Weighted multi-channel BCS mean-field theory with finite effective range

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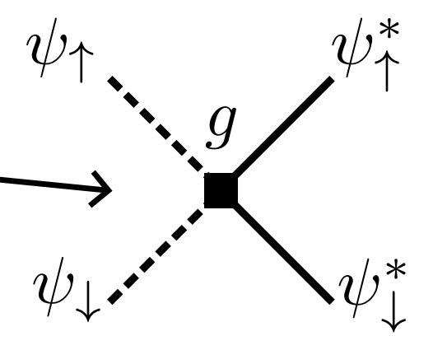
Multi-channel Hubbard-Stratonovich transformation

$$\mathcal{H}-\mu\mathcal{N}=\sum_{\sigma\in\mathcal{S}}\psi^*_\sigma(x)\Big[-\frac{\nabla^2}{2m}-\mu\Big]\psi_\sigma(x)+g\psi^*_\uparrow(x)\psi^*_\downarrow(x)\psi_\downarrow(x)\psi_\uparrow(x)$$

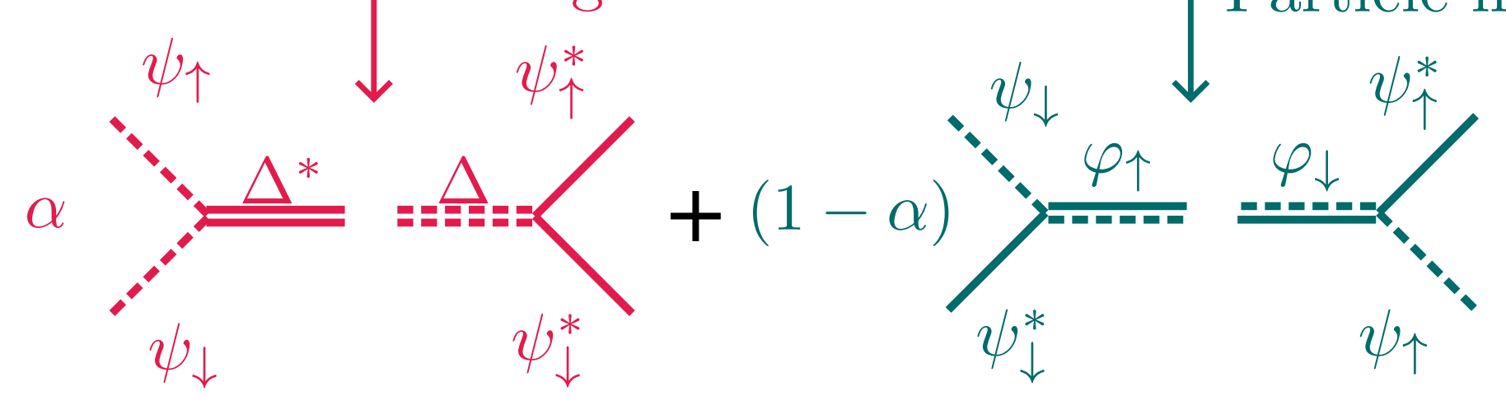
$$=\mathcal{H}_{\text{kin}}+\alpha g\psi^*_\uparrow(x)\psi^*_\downarrow(x)\psi_\downarrow(x)\psi_\uparrow(x)+(1-\alpha)g\psi^*_\uparrow(x)\psi_\uparrow(x)\psi^*_\downarrow(x)\psi_\downarrow(x)$$



$x=(\mathbf{x},\tau)$
$\hbar=1=k_{\text{B}}$
$V\subset\mathbb{R}^3$, $\beta=T^{-1}$

Pairing channel

Particle-hole channel



Partition function from action

$$S=\int_V\oint_{[0,\beta]}\mathrm{d}x\,\Big\{\sum_\sigma\psi^*_\sigma(x)\Big[\partial_\tau+\frac{\nabla^2}{2m}+\mu\Big]\psi_\sigma(x)-\alpha\mathcal{H}^{\Delta}_{\text{Int}}-(1-\alpha)\mathcal{H}^{\varphi}_{\text{Int}}\Big\}$$

$$\Rightarrow\ \mathcal{Z}=\oint\mathcal{D}\psi^*_\sigma\mathcal{D}\psi_\sigma\exp\big(-\underset{\downarrow}{S}[\psi^*_\sigma,\psi_\sigma]\big)=\oint\mathcal{D}\psi^*_\sigma\mathcal{D}\psi_\sigma\oint\mathcal{D}\Delta^*\mathcal{D}\Delta\mathcal{D}\varphi_\sigma\exp\big(-\underset{\downarrow}{S}[\psi^*_\sigma,\psi_\sigma,\Delta^*,\Delta,\varphi_\sigma]\big)$$

