Hartree-Fock Theory of Harmonically Trapped Dirty BEC via Replica Method

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Abstract: We present a non-perturbative approach towards the dirty boson problem relying on the Hartree-Fock theory which is worked out on the basis of the replica method. Self-consistency equations are obtained and solved. For weak disorder the results of Huang and Meng [1,2] are qualitatively reproduced.

BEC Model

- **Action of a Bose Gas**

$$\mathcal{A}=\int_0^{\hbar\beta}d\tau\int d\mathbf{r}\left\{\psi^*(\mathbf{r},\tau)\left[\hbar\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2M}\Delta+V(\mathbf{r})+U(\mathbf{r})-\mu\right]\psi(\mathbf{r},\tau)+\frac{g}{2}\psi^*(\mathbf{r},\tau)^2\psi(\mathbf{r},\tau)^2\right\}$$

- **Partition Function** $\quad \mathcal{Z}=\oint\mathcal{D}\psi^*\mathcal{D}\psi\, e^{-\mathcal{A}[\psi^*,\psi]/\hbar}$

- **Disorder**
  - Disorder Ensemble Average $\quad \overline{\bullet}=\int\mathcal{D}U\bullet P[U],\qquad \int\mathcal{D}U\,P[U]=1$
  - Assumption $\quad \overline{U(\mathbf{r})}=0,\qquad \overline{U(\mathbf{r_1})U(\mathbf{r_2})}=R^{(2)}(\mathbf{r_1}-\mathbf{r_2})$
  - Characteristic Functional

$$\overline{\exp\left\{i\int d\mathbf{r}J(\mathbf{r})U(\mathbf{r})\right\}}=\exp\left\{\sum_{n=2}^{\infty}\frac{i^n}{n!}\int d\mathbf{r}_1\cdots\int d\mathbf{r}_n R^{(n)}(\mathbf{r}_1,\dots,\mathbf{r}_n)J(\mathbf{r}_1)\cdots J(\mathbf{r}_n)\right\}$$

Method

- **Replica Trick [3,4]** $\quad \mathcal{F}=-\frac{1}{\beta}\lim_{N\to0}\frac{\overline{\mathcal{Z}^N}-1}{N}$

- **Disorder Averaged Replicated Partition Function**

$$\overline{\mathcal{Z}^N}=\overline{\left\{\prod_{\alpha'=1}^{N}\oint\mathcal{D}\psi^*_{\alpha'}\mathcal{D}\psi_{\alpha'}\right\}e^{-\sum_{\alpha=1}^{N}\mathcal{A}[\psi^*_\alpha,\psi_\alpha]/\hbar}}=\left\{\prod_{\alpha=1}^{N}\oint\mathcal{D}\psi^*_{\alpha}\mathcal{D}\psi_{\alpha}\right\}e^{-\mathcal{A}^{(N)}/\hbar}$$

- **Replicated Action**

$$\mathcal{A}^{(N)}=\int_0^{\hbar\beta}d\tau\int d\mathbf{r}\sum_{\alpha=1}^{N}\left\{\psi^*_\alpha(\mathbf{r},\tau)\left[\hbar\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2M}\Delta+V(\mathbf{r})-\mu\right]\psi_\alpha(\mathbf{r},\tau)+\frac{g}{2}|\psi_\alpha(\mathbf{r},\tau)|^4\right\}$$
$$+\sum_{n=2}^{\infty}\frac{1}{n!}\left(\frac{-1}{\hbar}\right)^{n-1}\int_0^{\hbar\beta}d\tau_1\cdots\int_0^{\hbar\beta}d\tau_n\int d\mathbf{r}_1\cdots\int d\mathbf{r}_n$$
$$\times\sum_{\alpha_1=1}^{N}\cdots\sum_{\alpha_n=1}^{N}R^{(n)}(\mathbf{r}_1,\dots,\mathbf{r}_n)\,|\psi_{\alpha_1}(\mathbf{r}_1,\tau_1)|^2\cdots|\psi_{\alpha_n}(\mathbf{r}_n,\tau_n)|^2$$

In the replica limit $N\to0$ higher-order disorder cumulants are negligible: only $R^{(2)}(\mathbf{r})$ contributes yielding attractive disorder induced interaction.

