m), \psi_i(\omega_m)\right] = \mathcal{F} - \frac{1}{\beta}\sum_i\sum_m \left[\psi_i^*(\omega_m) j_i(\omega_m) + \psi_i(\omega_m) j_i^*(\omega_m)\right]$$
$$= F_0 + \frac{1}{\beta}\sum_{ij}\Bigg\{ \sum_m \left(\frac{\delta_{ij}}{C_i^{(2)}(\omega_m)} - J_{ij}\right)\psi_i(\omega_m)\psi_j^*(\omega_m) - \sum_{\substack{m_1 m_2\\m_3 m_4}} \psi_i(\omega_{m_1})\psi_i(\omega_{m_2})$$
$$\times\, \psi_i^*(\omega_{m_3})\psi_i^*(\omega_{m_4}) \frac{\delta_{ij} C_i^{(4)}(\omega_{m_1}, \omega_{m_2}|\omega_{m_3}, \omega_{m_4})}{4C_i^{(2)}(\omega_{m_1})C_i^{(2)}(\omega_{m_2})C_i^{(2)}(\omega_{m_3})C_i^{(2)}(\omega_{m_4})} + \cdots \Bigg\}.$$

Static solution for equations of motion:

$$\frac{\partial\Gamma}{\partial\psi_i^*(\omega_m)} = 0 \quad , \quad \frac{\partial\Gamma}{\partial\psi_i(\omega_m)} = 0 \qquad \Rightarrow \qquad \psi_i(\omega_m) = \delta_{m,0}\psi_{\mathrm{eq}}(J, \mu, U)$$

Mott insulator phase: $\psi_{\mathrm{eq}}(J, \mu, U) = 0$; superfluid phase: $\psi_{\mathrm{eq}}(J, \mu, U) \neq 0$

Average particle number per site and compressibility:

$$\langle n\rangle = -\frac{1}{N_s}\frac{\partial\Gamma}{\partial\mu}\bigg|_{\psi=\psi_{\mathrm{eq}}} \quad , \quad \kappa = -\frac{1}{N_s}\frac{\partial^2\Gamma}{\partial\mu^2}\bigg|_{\psi=\psi_{\mathrm{eq}}}$$

Linearization around static solution:

$$\begin{pmatrix} \frac{\delta^2\Gamma}{\delta\psi_{\mathbf{k}}(\omega_m)\delta\psi_{-\mathbf{k}}(\omega_m)}\Big|_{\mathrm{eq}} & \frac{\delta^2\Gamma}{\delta\psi_{\mathbf{k}}(-\omega_m)\delta\psi_{\mathbf{k}}^*(-\omega_m)}\Big|_{\mathrm{eq}} \\ \frac{\delta^2\Gamma}{\delta\psi_{\mathbf{k}}^*(\omega_m)\delta\psi_{\mathbf{k}}(\omega_m)}\Big|_{\mathrm{eq}} & \frac{\delta^2\Gamma}{\delta\psi_{\mathbf{k}}^*(-\omega_m)\delta\psi_{\mathbf{k}}^*(\omega_m)}\Big|_{\mathrm{eq}} \end{pmatrix} \begin{pmatrix} \delta\psi_{\mathbf{k}}(\omega_m) \\ \delta\psi_{\mathbf{k}}^*(-\omega_m) \end{pmatrix} = 0$$

Excitation spectrum $\omega(\mathbf{k})$ is obtained after analytical continuation: $i\omega_m \to \omega + i\epsilon$.

Equation of motion in small U limit:

$$0 \stackrel{!}{=} \sum_j \Bigg\{ \left[(-\omega - \mu)\,\delta_{ij} - J_{ij}\right]\psi_j^*(\omega) + U\int_{-\infty}^{\infty}\mathrm{d}\omega_2\int_{-\infty}^{\infty}\mathrm{d}\omega_3\int_{-\infty}^{\infty}\mathrm{d}\omega_4\,\delta_{ij}$$
$$\times\,\delta(\omega + \omega_2 - \omega_3 - \omega_4)\psi_i(\omega_2)\psi_i^*(\omega_3)\psi_i^*(\omega_4)\Bigg\} \to \text{Gross-Pitaevskii}$$

