Degenerate Fermi Gases

Aristeu Lima[1], Kiel Howe[2], and Axel Pelster[1,3]

[1] Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, 14195 Berlin, Germany

[2] Department of Physics, University of Arizona, Tucson, AZ 85721, USA

[3] Fachbereich Physik, Universität Duisburg-Essen, Lotharstrasse 1, 47048 Duisburg, Germany

Spinor Fermi Gas

- **Ideal Spinor Fermi Gas**

  ⋆ Partition function

$$\mathcal{Z}^{(0)} = \oint \mathcal{D}\Psi^\dagger \oint \mathcal{D}\Psi \, \exp\left(-\mathcal{A}^{(0)}[\Psi^\dagger,\Psi]/\hbar\right)$$

  ⋆ Ensemble average

$$\langle \bullet \rangle = \frac{\oint \mathcal{D}\Psi^\dagger \oint \mathcal{D}\Psi \; \bullet \exp\left(-\mathcal{A}^{(0)}[\Psi^\dagger,\Psi]/\hbar\right)}{\oint \mathcal{D}\Psi^\dagger \oint \mathcal{D}\Psi \, \exp\left(-\mathcal{A}^{(0)}[\Psi^\dagger,\Psi]/\hbar\right)}$$

  ⋆ Free Action

$$\mathcal{A}^{(0)}[\Psi^\dagger,\Psi] = \int_0^{\hbar\beta} d\tau \int d^3x \, \psi_i^*(\mathbf{x},\tau)\mathcal{L}_{ij}\psi_j(\mathbf{x},\tau)$$

$$\mathcal{L}_{ij} = \left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2M}\Delta + U(\mathbf{x}) - \mu\right]\delta_{ij} - \eta F^z_{ij}$$

Harmonic trap: $U(\mathbf{x}) = \sum_{i=1}^{3} \frac{1}{2} M\omega_i^2 x_i^2$, Lagrange parameters

- **Perturbation Theory at T = 0: Contact Interaction [1,2]**

$$V^{(\text{contact})}_{ij i'j'}(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}') \sum c_n \left(\mathbf{F}_{ij}\cdot\mathbf{F}_{i'j'}\right)^n$$

- **First-Order Correction to Ground-State Energy**

  Example: $F = 3/2$, $E_F = \hbar\overline{\omega}(3N/2)^{1/3}$, $c_0 = \frac{-g_0+5g_2}{4}$, $c_1 = \frac{-g_0+g_2}{3}$

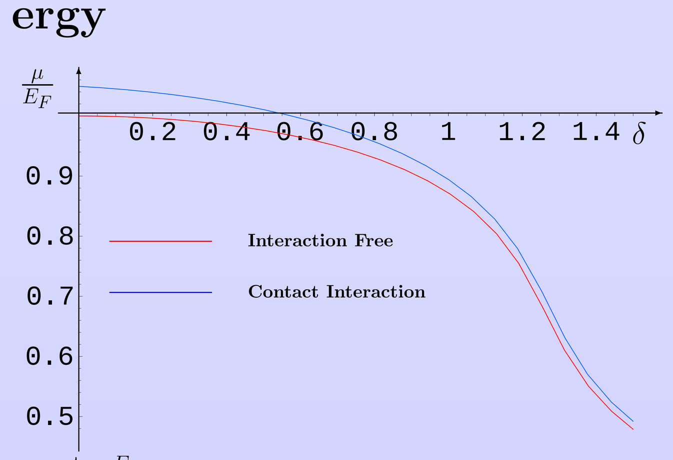
$$\frac{E}{NE_F} = \frac{3}{4} + 7\frac{N^{1/6}}{\sqrt{2\pi}}\left(\frac{3}{2}\right)^{7/6}\left\{\sum_{f,f'\neq f} A(y_<,y_>)\left[\frac{c_0'}{4\overline{a}^{\text{osc}}} + f(f'+1)\frac{c_1'}{3\overline{a}^{\text{osc}}}\right] + \frac{c_1'}{6\overline{a}^{\text{osc}}}\sum_{f=-3/2}^{3/2} \times\left[\left(\frac{3}{2}-f\right)\left(\frac{5}{2}+f\right)A(y_f,y_{f+1}) + \left(\frac{3}{2}+f\right)\left(\frac{5}{2}-f\right)A(y_{f-1},y_f)\right]\right\},$$

