Breakdown of Kohn Theorem Near Feshbach Resonance

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Motivation We study the collective excitation modes of a harmonically trapped Bose-Einstein condensate (BEC) in the vicinity of a Feshbach resonance at zero temperature [1]. To this end we solve the underlying Gross-Pitaevskii equation by using a Gaussian variational approach and obtain the coupled set of ordinary differential equations for the widths and the center of mass of the condensate. A linearization shows that the dipole mode frequency [2] changes when the bias magnetic field approaches the Feshbach resonance.

Near Feshbach Resonance

- **Gross-Pitaevskii (GP) Equation**

⋆ At zero temperature, BEC can be described by GP equation [3,4]

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t)=\left[-\frac{\hbar^2}{2M}\Delta+V(\mathbf{r})+gN\left|\Psi(\mathbf{r},t)\right|^2\right]\Psi(\mathbf{r},t)$$

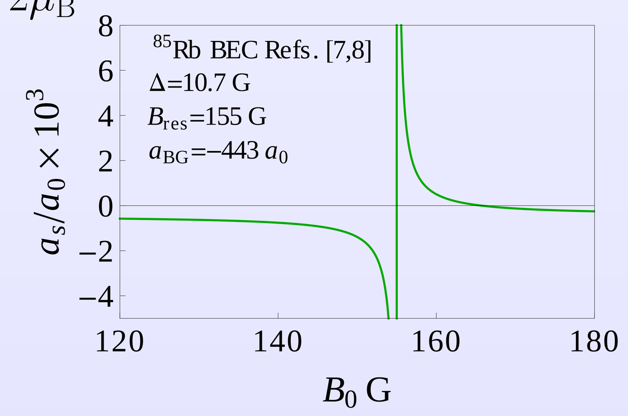
where $V(\mathbf{r})=V_0+\frac{1}{2}M\omega_\rho^2\left(\rho^2+\lambda^2z^2\right)$ is trap with trap aspect ratio λ and with bias potential $V_0=B_0\mu_\mathrm{B}$, and μ_B denotes the magnetic dipole moment and $g=4\pi\hbar^2a_s/M$ is two-body interaction strength.

⋆ Potential $V(\mathbf{r})$ generated by corresponding magnetic field in Ioffe-Pritchard trap [5]

$$B=B_0+\frac{M\omega_\rho^2}{2\mu_\mathrm{B}}\left(\rho^2+\lambda^2z^2\right)$$

- **Feshbach resonance [4,6]**

$$a_s=a_\mathrm{BG}\left[1-\frac{\Delta}{B-B_\mathrm{res}}\right]$$



⋆ Experimental values [7,8]: $N=4\times10^4$, $\omega_\rho=2\pi\times156$ Hz, $\omega_z=2\pi\times16$ Hz, and $\mu_\mathrm{B}=1$ Bohr magneton of the Hydrogen atom [9].

- **Problem**: How does Kohn mode frequency $\omega_D=\omega_z$ far away from the Feshbach resonance [2] changes near the Feshbach resonance?

Variational Approach

⋆ By using Gaussian variational ansatz [10–13], we obtain equations for condensate widths u_ρ, u_z and center of mass coordinate z_0:

$$\ddot{u}_\rho+\omega_\rho^2u_\rho-\frac{\hbar^2}{M^2u_\rho^3}-\sqrt{\frac{2}{\pi}}\frac{a_\mathrm{BG}N\hbar^2}{M^2u_zu_\rho^3}\left[1-\frac{16\Delta f}{\sqrt{2\pi}u_\rho^2u_z}+\frac{4\Delta}{\sqrt{2\pi}u_\rho u_z}\frac{\partial f}{\partial u_\rho}\right]=0$$

$$\ddot{u}_z+\lambda^2\omega_\rho^2u_z-\frac{\hbar^2}{M^2u_z^3}-\sqrt{\frac{2}{\pi}}\frac{a_\mathrm{BG}N\hbar^2}{M^2u_z^2u_\rho^2}\left[1-\frac{16\Delta f}{\sqrt{2\pi}u_\rho^2u_z}+\frac{8\Delta}{\sqrt{2\pi}u_\rho^2}\frac{\partial f}{\partial u_z}\right]=0$$

$$\ddot{z}_0+z_0\left[\lambda^2\omega_\rho^2-\frac{4\hbar^2Na_\mathrm{BG}\Delta}{\pi M^2u_\rho^4u_z^2}\frac{\partial f}{\partial z_0}\right]=0$$