2. Application: Collapse and Revival

Site-dependent chemical potential due to trap:

$$\mu_i = \mu - \frac{m\omega^2}{2}\mathbf{r}_i^2$$

Wick rotation $\tau = it$ leads to equations of motion

$$\frac{\delta\Gamma\left[\psi_i^*(t), \psi_i(t)\right]}{\delta\psi_i^*(t)} = 0 \quad , \quad \frac{\delta\Gamma\left[\psi_i^*(t), \psi_i(t)\right]}{\delta\psi_i(t)} = 0$$

General zeroth-order solution in Mott phase:

$$\psi_i(t) = A_i^+ e^{-it(Un_i - \mu_i)} + A_i^- e^{-it[U(n_i-1)-\mu_i]}$$

A_i^+ and A_i^- follow from initial conditions $\psi_i(0)$ and $\dot{\psi}_i(0)$: lattice Gross-Pitaevskii equation deep in superfluid phase

$$i\dot{\psi}_i = -J\sum_{<ij>}\psi_j + U\psi_i|\psi_i|^2 - \mu_i^{(0)}\psi_i$$

Exact solution in zeroth hopping order:

$$\psi_i(t) = \psi_i(0)e^{-i\left(U|\psi_i(0)|^2 - \mu_i\right)t}$$

Thomas-Fermi is used as initial static solution: $\psi_i(0) = \sqrt{(\mu_i^{(0)} + zJ)/U}$

Momentum space distribution:

$$|\psi(\mathbf{k}, t)|^2 = |w(\mathbf{k})|^2\sum_{ij}\psi_i^*(t)\psi_j(t)e^{i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)}$$

Integration in k_z defines planar distribution:

$$\rho(\mathbf{k}_\perp) = |w(\mathbf{k}_\perp)|^2\sum_{ij} G(\mathbf{r}_{\perp j}; \mathbf{r}_{\perp i}; t)e^{i\frac{m\omega^2}{2}t(\mathbf{r}_{\perp i}^2 - \mathbf{r}_{\perp j}^2)}e^{i\mathbf{k}_\perp\cdot(\mathbf{r}_{\perp i} - \mathbf{r}_{\perp j})}$$

$$G(\mathbf{r}_{\perp j}; \mathbf{r}_{\perp i}; t) = \sum_{z_l}\Big[A^+(\mathbf{r}_{\perp j}, z_l)^2 + A^-(\mathbf{r}_{\perp j}, z_l)^2$$
$$+2\cos(Ut)A^+(\mathbf{r}_{\perp j}, z_l)A^-(\mathbf{r}_{\perp j}, z_l)\Big]$$

Long-time limit $t \to \infty$:

$$\rho(\mathbf{k}_\perp) = |w(\mathbf{k}_\perp)|^2 G(\mathbf{0}; \mathbf{0}; t)\left|\sum_i e^{i\frac{m\omega^2 t}{2}\mathbf{r}_{\perp i}^2}\right|^2 \approx w(\mathbf{k}_\perp)|^2 G(\mathbf{0}; \mathbf{0}; t)\left(\frac{2\pi}{m\omega^2 a^2 t}\right)^2$$

Experiments are restricted to small region in $\mathbf{k}_\perp$ space:

$$N_{\mathrm{coh}} \approx \rho(\mathbf{k}_\perp = 0)(\delta k)^2 = (\delta k)^2|w(\mathbf{0})|^2 G(\mathbf{0}; \mathbf{0}; t)\left(\frac{2\pi}{m\omega^2 a^2 t}\right)^2$$

δk relates to time-of-flight through: $\delta k = \frac{m\delta x}{\hbar t_f}$

Wannier function in harmonic approximation:

$$w(k_x) = \sqrt{\frac{a}{\pi^2}}\left(\frac{\pi^2}{V_0/E_r}\right)^{1/8}\exp\left(-\frac{a^2 k_x^2}{2\pi^2\sqrt{V_0/E_r}}\right)$$