- **Bogoliubov Background Method** $\quad \psi_\alpha(\mathbf{r},\tau)=\Psi_\alpha(\mathbf{r},\tau)+\delta\psi_\alpha(\mathbf{r},\tau)$
- **Hartree-Fock Theory [5] and Semiclassical Approximations**
- **Replica Symmetry** $\quad \Psi_\alpha(\mathbf{r},\tau)=\sqrt{n_0(\mathbf{r})}$

$$\langle\delta\psi_\alpha(\mathbf{r},\tau)\,\delta\psi_{\alpha'}(\mathbf{r}',\tau')\rangle=\mathcal{Q}\left(\mathbf{r}-\mathbf{r}',\tfrac{\mathbf{r}+\mathbf{r}'}{2},\tau-\tau'\right)\delta_{\alpha\alpha'}+q\left(\tfrac{\mathbf{r}+\mathbf{r}'}{2},\tau-\tau'\right)$$
$$n(\mathbf{r})=\Psi_\alpha(\mathbf{r},\tau)\Psi^*_\alpha(\mathbf{r},\tau)+\langle\delta\psi_\alpha(\mathbf{r},\tau)\,\delta\psi_\alpha(\mathbf{r},\tau)\rangle$$

Self-Consistency Equations

Decomposition of particle density $\quad n(\mathbf{r})=n_0(\mathbf{r})+q(\mathbf{r})+n_{\mathrm{th}}(\mathbf{r})$

$$\left\{-gn_0(\mathbf{r})+\left[\sqrt{-\mu+d^2+2gn(\mathbf{r})+V(\mathbf{r})}+d\right]^2-\frac{\hbar^2}{2M}\Delta\right\}\sqrt{n_0(\mathbf{r})}=0$$

$$q(\mathbf{r})=\frac{dn_0(\mathbf{r})}{\sqrt{-\mu+d^2+2gn(\mathbf{r})+V(\mathbf{r})}}$$

$$n_{\mathrm{th}}(\mathbf{r})=\left(\frac{M}{2\pi\beta\hbar^2}\right)^{3/2}\varsigma_{3/2}\left(e^{\beta\,[\mu-d^2-2gn(\mathbf{r})-V(\mathbf{r})]}\right)$$

where $R^{(2)}(\mathbf{r}_1-\mathbf{r}_2)=D\delta(\mathbf{r}_1-\mathbf{r}_2)$, $d=\sqrt{\pi}D\left(\frac{M}{2\pi\hbar^2}\right)^{3/2}$, $g=\frac{4\pi\hbar^2 a}{M}$

Anisotropic Trap $V(\mathbf{r})=\frac{M}{2}\left(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2\right)$ at $T=0$

- **Definitions**
  - Energy scale $\bar{\mu}=\frac{15^{2/5}}{2}\left(\frac{aN}{(l_xl_yl_z)^{1/3}}\right)^{2/5}\hbar\,(\omega_x\omega_y\omega_z)^{1/3}$
  - Length scales $l_i=\sqrt{\frac{\hbar}{M\omega_i}},\ i=x,y,z$
  - Trap aspect ratios $k=\omega_y/\omega_x$ and $\lambda=\omega_z/\omega_x$

- **Self-Consistency Equations in Thomas-Fermi Approximation**
  - **Superfluid Region:** $n_0(x,y,z)\neq0$ and $q(x,y,z)\neq0$
    - Condensate Density

$$n_0(x,y,z)=\frac{1}{g}\left[\sqrt{-\mu+d^2+2gn(x,y,z)+\tfrac{1}{2}M\omega_x^2\left(x^2+k^2y^2+\lambda^2z^2\right)}+d\right]^2$$

    - Depletion

$$q(x,y,z)=\frac{d\left[\sqrt{-\mu+d^2+2gn(x,y,z)+\tfrac{1}{2}M\omega_x^2\left(x^2+k^2y^2+\lambda^2z^2\right)}+d\right]^2}{g\sqrt{-\mu+d^2+2gn(x,y,z)+\tfrac{1}{2}M\omega_x^2\left(x^2+k^2y^2+\lambda^2z^2\right)}}$$