⋆ Integral:

$$f=\int_0^\infty d\rho\int_{-\infty}^\infty dz\,\frac{\rho\exp\left[-2\rho^2/u_\rho^2-2(z-z_0)^2/u_z^2\right]}{B_0-B_\mathrm{res}+\frac{M\omega_\rho^2}{2\mu_\mathrm{B}}\left(\rho^2+\lambda^2z^2\right)}$$

⋆ Schwinger trick [14] and expansion up to second order of z_0 yields following integral representation with $\mathcal{H}=B_0-B_\mathrm{res}$

$$f=\int_0^\infty d\rho\int_{-\infty}^\infty dz\int_0^\infty ds\,\rho\left[1+\frac{4zz_0}{u_z^2}-\frac{2z_0^2}{u_z^2}+\frac{8z^2z_0^2}{u_z^4}+\mathcal{O}[z_0]^3\right]\times\exp\left\{\left[-\frac{2\rho^2}{u_\rho^2}-\frac{2z^2}{u_z^2}\right]-s\left[\mathcal{H}+\frac{M\omega_\rho}{2\mu_\mathrm{B}}\left(\rho^2+\lambda^2z^2\right)\right]\right\}$$

⋆ Equilibrium positions: $u_\rho=u_{\rho0}$, $u_z=u_{z0}$, $z_0=z_{00}=0$

⋆ Frequencies of collective modes follow from linearizing the equations of motion around equilibrium positions:

⋆ Dipole mode

$$\omega_D^2=\lambda^2\omega_\rho^2-\frac{4\hbar^2Na_\mathrm{BG}\Delta}{\pi M^2u_\rho^4u_z^2}\frac{\partial f}{\partial z_0}\bigg|_{z_0=0}$$

⋆ Breathing and quadrupole modes

$$\omega_{B,Q}^2=\frac{m_1+m_3\pm\sqrt{(m_1-m_3)^2+8m_2^2}}{2}$$

where m_1, m_2, and m_3 are calculated by using Mathematica.

On Top of Feshbach Resonance

⋆ Equations of motions in dimensionless form:

$$\ddot{u}_\rho+u_\rho-\frac{1}{u_\rho^3}-\frac{P_\mathrm{BG}}{u_zu_\rho^3}\left[1-\varepsilon\frac{\left(12u_z^2\lambda^2-16u_\rho^2\right)\mathrm{ArcSec}\left[\frac{u_z\lambda}{u_\rho}\right]}{u_\rho\left(u_z^2\lambda^2-u_\rho^2\right)^{3/2}}-\frac{4\varepsilon}{\left(u_z^2\lambda^2-u_\rho^2\right)}\right]=0$$

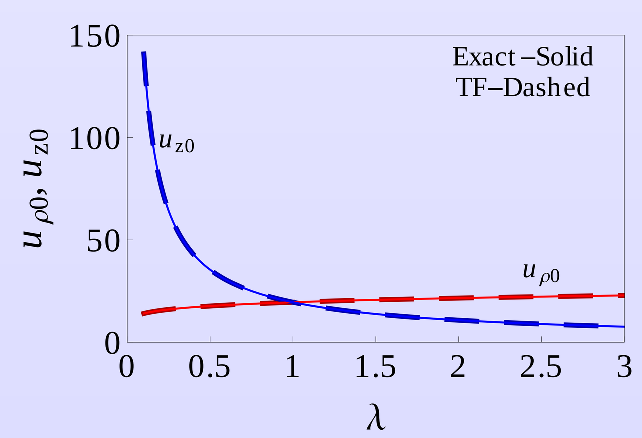
$$\ddot{u}_z+\lambda^2u_z-\frac{1}{u_z^3}-\frac{P_\mathrm{BG}}{u_z^2u_\rho^2}\left[1+\frac{8\varepsilon\left(u_\rho^2-2u_z^2\lambda^2\right)\mathrm{ArcSec}\left[\frac{u_z\lambda}{u_\rho}\right]}{u_\rho\left(u_z^2\lambda^2-u_\rho^2\right)^{3/2}}+\frac{8\varepsilon}{\left(u_z^2\lambda^2-u_\rho^2\right)}\right]=0$$

