Critical Temperature of Dirty Bosons

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1. Finite-Size Effects of Bosons in Harmonic Trap

- Grand-canonical partition function:

$$Z = \oint \mathcal{D}\psi^* \oint \mathcal{D}\psi \; e^{-\mathcal{A}/\hbar}$$

$$\mathcal{A} = \int_0^{\hbar\beta} d\tau \int d^3x \; \psi^*(\mathbf{x},\tau) \left[\hbar \frac{\partial}{\partial\tau} + \hat{\mathrm{H}}(\mathbf{x}) - \mu\right] \psi(\mathbf{x},\tau)$$

$$\hat{\mathrm{H}}(\mathbf{x}) = -\frac{\hbar^2}{2M}\,\Delta + \frac{M}{2}\,\omega^2 \mathbf{x}^2$$

- Evaluation of functional integral [1]:

$$\int d^3x \; \psi_0^*(\mathbf{x})\psi(\mathbf{x},\tau) = 0 \quad \Rightarrow \quad \psi(\mathbf{x},\tau) = \sum_{\mathbf{n}\neq 0} \sum_{m=-\infty}^{\infty} c_{\mathbf{n},m} \psi_{\mathbf{n}}(\mathbf{x}) \; e^{-i\omega_m \tau}$$

$$\hat{\mathrm{H}}(\mathbf{x})\psi_{\mathbf{n}}(\mathbf{x}) = E_{\mathbf{n}}\psi_{\mathbf{n}}(\mathbf{x}) \,, \qquad \omega_m = \frac{2\pi}{\hbar\beta} m$$

- Effective potential:

$$\Gamma[\Psi^*,\Psi] = \frac{1}{\beta}\,\mathrm{Tr}\,\ln G_0^{-1} + \frac{1}{\hbar\beta}\,\mathcal{A}[\Psi^*,\Psi] \quad \overset{\text{extremalization}}{\Rightarrow} \quad \Omega(T,V,\mu) = \Gamma[\Psi_e^*,\Psi_e]$$

- Gross-Pitaevskii equation:

$$\left[\hbar \frac{\partial}{\partial\tau} + \hat{\mathrm{H}}(\mathbf{x}) - \mu\right] \Psi_e(\mathbf{x},\tau) = 0 \,, \quad \Psi_e(\mathbf{x},\tau) = \sqrt{N_0}\,\psi_0(\mathbf{x}) \quad \Rightarrow \quad \begin{cases} N_0 = 0 & \text{gas phase} \\ \mu = E_0 & \text{BEC phase} \end{cases}$$

- Particle number equation:

$$N = N_0 + \sum_{k=1}^{\infty} e^{\beta(\mu - E_0)} \left[\frac{1}{(1-e^{-\hbar\omega\beta k})^3} - 1\right]$$

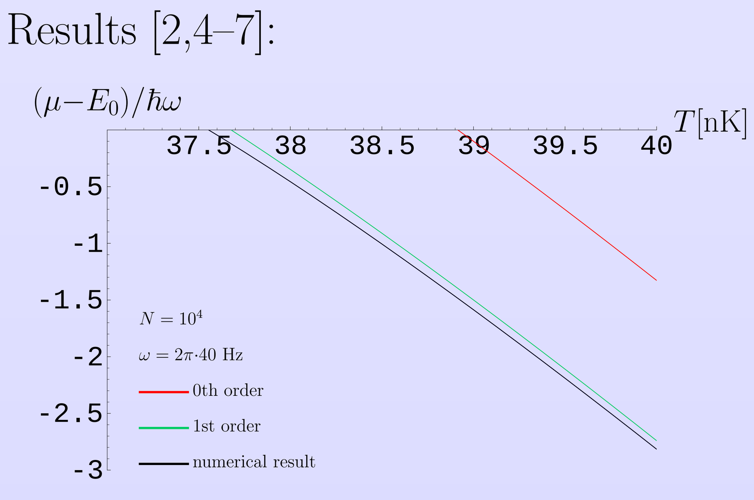
- Semiclassical approximation $0<b=\hbar\omega\beta\ll 1$ [2]:

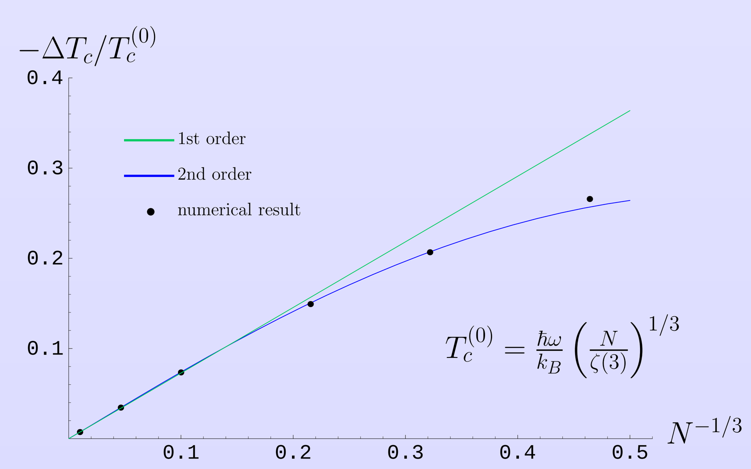
$$\sum_{k=1}^{\infty} \frac{e^{-Abk}}{(1-e^{-bk})^3} = \frac{1}{b^3}\zeta_3(e^{-Ab}) + \frac{3}{2b^2}\zeta_2(e^{-Ab}) + \frac{1}{b}\left\{\frac{1}{2}(A-1)(A-2)\left[\ln A - \psi_0(A-2)\right] - \frac{5}{4}A + \frac{53}{24} + \zeta_1(e^{-Ab})\right\} + \mathcal{O}(b^0)$$

- Robinson formula for natural indices l [3]:

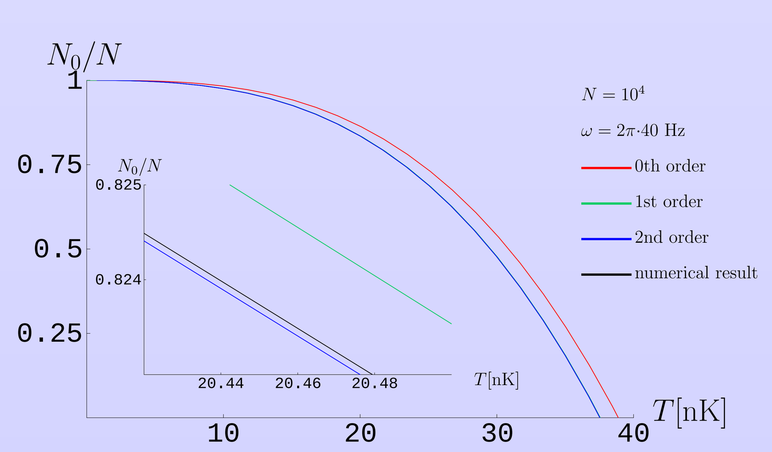
$$\zeta_l(e^{-a}) = \frac{(-a)^{l-1}}{(l-1)!}\left[\sum_{k=1}^{l-1}\frac{1}{k} - \ln a\right] + \sum_{\substack{k=0 \\ k\neq l-1}}^{\infty} \frac{(-a)^k}{k!}\zeta(l-k)$$

