Bose-Einstein Condensation with Strong Disorder: Replica Method

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Abstract

A recent non-perturbative approach towards the dirty boson problem relies on applying the replica method [1]. Here we extend this Hartree-Fock theory for a weakly interacting Bose gas in a quenched δ-correlated disorder potential from the homogeneous case to a harmonic confinement within the Thomas-Fermi approximation. In this way we obtain and solve coupled self-consistency equations which involve the particle and the condensate density as well as the density of fragmented local Bose-Einstein condensates, which emerge in the respective minima of the random potential landscape. Whereas for weak disorder the results of Huang and Meng from a Bogoliubov theory [2,3] are reproduced only qualitatively, we yield for strong disorder a quantum phase transition to a Bose-glass phase.

Model

- **Action of a Bose gas**

$$\mathcal{A}=\int_0^{\hbar\beta} d\tau \int d\mathbf{r}\left\{\psi^*(\mathbf{r},\tau)\left[\hbar\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2M}\Delta+V(\mathbf{r})+U(\mathbf{r})-\mu\right]\psi(\mathbf{r},\tau)+\frac{g}{2}\psi^*(\mathbf{r},\tau)^2\psi(\mathbf{r},\tau)^2\right\}$$

- **Partition function**

$$\mathcal{Z}=\oint \mathcal{D}\psi^*\mathcal{D}\psi e^{-\mathcal{A}[\psi^*,\psi]/\hbar}$$

- **Disorder**
  - Disorder Ensemble Average

$$\overline{\bullet}=\int \mathcal{D}U\bullet P[U], \qquad \int \mathcal{D}U P[U]=1$$

  - Assumption

$$\overline{U(\mathbf{r})}=0, \qquad \overline{U(\mathbf{r_1})U(\mathbf{r_2})}=R^{(2)}(\mathbf{r_1}-\mathbf{r_2})$$

  - Disorder Ensemble Average

$$\overline{\exp\left\{i\int d\mathbf{r}J(\mathbf{r})U(\mathbf{r})\right\}}=\exp\left\{\sum_{n=2}^{\infty}\frac{i^n}{n!}\int d\mathbf{r}_1\cdots\int d\mathbf{r}_n R^{(n)}(\mathbf{r}_1\cdots\mathbf{r}_n)J(\mathbf{r}_1)\cdots J(\mathbf{r}_n)\right\}$$

Method

- **Replica Trick [4, 5]**

$$F=-\frac{1}{\beta}\lim_{N\to 0}\frac{\overline{\mathcal{Z}^N}-1}{N}$$

- **Disorder Averaged Replicated Partition Function**

$$\overline{\mathcal{Z}^N}=\overline{\left\{\prod_{\alpha'=1}^{N}\oint\mathcal{D}\psi^*_{\alpha'}\mathcal{D}\psi_{\alpha'}\right\}e^{-\sum_{\alpha=1}^{N}\mathcal{A}[\psi^*_\alpha,\psi_\alpha]/\hbar}}=\left\{\prod_{\alpha=1}^{N}\oint\mathcal{D}\psi^*_{\alpha}\mathcal{D}\psi_{\alpha}\right\}e^{-\mathcal{A}^{(N)}/\hbar}$$

- **Replicated Action**

$$\mathcal{A}^{(N)}=\int_0^{\hbar\beta}d\tau\int d\mathbf{r}\sum_{\alpha=1}^{N}\left\{\psi^*_\alpha(\mathbf{r},\tau)\left[\hbar\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2M}\Delta+V(\mathbf{r})-\mu\right]\psi_\alpha(\mathbf{r},\tau)+\frac{g}{2}|\psi_\alpha(\mathbf{r},\tau)|^4\right\}$$
$$+\sum_{n=2}^{\infty}\frac{1}{n!}\left(\frac{-1}{\hbar}\right)^{n-1}\int_0^{\hbar\beta}d\tau_1\cdots\int_0^{\hbar\beta}d\tau_n\int d\mathbf{r}_1\cdots\int d\mathbf{r}_n$$
$$\times\sum_{\alpha_1=1}^{N}\cdots\sum_{\alpha_n=1}^{N}R^{(n)}(\mathbf{r}_1\cdots\mathbf{r}_n)\,|\psi_{\alpha_1}(\mathbf{r}_1,\tau_1)|^2\cdots|\psi_{\alpha_n}(\mathbf{r}_n,\tau_n)|^2$$