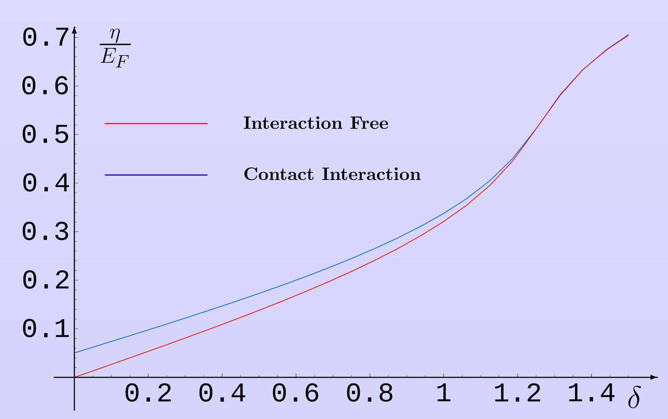
$$\overline{a}^{\text{osc}} = \sqrt{\hbar/M\overline{\omega}}, \quad \overline{\omega} = (\omega_x\omega_y\omega_z)^{1/3}, \quad c_0' = 5a_2 - a_0, \quad c_1' = a_2 - a_0$$

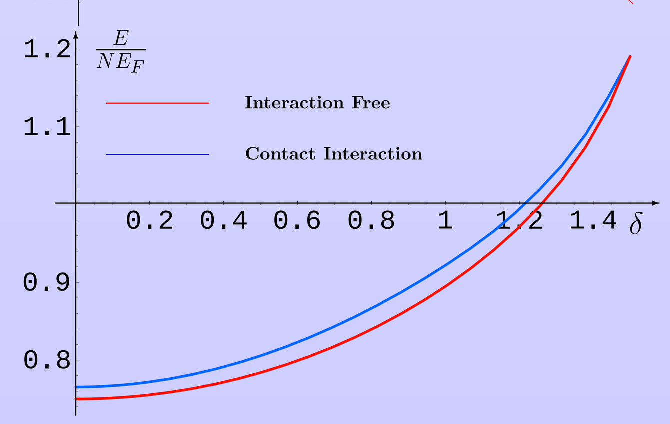
$$A(y_<,y_>) = {}_2F_1\left(-\tfrac{3}{2},\tfrac{3}{2};4;\tfrac{y_<}{y_>}\right)\frac{y_<^3}{\Gamma(4)}\frac{y_>^{3/2}}{\Gamma(5/2)}\Theta(y_<)$$

$y_<$ ($y_>$) is the smaller (larger) of y_f and $y_{f'}$; $y = \mu + f\eta$

- **Chemical, Magneto-Chemical Potential and Ground-State Energy**







$F = 3/2$, ^{53}Cr

$\delta = \langle\Psi^\dagger F^z\Psi\rangle / \langle\Psi^\dagger\Psi\rangle$

$\overline{\omega} \approx 2\pi \times 100$ Hz [3]

$c_0' \approx 10^{-9}$ m

$c_1' \approx 0$

$N \approx 10^5$ [4]

Polarized Dipolar Fermi Gas [5]

- **Thomas-Fermi Mean-Field Hamiltonian [6]**

$$E^{\text{TF}}[n] = \int d^3x \left\{\frac{\hbar^2}{20M\pi^2}[6\pi n(\mathbf{x})]^{5/3} + \frac{1}{2}M\omega_x^2\left(x^2+y^2+\gamma^2z^2\right)n(\mathbf{x}) + \frac{1}{2}\int d^3x' n(\mathbf{x})\frac{\mu_0 m^2}{4\pi|\mathbf{x}-\mathbf{x}'|^3}\left[1 - 3\frac{(z-z')^2}{|\mathbf{x}-\mathbf{x}'|^2}\right]n(\mathbf{x}')\right\}$$

- **Parabolic Ansatz**

$$n(\mathbf{x}) = \frac{8N}{\pi^2 R_x R_y R_z}\left(1 - \frac{x^2+y^2}{R_x^2} - \frac{z^2}{R_z^2}\right)^{3/2}$$

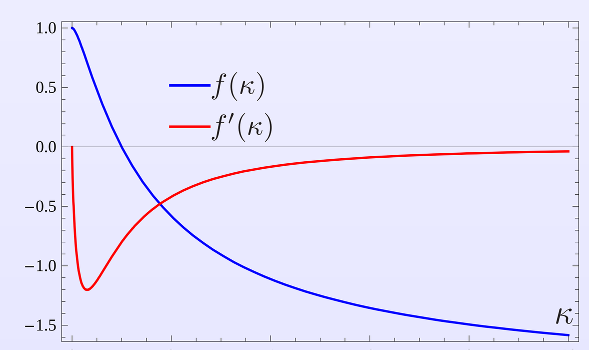
- **Thomas-Fermi Mean-Field Energy:** $\kappa = R_x/R_z$

$$E^{\text{TF}}(R_z,\kappa) = \frac{\hbar^2(48N)^{5/3}}{256M\kappa^{4/3}R_z^2} + \frac{MN\omega_x^2}{16}\left(2\kappa^2+\gamma^2\right) - \frac{32768\sqrt{2}\mu_0 m^2 N^2}{6615\pi^{3/2}\kappa^2 R_z^3}f(\kappa)$$