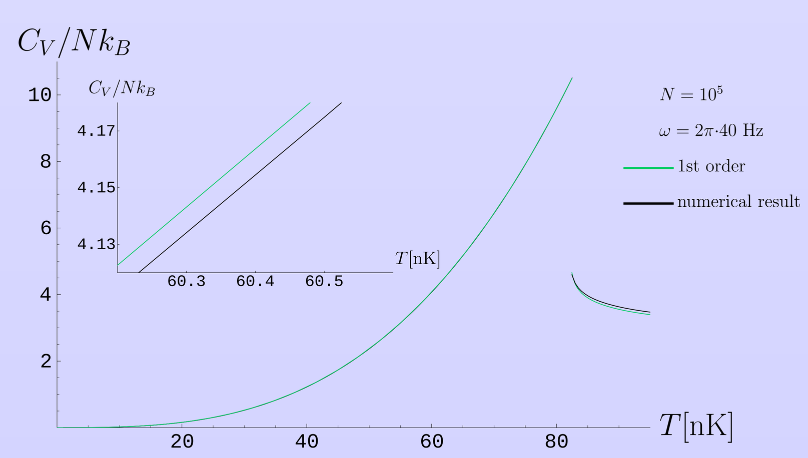
- Results [2,4–7]:





$$\frac{T_c}{T_c^{(0)}} = 1 - \left(\frac{\zeta(3)}{N}\right)^{\frac{1}{3}} \frac{\zeta(2)}{2\zeta(3)} - \left(\frac{\zeta(3)}{N}\right)^{\frac{2}{3}} \frac{1}{3\zeta(3)}\left[-\frac{1}{3}\ln\left(\frac{\zeta(3)}{N}\right) + \gamma - \frac{19}{24} - \frac{3\zeta(2)^2}{4\zeta(3)}\right] + \dots$$





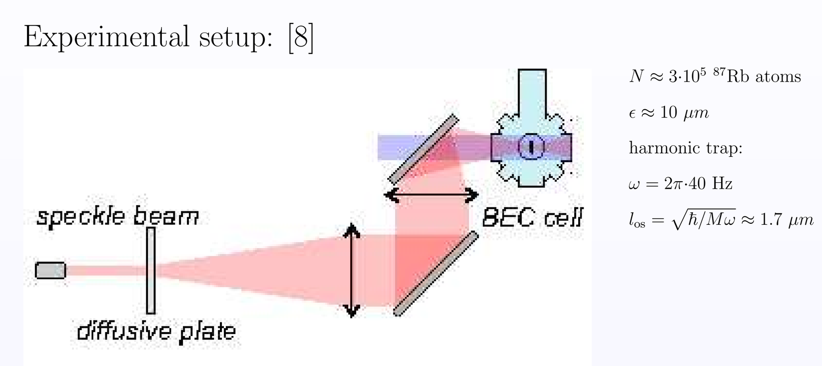
No divergence of heat capacity at critical point:

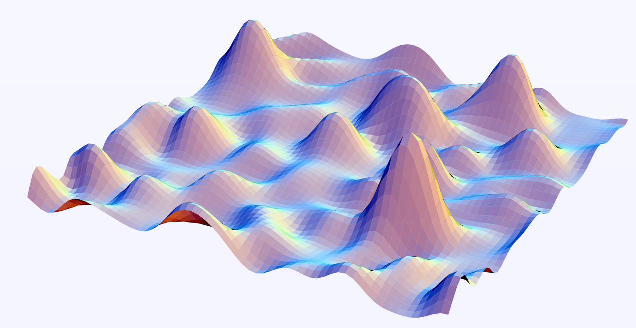
$$\lim_{T\uparrow T_c} C_{V,T\le T_c} = 3Nk_B\left[\frac{4\,\zeta(4)}{\zeta(3)} + \left(\frac{\zeta(3)}{N}\right)^{\frac{1}{3}}\left(3 - \frac{6\,\zeta(4)\zeta(2)}{\zeta(3)^2}\right)\right] + \dots$$