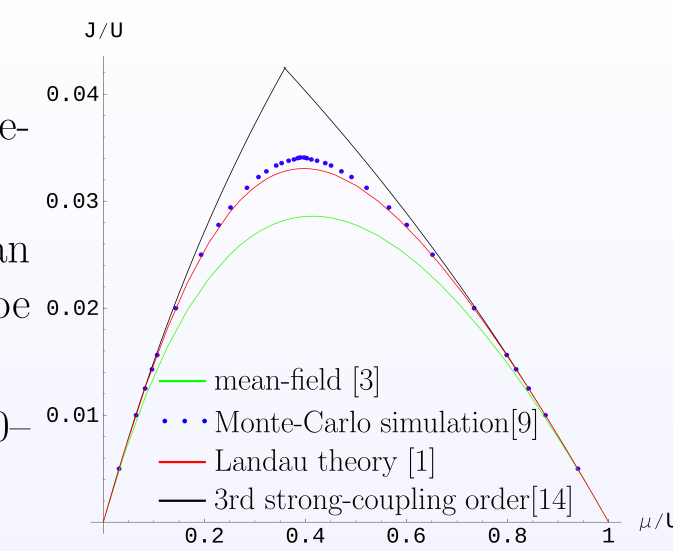
Observed condensate loss in terms of experimental parameters [7,8]:

$$N_{\mathrm{coh}} \approx G(\mathbf{0}; \mathbf{0}; t)(t/T)^{-2} \quad , \quad T = \frac{2\delta k}{\sqrt{\pi}am\omega^2(V_0/E_r)^{1/4}}$$

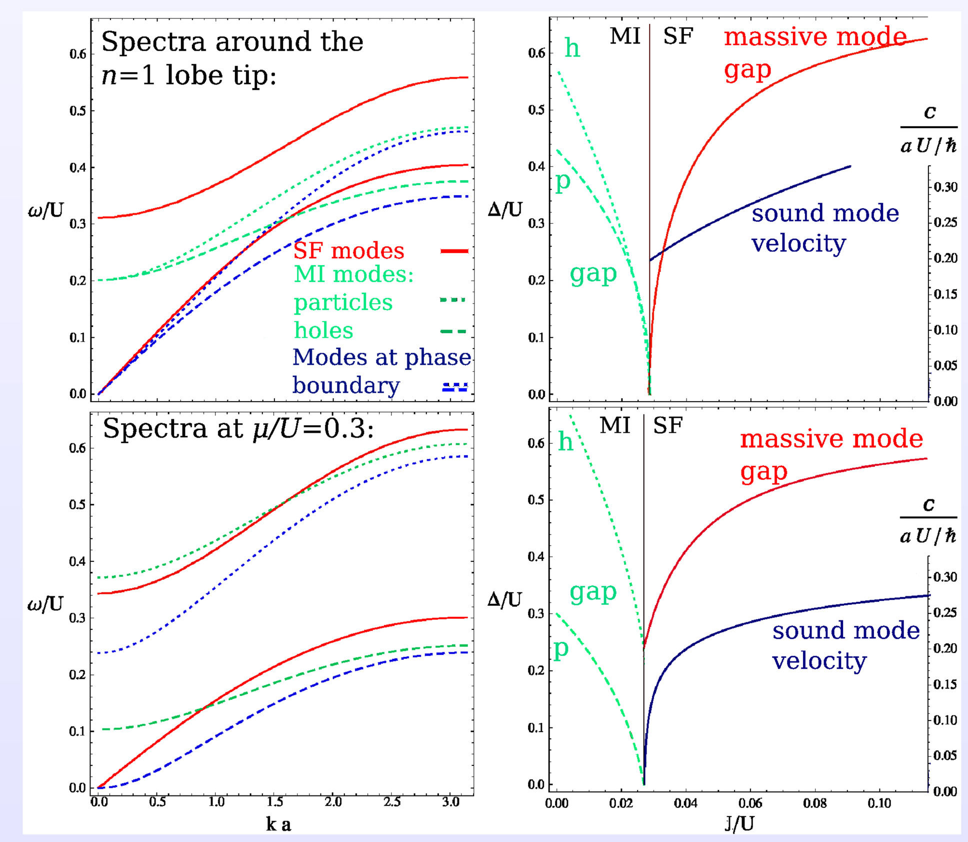
3. Picture Gallery

Phase diagram for zero temperature:

- Recent Monte-Carlo simulation is believed to be very precise [9].
- Landau theory gives a difference less than 3% from the Monte-Carlo data at the lobe tip [1].
- Extension to higher hopping orders [10–12] and $T > 0$ [13].