    - Particle Density

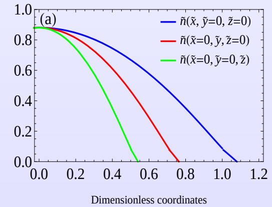
$$n(x,y,z)=\frac{\left[\sqrt{-\mu+d^2+2gn(x,y,z)+\tfrac{1}{2}M\omega_x^2\left(x^2+k^2y^2+\lambda^2z^2\right)}+d\right]^3}{g\sqrt{-\mu+d^2+2gn(x,y,z)+\tfrac{1}{2}M\omega_x^2\left(x^2+k^2y^2+\lambda^2z^2\right)}}$$

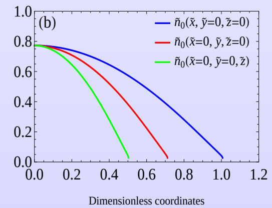
  - **Bose-glass Region:** $n_0(x,y,z)=0$ and $q(x,y,z)=n(x,y,z)\neq0$

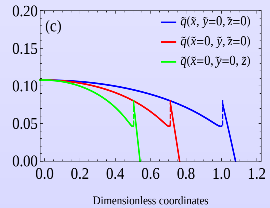
$$-\mu+d^2+2n(x,y,z)+x^2+k^2y^2+\lambda^2z^2=0$$

- **Densities**

Fig. 1: Dimensionless total density $\tilde{n}(\tilde{x},\tilde{y},\tilde{z})=n(x,y,z)/\bar{\mu}$ (a), dimensionless condensate $\tilde{n}_0(\tilde{x},\tilde{y},\tilde{z})=n_0(x,y,z)/\bar{\mu}$ (b), dimensionless condensate depletion $\tilde{q}(\tilde{x},\tilde{y},\tilde{z})=q(x,y,z)/\bar{\mu}$ in disconnected local minicondensates (c), along $\tilde{x}=\sqrt{\frac{M\omega_x^2}{2\bar{\mu}}}x$, $\tilde{y}=\sqrt{\frac{M\omega_x^2}{2\bar{\mu}}}y$ and $\tilde{z}=\sqrt{\frac{M\omega_x^2}{2\bar{\mu}}}z$ axis for ^{87}Rb, $N=10^6$, $\tilde{d}=\frac{d}{\sqrt{\bar{\mu}}}=0.107$, $\tilde{\mu}=\frac{\mu-d^2}{\bar{\mu}}=1.165$, $\omega_x=320\pi$ Hz, $k=\sqrt{2}$, $\lambda=2$, and $a=5.29$ nm.



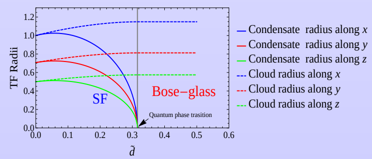




- **Thomas-Fermi Radii [6]**

Both condensate and cloud Thomas-Fermi radii along y and z axis are proportional to the ones along x axis with the proportionality constants $1/k$ and $1/\lambda$ respectively.

Fig. 2: Dimensionless condensate radii and dimensionless cloud radii along the $\tilde{x}=\sqrt{\frac{M\omega_x^2}{2\bar{\mu}}}x$, $\tilde{y}=\sqrt{\frac{M\omega_x^2}{2\bar{\mu}}}y$ and $\tilde{z}=\sqrt{\frac{M\omega_x^2}{2\bar{\mu}}}z$ as function of the dimensionless disorder strength $\tilde{d}=\frac{d}{\sqrt{\bar{\mu}}}$ for ^{87}Rb, $N=10^6$, $\omega_x=320\pi$ Hz, $k=\sqrt{2}$, $\lambda=2$, and $a=5.29$ nm.



- **Comparison with Huang-Meng Theory**

For weak disorder and at zero temperature, depletion is proportional to the disorder strength [1,2] $\quad q(0)=\frac{\sqrt{2}M^2}{8\pi^{3/2}\hbar^4}\sqrt{\frac{n_0(0)}{a}}D$

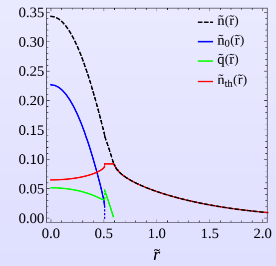
Isotropic Trap $V(\mathbf{r})=\frac{1}{2}M\Omega^2r^2$ at $T>0$

- **Definitions**
  - Energy scale $\mu_0=\frac{15^{2/5}}{2}\left(\frac{aN}{l}\right)^{2/5}\hbar\Omega$
  - Length scale $l=\sqrt{\frac{\hbar}{M\Omega}}$