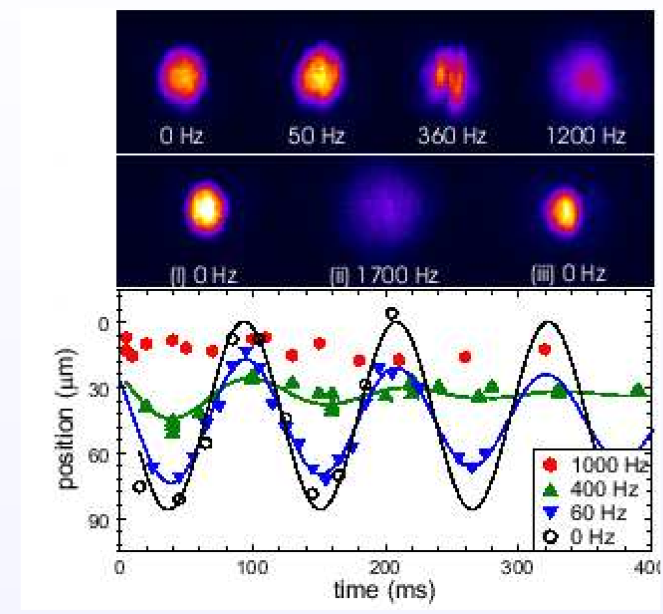
$$\lim_{T\downarrow T_c} C_{V,T>T_c} = 3Nk_B\left(\frac{4\zeta(4)}{\zeta(3)} - \frac{3\zeta(3)}{\zeta(2)} + \left(\frac{\zeta(3)}{N}\right)^{\frac{1}{3}}\left\{\frac{3}{2} - \frac{6\zeta(4)\zeta(2)}{\zeta(3)^2} + \frac{3\zeta(3)}{\zeta(2)^2}\left[-\frac{1}{2}\ln\left(\frac{\zeta(3)}{N}\right) + \frac{5}{4} + \zeta(2) + \frac{3}{2}\gamma\right]\right\}\right) + \dots$$

2. Critical Temperature of Dirty Bosons in Harmonic Trap

- Experimental realization: laser speckles [8,9]







Top: Density profiles of BEC for different speckle intensities after 28 ms of expansion.
Bottom: Dipole oscillations for varying speckle intensity.

- Properties of random potential $U(\mathbf{x})$ [6]:

Ensemble average: $\overline{\bullet} = \int \mathcal{D}U \bullet P[U] \,, \qquad \overline{1} = 1$

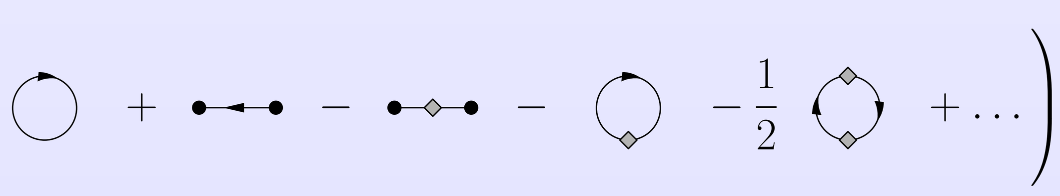
Assumptions: $\overline{U(\mathbf{x}_1)} = 0 \,, \qquad \overline{U(\mathbf{x}_1)U(\mathbf{x}_2)} = \frac{R}{(2\pi\epsilon^2)^{\frac{3}{2}}} e^{-\frac{(\mathbf{x}_1-\mathbf{x}_2)^2}{2\epsilon^2}}$

- Grand-canonical partition function:

$$Z = \oint \mathcal{D}\psi^* \oint \mathcal{D}\psi \; e^{-\mathcal{A}/\hbar}$$

$$\mathcal{A} = \int_0^{\hbar\beta} d\tau \int d^3x \; \psi^*(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau} + \frac{\hbar^2}{2M}\Delta + \frac{M}{2}\omega^2\mathbf{x}^2 + U(\mathbf{x}) - \mu\right]\psi(\mathbf{x},\tau)$$

- Effective potential: perturbative calculation

$\Gamma[\Psi^*,\Psi] = \frac{1}{\beta}\Big($  $\Big)$

$G_0(\mathbf{x}_1,\tau_1;\mathbf{x}_2,\tau_2) =$ 1 —►— 2 , $\quad \frac{-1}{\hbar}\int_0^{\hbar\beta} d\tau_1 \int d^3x_1 U(\mathbf{x}_1) =$ [vertex diagram 1] ,