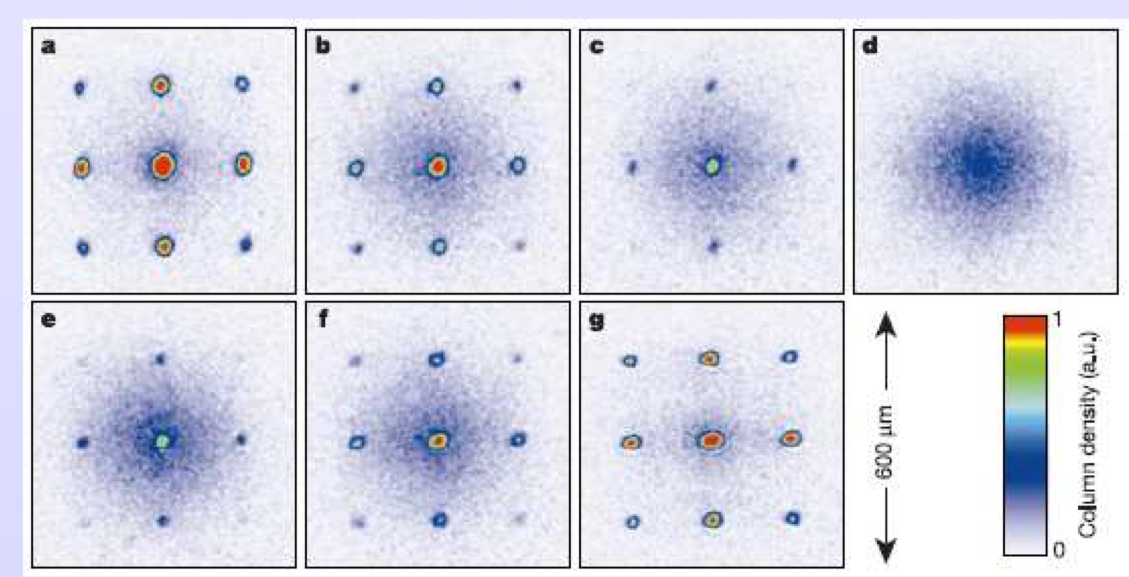


Zero temperature spectra:



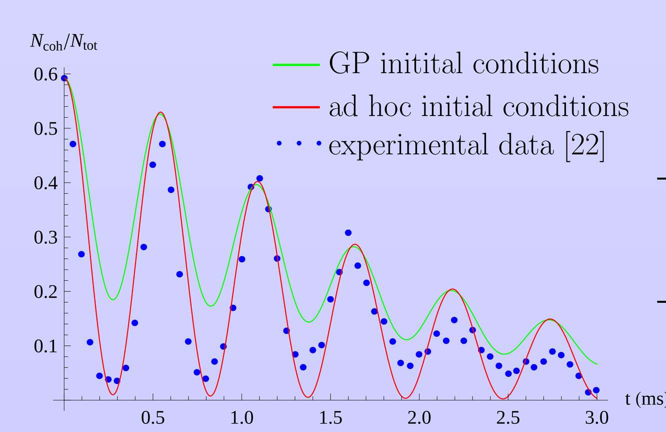
- Goldstone sound mode [15–18].
- Gapped mode [19–21].
- Both modes map onto MI particle/hole spectra.
- Special behavior at lobe tip [3]: dynamic critical exponent is $z = 1$ and $\Delta \sim \sqrt{J - Jc}$.
- Off the tip: $z = 2$ and $\Delta \sim (J - Jc)^1$.

Time-of-flight pictures [22]:



- Trapping potential is suddenly changed from $V_A = 8\ E_r$ to $V_B = 22\ E_r$. After a subsequent hold time (0 μs - 550 μs) the interfence pattern is observed.
- Condensed fraction is extracted from squares around interference peaks.

Comparison with experiments:



- Calculated revival time $h/U = 0.55$ ms coincides with experimental value.
- Interpolation between large- and small-time asymptotic limits was obtained from first Padé approximant.

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