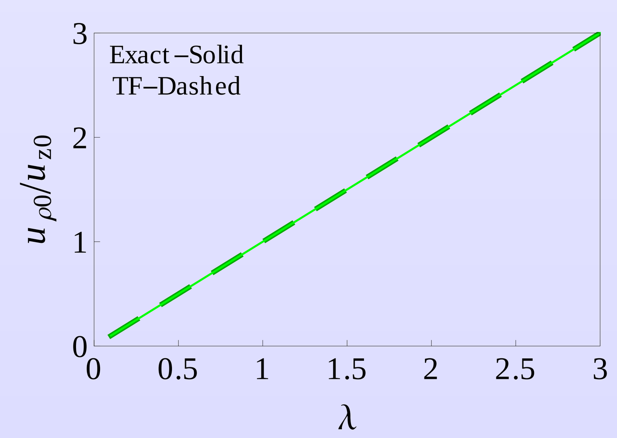
$$\ddot{z}_0+\lambda^2z_0+\frac{16\varepsilon P_\mathrm{BG}}{u_\rho^3u_z^2}\left[\frac{u_z\lambda^2\mathrm{ArcSec}\left[\frac{u_z\lambda}{u_\rho}\right]}{\left(u_\rho^2-u_z^2\lambda^2\right)^{3/2}}+\frac{u_\rho}{u_z\left(u_\rho^2-u_z^2\lambda^2\right)^2}\right]z_0=0$$

with $P_\mathrm{BG}=\sqrt{2/\pi}Na_\mathrm{BG}/l=-856.732$ and $\varepsilon=\frac{\Delta\mu_\mathrm{B}}{\hbar\omega_\rho}=0.096052\times10^6$.

⋆ Thomas-Fermi (TF) approximation:

$$u_{\rho0}^5-\lambda P_\mathrm{BG}\left(1-\frac{40\varepsilon}{3u_{\rho0}^2}\right)=0,\qquad u_{\rho0}=\lambda u_{z0}$$





⋆ Dipole mode frequency:

⋆ For $\varepsilon=0$: $\omega_D=\lambda$

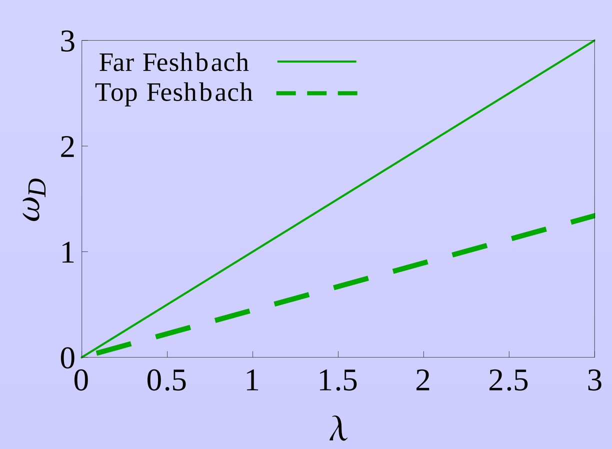
⋆ TF: $\omega_D^2=\lambda^2+\frac{32\varepsilon\lambda^3P_\mathrm{BG}}{3u_{\rho0}^7}$

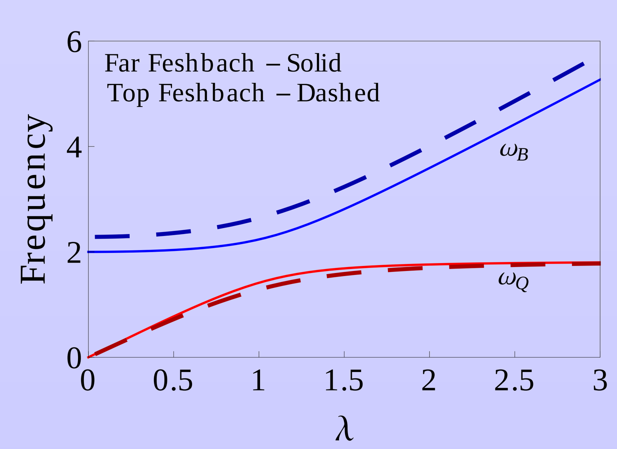
⋆ Breathing and quadrupole modes in the TF:

⋆ For $\varepsilon=0$: $\omega_{B,Q}^2=2+\frac{3}{2}\lambda^2\pm\frac{1}{2}\sqrt{16-16\lambda^2+9\lambda^4}$

⋆ TF $m_1=1+\frac{3\lambda P_\mathrm{BG}}{u_{\rho0}^5}\left(1-\frac{5401\varepsilon}{288u_{\rho0}^2}\right)$, $m_2=\frac{P_\mathrm{BG}\lambda^2}{u_{\rho0}^5}\left(1-\frac{2279\varepsilon}{96u_{\rho0}^2}\right)$,