- **Self-Consistency Equations in Thomas-Fermi Approximation**
  - **Superfluid Region:** $n_0(r)\neq0$, $q(r)\neq0$ and $n_{\mathrm{th}}(r)\neq0$
    - Condensate Density $\quad n_0(r)=\frac{1}{g}\left[\sqrt{-\mu+d^2+2gn(r)+\frac{1}{2}M\Omega^2r^2}+d\right]^2$
    - Depletion $\quad q(r)=\frac{dn_0(r)}{\sqrt{-\mu+d^2+2gn(r)+\frac{1}{2}M\Omega^2r^2}}$
    - Thermal Density $\quad n_{\mathrm{th}}(r)=\left(\frac{M}{2\pi\beta\hbar^2}\right)^{3/2}\varsigma_{3/2}\left(e^{\beta[\mu-d^2-2gn(r)-\frac{1}{2}M\Omega^2r^2]}\right)$
    - Particle Density $\quad n(r)=n_0(r)+q(r)+n_{\mathrm{th}}(r)$
  - **Bose-glass Region:** $n_0(r)=0$, $q(r)\neq0$ and $n_{\mathrm{th}}(r)\neq0$
    - Depletion $\quad q(r)=\frac{1}{2g}\left(\mu-d^2-\frac{1}{2}M\Omega^2r^2\right)-\left(\frac{M}{2\pi\beta\hbar^2}\right)^{3/2}\varsigma(3/2)$
    - Thermal Density $\quad n_{\mathrm{th}}(r)=\left(\frac{M}{2\pi\beta\hbar^2}\right)^{3/2}\varsigma(3/2)$
    - Particle Density $\quad n(r)=\frac{1}{2g}\left(\mu-d^2-\frac{1}{2}M\Omega^2r^2\right)$
  - **Thermal Region:** $n_0(r)=q(r)=0$ and $n_{\mathrm{th}}(r)=n_{\mathrm{th}}(r)\neq0$
    - Thermal Density $\quad n_{\mathrm{th}}(r)=\left(\frac{M}{2\pi\beta\hbar^2}\right)^{3/2}\varsigma_{3/2}\left(e^{\beta[\mu-d^2-2gn_{\mathrm{th}}(r)-\frac{1}{2}M\Omega^2r^2]}\right)$

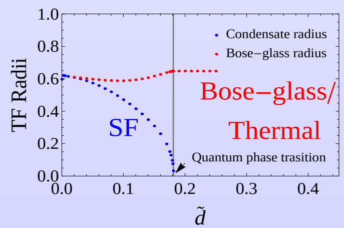
- **Densities**

Fig. 3: Dimensionless total density $\tilde{n}(\tilde{r})=\frac{gn(r)}{\mu_0}$ (dashed, black), dimensionless condensate $\tilde{n}_0(\tilde{r})=\frac{gn_0(r)}{\mu_0}$ (blue), dimensionless depletion $\tilde{q}(\tilde{r})=\frac{gq(r)}{\mu_0}$ (green) in disconnected local minicondensates, dimensionless thermal density $\tilde{n}_{\mathrm{th}}(\tilde{r})=\frac{n_{\mathrm{th}}(r)}{\mu_0}$ (red) as function of dimensionless radial coordinate $\tilde{r}=\sqrt{\frac{M\Omega^2}{2\mu_0}}r$, for ^{87}Rb, $N=10^6$, $\tilde{d}=\frac{d}{\sqrt{\mu_0}}=0.088$, $\tilde{\mu}=\frac{\mu-d^2}{\mu_0}=0.535$, $\Omega=100$ Hz, $T=60$ nK and $a=5.29$ nm.



- **Thomas-Fermi Radii [6]**

Fig. 4: Dimensionless condensate radius (blue) and dimensionless Bose-glass radius (red) as function of dimensionless disorder strength $\tilde{d}=\frac{d}{\sqrt{\mu_0}}$ for ^{87}Rb, $N=10^6$, $\Omega=100$ Hz, $T=60$ nK and $a=5.29$ nm.



- **Perspectives**
  - Beyond Thomas-Fermi approximation
  - Finite temperature with anisotropic trap potential
  - General interaction potential
  - Time dependence of densities and Thomas-Fermi radii [7]
  - Replica Symmetry Breaking?

- **Acknowledgment**

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