In the replica limit $N\to 0$ higher-order disorder cumulants are negligible: only $R^{(2)}(\mathbf{r})$ contributes.

Assumptions

- **Bogoliubov background method**

$$\psi_\alpha(\mathbf{r},\tau)=\Psi_\alpha(\mathbf{r},\tau)+\delta\psi_\alpha(\mathbf{r},\tau)$$

- **Hartree-Fock theory**
- **Semiclassical approximations**
- **Replica symmetry**

$$\Psi_\alpha(\mathbf{r},\tau)=\sqrt{n_0(\mathbf{r})}$$
$$\langle\delta\psi_\alpha(\mathbf{r},\tau)\,\delta\psi_{\alpha'}(\mathbf{r}',\tau')\rangle=\mathcal{Q}\left(\mathbf{r}-\mathbf{r}',\frac{\mathbf{r}+\mathbf{r}'}{2},\tau-\tau'\right)\delta_{\alpha\alpha'}+q\left(\frac{\mathbf{r}+\mathbf{r}'}{2},\tau-\tau'\right)$$
$$n(\mathbf{r})=\Psi_\alpha(\mathbf{r},\tau)\Psi^*_\alpha(\mathbf{r},\tau)+\langle\delta\psi_\alpha(\mathbf{r},\tau)\,\delta\psi_\alpha(\mathbf{r},\tau)\rangle$$

Self-Consistency equations

$$n(\mathbf{r})=n_0(\mathbf{r})+q(\mathbf{r})+\left(\frac{M}{2\pi\beta\hbar^2}\right)^{3/2}\varsigma_{3/2}\left(e^{\beta\,[\mu-d^2-2gn(\mathbf{r})-V(\mathbf{r})]}\right)$$

$$q(\mathbf{r})=\frac{d\left[n(\mathbf{r})-\left(\frac{M}{2\pi\beta\hbar^2}\right)^{3/2}\varsigma_{3/2}\left(e^{\beta\,[\mu-d^2-2gn(\mathbf{r})-V(\mathbf{r})]}\right)\right]}{d+\sqrt{-\mu+d^2+2gn(\mathbf{r})+V(\mathbf{r})}}$$

$$\left\{-gn_0(\mathbf{r})+\left[\sqrt{-\mu+d^2+2gn(\mathbf{r})+V(\mathbf{r})}+d\right]^2-\frac{\hbar^2}{2M}\Delta\right\}\sqrt{n_0(\mathbf{r})}=0$$

where $R^{(2)}(\mathbf{r_1}-\mathbf{r_2})=D\delta(\mathbf{r_1}-\mathbf{r_2})$, $d=\sqrt{\pi}D\left(\frac{M}{2\pi\hbar^2}\right)^{3/2}$

- **Assumptions**
  - $T=0$
  - $V(\mathbf{r})=\frac{1}{2}M\Omega^2r^2$
  - Repulsive interaction $g=\frac{4\pi\hbar^2 a}{M}$
  - Thomas-Fermi approximation
  - Length scale $l=\sqrt{\frac{\hbar}{M\Omega}}$
  - Energy scale $\mu_0=\frac{15^{2/5}}{2}\left(\frac{aN}{l}\right)^{2/5}\hbar\Omega$