$m_3=\lambda^2+\frac{2\lambda^3P_\mathrm{BG}}{u_{\rho0}^5}\left(1-\frac{1561\varepsilon}{96u_{\rho0}^2}\right)$





Right-Hand Side of Feshbach Resonance

⋆ Equations of motions in dimensionless form:

$$\ddot{u}_\rho+u_\rho-\frac{1}{u_\rho^3}-\frac{P_\mathrm{BG}}{u_zu_\rho^3}\left[1-16\varepsilon\,\varepsilon_1^{1/2}\int_0^\infty\frac{d\mathcal{S}\,e^{-\mathcal{S}}\left(2\varepsilon_1+\mathcal{S}u_\rho^2\right)}{\left(4\varepsilon_1+\mathcal{S}u_\rho^2\right)^2\sqrt{4\varepsilon_1+\mathcal{S}u_z^2\lambda^2}}\right]=0$$

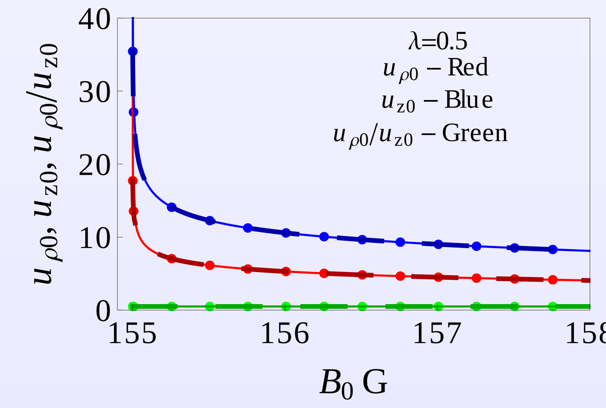
$$\ddot{u}_z+\lambda^2u_z-\frac{1}{u_z^3}-\frac{P_\mathrm{BG}}{u_z^2u_\rho^2}\left[1-16\varepsilon\,\varepsilon_1^{1/2}\int_0^\infty\frac{d\mathcal{S}\,e^{-\mathcal{S}}\left(2\varepsilon_1+\mathcal{S}u_z^2\lambda^2\right)}{\left(4\varepsilon_1+\mathcal{S}u_\rho^2\right)\left(4\varepsilon_1+\mathcal{S}u_z^2\lambda^2\right)^{3/2}}\right]=0$$

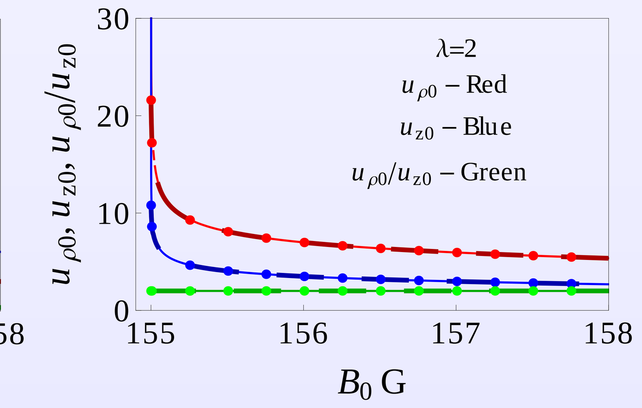
$$\ddot{z}_0+\lambda^2\left[1+\frac{16P_\mathrm{BG}}{u_\rho^2u_z}\varepsilon\,\varepsilon_1^{1/2}\int_0^\infty\frac{d\mathcal{S}\,e^{-\mathcal{S}}\mathcal{S}}{\left(4\varepsilon_1+\mathcal{S}u_\rho^2\right)\left(4\varepsilon_1+\mathcal{S}u_z^2\lambda^2\right)^{3/2}}\right]z_0=0$$

with $\varepsilon_1=\frac{\mathcal{H}\mu_\mathrm{B}}{\hbar\omega_\rho}$ and $\mathcal{S}=\mathcal{H}s$.