Quartic in the fermionic fields ψ^4 — Quadratic in the fermionic fields ψ^2

$\Rightarrow$ Gaussian functional integral eliminates fermionic degrees of freedom

Effective range saddle-point equations

Integrating out fermions yields Nambu-Gorkov action

$$S_{\text{Eff}}[\Delta^*_0,\Delta_0,\varphi_\sigma]=\beta\sum_{\mathbf{k}}\Big\{-\frac{|\Delta_0|^2}{\alpha g}-\frac{\varphi_0^2}{(1-\alpha)g}+\boxed{\frac{k^2}{2m}-\mu+\varphi_0}\Big\}-\sum_{\mathbf{k}}\sum_{ik_n}\operatorname{tr}\ln\mathbf{G}^{-1}_{k,k'}[\Delta^*_0,\Delta_0,\varphi_0]$$

$\xi_{\mathbf{k}}=\varepsilon_{\mathbf{k}}-\mu+\varphi_0$

Bogoliubov Dispersion

$$E(\mathbf{k})=\sqrt{\xi^2_{\mathbf{k}}+|\Delta_0|^2}$$

Saddle Point (Mean-Field) approximation for static, uniform auxilliary fields

$$(1)\,0=\frac{\partial S_{\text{Eff}}}{\partial\alpha}\Rightarrow\alpha=\frac{|\Delta_0|}{|\Delta_0|+|\varphi_0|}$$

$$(2)\ 0=\frac{\delta S_{\text{Eff}}}{\delta\Delta^*_0}\Rightarrow 0=|\Delta_0|\Big[\frac{1}{\alpha g}+\frac{1}{V}\sum_{\mathbf{k}}\frac{\tanh(\frac{\beta}{2}E(\mathbf{k}))}{2E(\mathbf{k})}\Big]$$

$$(3)\ 0=\frac{\delta S_{\text{Eff}}}{\delta\varphi_0}\Rightarrow\varphi_0=\frac{(1-\alpha)g}{2}\frac{1}{V}\sum_{\mathbf{k}}\Big[1-\frac{\xi_{\mathbf{k}}}{E(\mathbf{k})}\tanh(\frac{\beta}{2}E(\mathbf{k}))\Big]=\boxed{\frac{(1-\alpha)g}{2}}\frac{1}{V}\sum_{\mathbf{k}}n_{\mathbf{k}}$$

Explicitly bare coupling depending
$\Rightarrow$ effective range effects

Finite effective range scattering theory

Low energy s-wave scattering phase shift $q\cot\delta_0(q)=-\frac{1}{a_{\text{S}}}+\frac{1}{2}r_{\text{eff}}q^2+\mathcal{O}(q^4)$ [1]

Lippmann-Schwinger equation for δ-potential yields

$$\mathcal{O}(q^0):\ \frac{1}{g}=\frac{m}{4\pi a_{\text{S}}}-\sum_{|\mathbf{k}|\le k_{\text{c}}}\frac{1}{2\varepsilon_{\mathbf{k}}}$$

$$\mathcal{O}(q^2):k_{\text{c}}=\frac{4}{\pi}\frac{1}{r_{\text{eff}}}$$

Range dependent UV Cutoff

δ-potential with finite k_{c}
mimics short ranged potential
up to second order in q