$\Psi^*(\mathbf{x}_1,\tau_1) =$ ●— 1 , $\quad \Psi(\mathbf{x}_1,\tau_1) =$ 1 —●

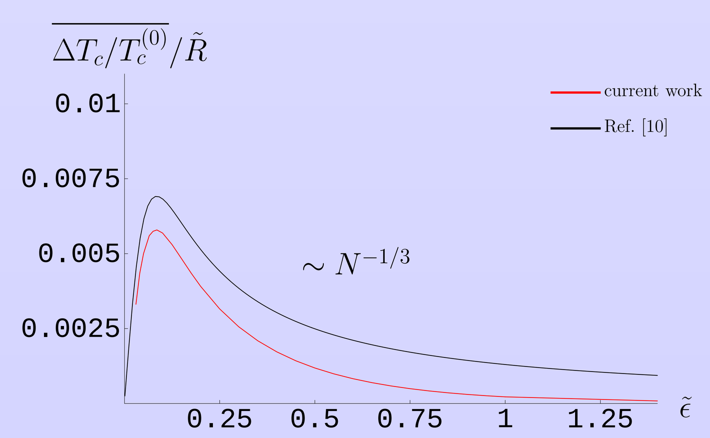
Extremalization: $\frac{\delta\Gamma[\Psi^*,\Psi]}{\delta\Psi^*(\mathbf{x},\tau)} = 0 \quad \Rightarrow \quad \Omega(T,V,\mu) = \Gamma[\Psi_e^*,\Psi_e]$

Disorder expansion: $\mu = \mu^{(0)} + \mu^{(1)} + \mu^{(2)} + \dots \,, \quad \Psi_e = \Psi_e^{(0)} + \Psi_e^{(1)} + \Psi_e^{(2)} + \dots$

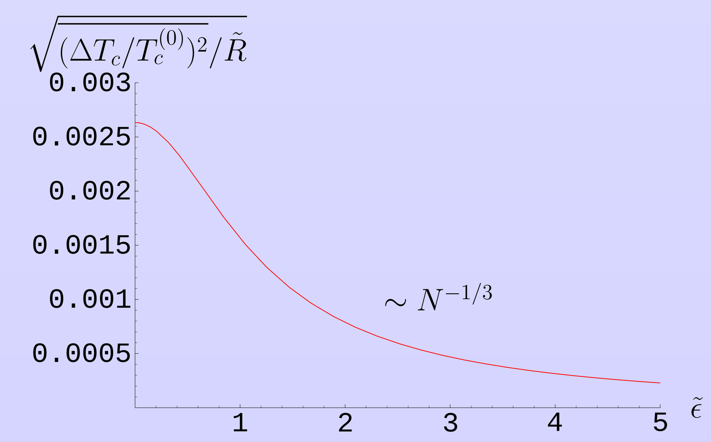
- Thermodynamic limit: $\omega \to 0$, $\epsilon \to \infty$, and $N \to \infty$

$$T_c^{(0)} \sim \omega \cdot N^{1/3} = \text{const.} \quad \Rightarrow \quad l_{os} = \sqrt{\frac{\hbar}{M\omega}} \sim N^{1/6} \,, \quad \tilde{\epsilon} = \frac{\epsilon}{l_{os}} = \text{const.} \quad \Rightarrow \quad \epsilon \sim N^{1/6}$$

- Results for $N = 10^5$ and $\tilde{R} = R/(\hbar\omega)^2\, l_{os}^3$:



Leading shift of critical temperature



Leading mean deviation

- Remark: interaction-induced shift of critical temperature experimentally resolved [11,12]

$$\frac{\Delta T_c}{T_c^{(0)}} = -3.426\,\frac{a}{\lambda_c^{(0)}}$$

a: s-wave scattering length, $\lambda_c^{(0)} = \sqrt{Mk_B T_c^{(0)}/2\pi\hbar^2}$: de Broglie wavelength

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