⋆ Thomas-Fermi (TF) approximation:

$$u_{\rho0}^5-P_\mathrm{BG}\lambda\left[1+\kappa_0(u_{\rho0},\varepsilon,\varepsilon_1)\right]=0,\quad\lambda u_{z0}=u_{\rho0}$$

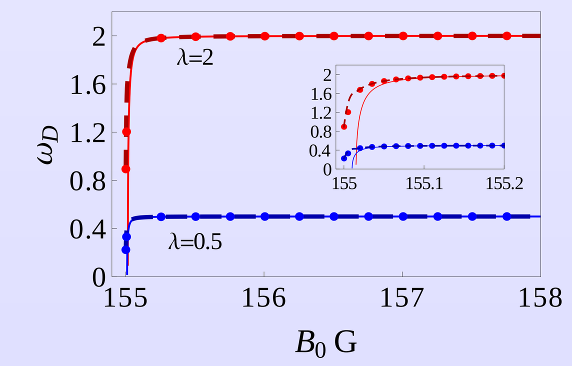




⋆ Dipole Mode:

⋆ Exact: $\omega_D^2=\lambda^2\left[1+16\varepsilon\,\varepsilon_1^{1/2}\frac{P_\mathrm{BG}}{u_{\rho0}^2u_{z0}}\int_0^\infty\frac{d\mathcal{S}\,e^{-\mathcal{S}}\mathcal{S}}{\left(4\varepsilon_1+\mathcal{S}u_{\rho0}^2\right)\left(4\varepsilon_1+\mathcal{S}u_{z0}^2\lambda^2\right)^{3/2}}\right]$

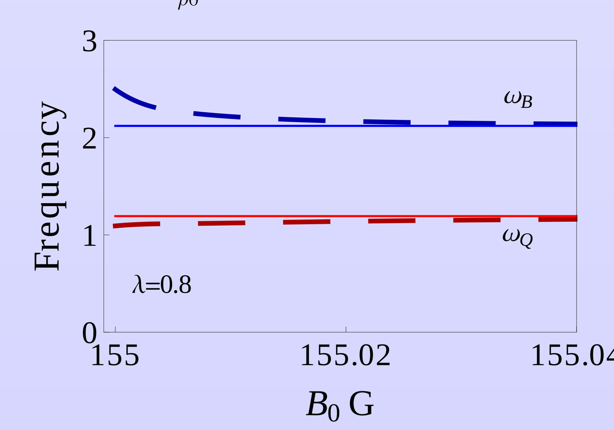
⋆ TF: $\omega_D^2=\lambda^2+\frac{32P_\mathrm{BG}\lambda^3\varepsilon\,\kappa_1(\rho0,\varepsilon,\varepsilon_1)}{3u_{\rho0}^{10}}$

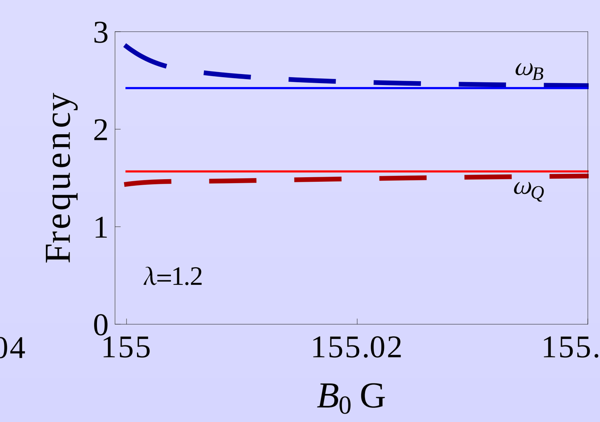


⋆ Breathing and quadrupole modes in the TF:

$m_1=1+\frac{3P_\mathrm{BG}\lambda}{u_{\rho0}^5}\left[1+\kappa_2(u_{\rho0},\varepsilon,\varepsilon_1)\right]$, $m_2=\frac{P_\mathrm{BG}\lambda^2}{u_{\rho0}^5}\left[1+\kappa_3(u_{\rho0},\varepsilon,\varepsilon_1)\right]$,

$m_3=\lambda^2+\frac{2P_\mathrm{BG}\lambda^3}{u_{\rho0}^5}\left[1+\kappa_4(u_{\rho0},\varepsilon,\varepsilon_1)\right]$





- **Note:** Figures above show exact result (dotted), approximation of Ref. [1] (solid), and TF approximation (dashed), respectively.

Summary and Outlook

⋆ We have studied in detail how dipole mode frequency changes on top of the Feshbach resonance and on the right-hand side of the Feshbach resonance.

⋆ Also quadrupole and breathing modes have been discussed.

⋆ We showed that Ref. [1] is not valid in the vicinity of the Feshbach resonance.

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