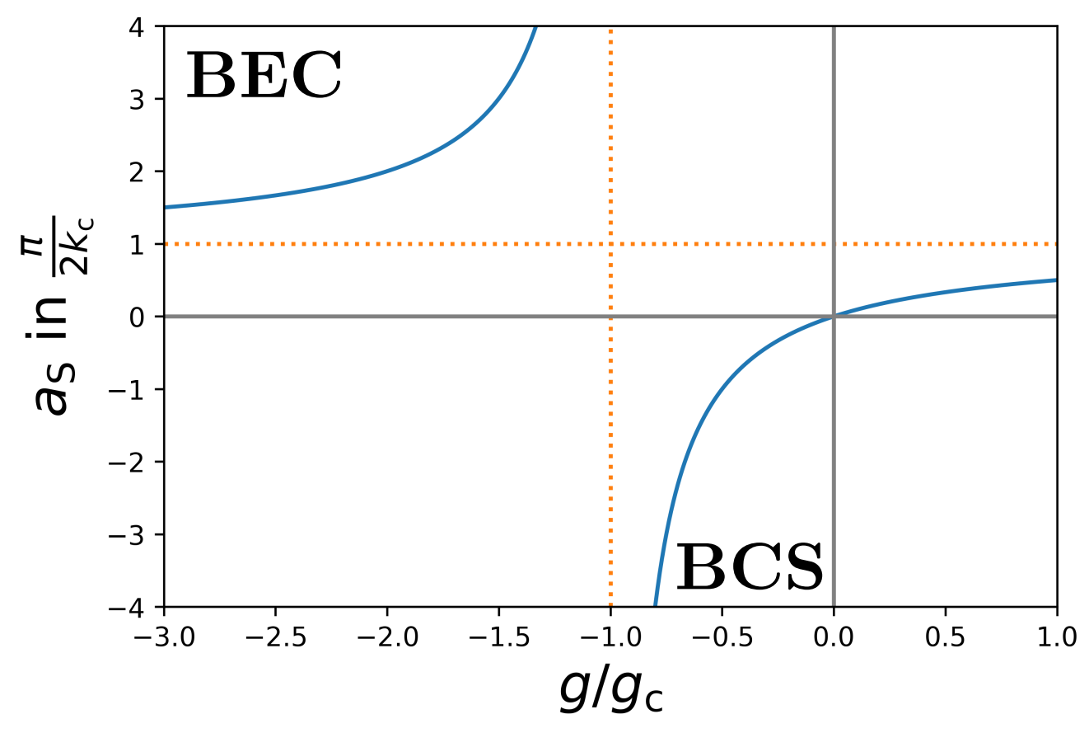


Fig: Visualization of $\mathcal{O}(q^0)$ of two body scattering with finite UV momentum Cutoff k_{c} inspired from [2]

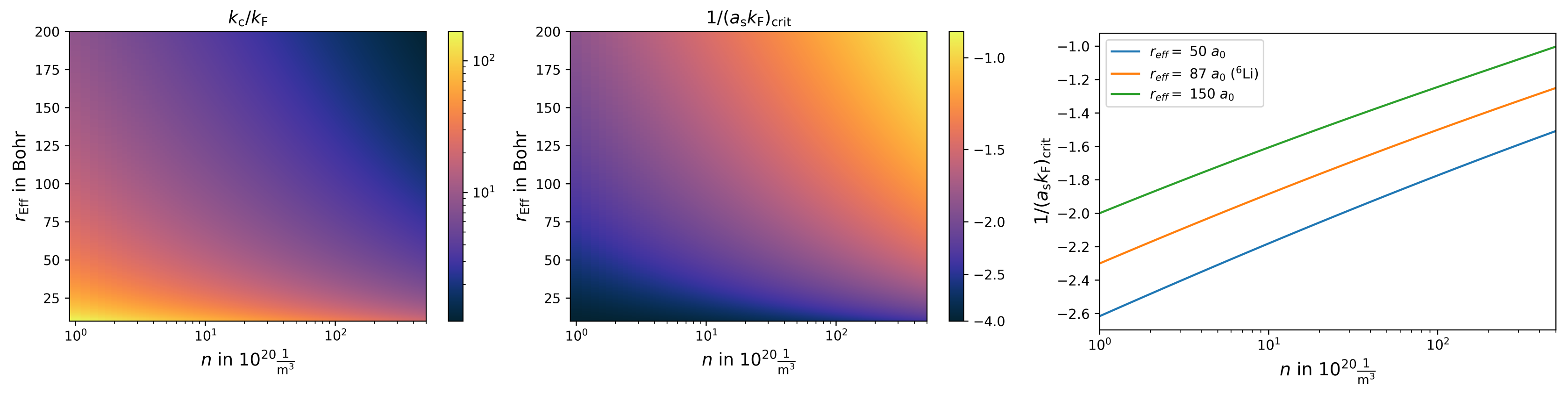
Hartree existence condition and ground state boundaries