- **Bose-glass Phase** $n_0(r)=0$ **and** $q(r)=n(r)\neq 0$

$$-\mu+d^2+2gn(r)+\frac{1}{2}M\Omega^2r^2=0$$

- **Superfluid Phase** $n_0(r)\neq 0$ **and** $q(r)\neq 0$
  - Condensate density

$$n_0(r)=\frac{1}{g}\left[\sqrt{-\mu+d^2+2gn(r)+\frac{1}{2}M\Omega^2r^2}+d\right]^2$$

  - Depletion

$$q(r)=\frac{d}{g\sqrt{-\mu+d^2+2gn(r)+\frac{1}{2}M\Omega^2r^2}}\left[\sqrt{-\mu+d^2+2gn(r)+\frac{1}{2}M\Omega^2r^2}+d\right]^2$$

  - Particle density

$$n(r)=\frac{1}{g\sqrt{-\mu+d^2+2gn(r)+\frac{1}{2}M\Omega^2r^2}}\left[\sqrt{-\mu+d^2+2gn(r)+\frac{1}{2}M\Omega^2r^2}+d\right]^3$$

Results

- **Densities**

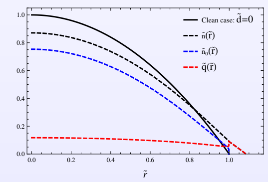

Fig. 1: Dimensionless total density $\tilde{n}(\tilde{r})=\frac{gn(r)}{\mu_0}$ in absence of disorder (solid, black) and in presence of disorder (dashed, black), dimensionless condensate $\tilde{n}_0(\tilde{r})=\frac{gn_0(r)}{\mu_0}$ (dashed, blue), dimensionless depletion $\tilde{q}(\tilde{r})=\frac{gq(r)}{\mu_0}$ (dashed, red) in the disconnected local minicondensates as a function of dimensionless radial coordinate $\tilde{r}=\sqrt{\frac{M\Omega^2}{2\mu_0}}r$, for ^{87}Rb, $N=10^6$, $\tilde{d}=\frac{d}{\sqrt{\mu_0}}=0.117$, $\tilde{\mu}=\frac{\mu-d^2}{\mu_0}=1.177$, $\Omega=200\pi$ Hz and $a=5.29$ nm.

- **Thomas-Fermi Radii**

Existence of the Bose-glass phase

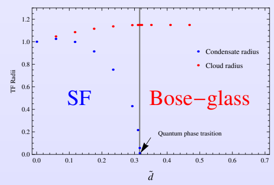

Fig. 2: Dimensionless condensate radius (blue) and dimensionless cloud radius (red) as a function of the dimensionless disorder strength $\tilde{d}=\frac{d}{\sqrt{\mu_0}}$ for ^{87}Rb, $N=10^6$, $\Omega=200\pi$ Hz and $a=5.29$ nm.

- **Comparison with Huang-Meng theory for weak disorder**

Depletion is proportional to the disorder strength [2,3,6–8]

$$q(0)=\frac{\sqrt{2}M^2}{8\pi^{3/2}\hbar^4}\sqrt{\frac{n_0(0)}{a}}D$$

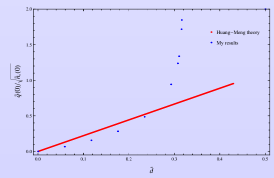

Fig. 3: Fraction of dimensionless depletion $\tilde{q}(0)=\frac{gq(0)}{\mu_0}$ over $\sqrt{\tilde{n}_0(0)}=\sqrt{\frac{gn_0(0)}{\mu_0}}$ in the center of the BEC ($\tilde{r}=0$) as a function of the dimensionless disorder strength $\tilde{d}=\frac{d}{\sqrt{\mu_0}}$ for ^{87}Rb, $N=10^6$, $\Omega=200\pi$ Hz and $a=5.29$ nm.

Perspectives

- Perturbation method
- Beyond Thomas-Fermi approximation
- Finite temperature
- Anisotropic trap potential
- General interaction potential
- Time dependence of densities and Thomas-Fermi radii [9]
- Replica Symmetry Breaking?

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DAAD

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