Sign of Hartree shift must always coincide the sign of interaction strength g

$$\left.\begin{aligned}&\varphi_0\propto g\Rightarrow\varphi_0\le0\\&\varphi_0=\tfrac{gn}{2}+|\Delta_0|\end{aligned}\right\}\ \varphi_0\neq0\text{ if and only if }|\Delta_0|<\frac{|g|n}{2}\xrightarrow{T=0}\frac{1}{a_{\text{S}}k_{\text{F}}}<\underbrace{\frac{2}{\pi}\Big(\frac{k_{\text{c}}}{k_{\text{F}}}+\mathrm{W}_{-1}\big(-\frac{e^2}{12}e^{-\frac{k_{\text{c}}}{k_{\text{F}}}}\big)\Big)}_{\frac{1}{a_{\text{crit}}k_{\text{F}}}}$$

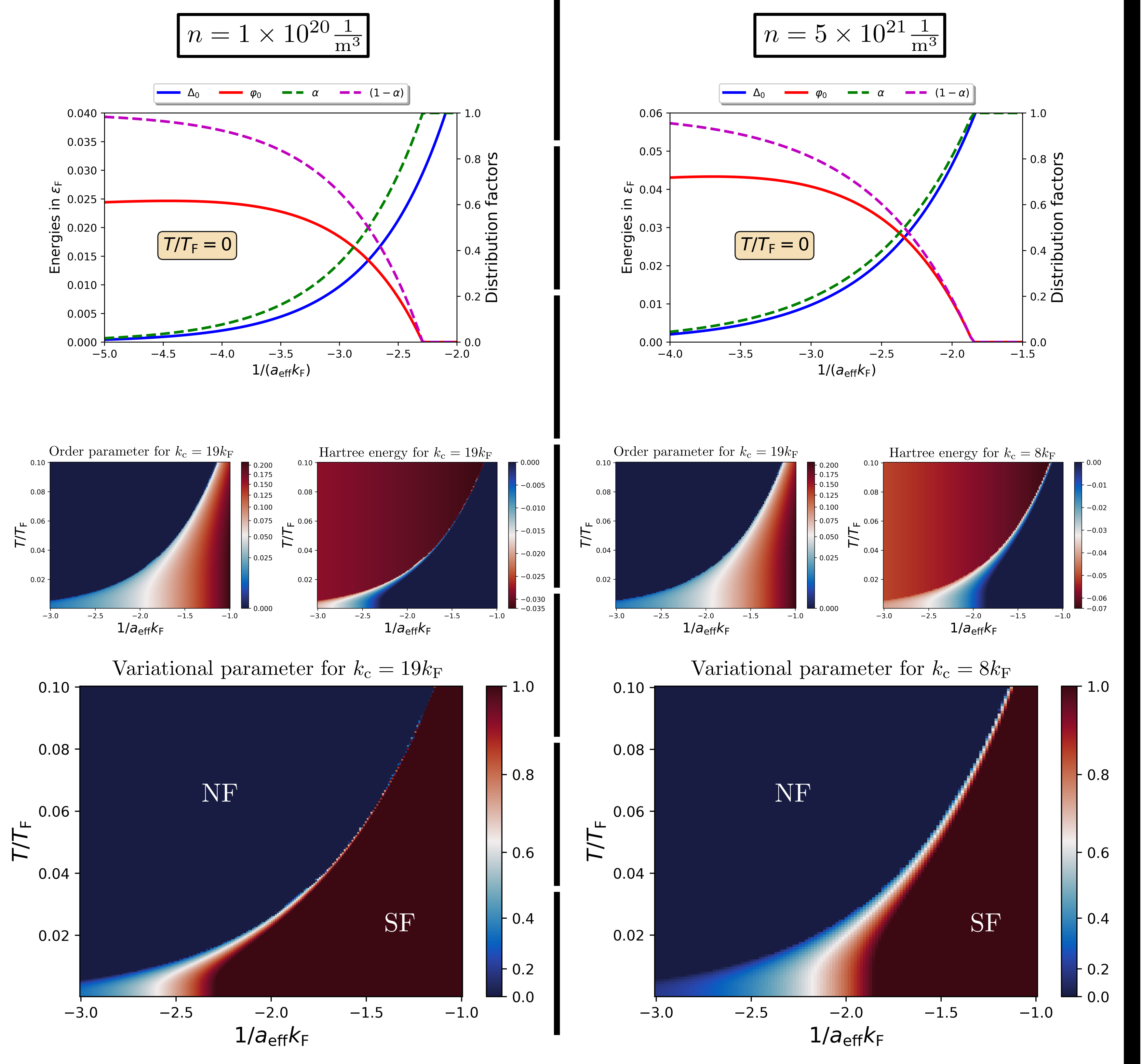
Lambert-W-function

When Δ_0 is large: Regular superfluid. When Δ_0 is small: Hartree term arises



Self-consistent results and phase diagrams

^{6}Li: $r_{\text{eff}}\approx87a_0$ [3]

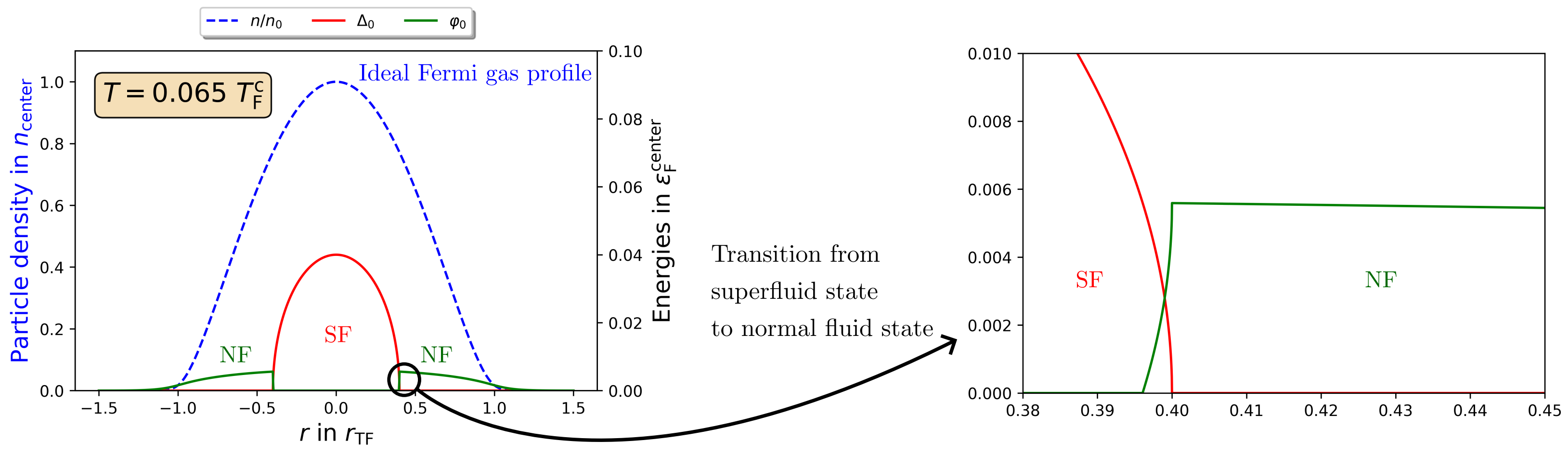


Inclusion of direct channel destroys universality of BCS mean-field theory!

Harmonically trapped clouds: qualitative discussion

- Hartree Energy depends strongly on local Fermi-momentum
- Hence harmonically confined clouds should exhibit two regions
- NF and SF phase should be connected by small intersection region
- Assuming constant temperature one sweeps horizontally through phase diagram

Schematic representation for ideal Fermi gas approximation (3D spheric trap)



Conclusion and outlook

- BCS mean-field theory [4] can straight-forwardly be extended by multi-channel Hubbard-Stratonovich transformation including direct channel
- Homogeneous case divides NF and SF phase by intersection region
- Presence breaks universality of theory $\Rightarrow$ direct density dependence
- In traps we expect phases to be separated by small intersection region
- Further calculations and fits need to be done in traps
- Compare to machine learning phase diagram in harmonic traps [5]

References:

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