Deformation Function [7,8]

$$f(\kappa) = \frac{2+4\kappa^2-3\kappa^2\,\Xi(\kappa)}{2(1-\kappa^2)}$$

$$\Xi(\kappa) \equiv \begin{cases}\frac{2}{\sqrt{\kappa^2-1}}\arctan(\sqrt{\kappa^2-1}); & \kappa \geq 1\\ \frac{1}{\sqrt{1-\kappa^2}}\log\left(\frac{1+\sqrt{1-\kappa^2}}{1-\sqrt{1-\kappa^2}}\right); & \kappa < 1\end{cases}$$



- **Extremization of the MFE**

$$\frac{\partial E^{\text{TF}}(R_z,\kappa)}{\partial R_z} \equiv 0 \qquad \frac{\partial E^{\text{TF}}(R_z,\kappa)}{\partial\kappa} \equiv 0$$

- **Thomas Fermi Radii**

$$R_z = \frac{\sqrt{2}(6N)^{1/6}}{\gamma^{5/6}}\sqrt{\frac{\hbar}{M\omega_x}}W(\kappa,\gamma)^{1/4}$$

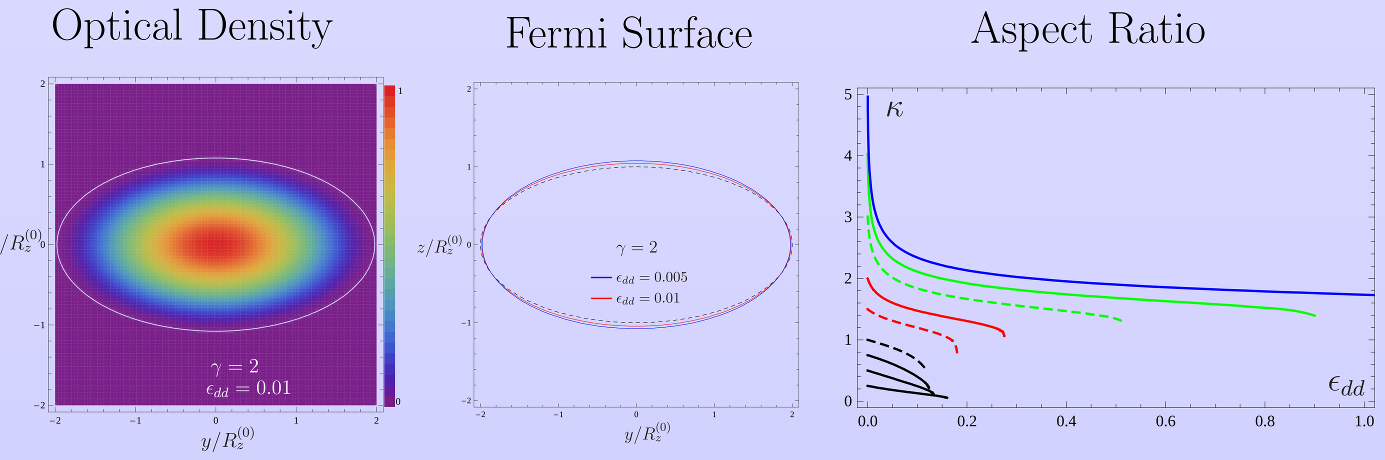
$$[W(\kappa,\gamma)]^{5/4} - \left(\frac{\gamma}{\kappa}\right)^{10/3}[W(\kappa,\gamma)]^{1/4} + \frac{2\epsilon_{dd}\gamma^{25/6}}{\kappa^4}[2f(\kappa)+f'(\kappa)] = 0$$

$$W(\kappa,\gamma) = \left(\frac{\gamma}{\kappa}\right)^{10/3}\frac{3\kappa^2 f'(\kappa) - 6\kappa^2(\kappa^2-1)f(\kappa)}{(2\kappa^2+\gamma^2)f'(\kappa) - f(\kappa)\left(6\kappa^4-4\kappa^2-2\gamma^2\right)}$$

$$\epsilon_{dd} = \frac{8192}{19845\pi^{3/2}}\frac{M(6N)^{1/6}\mu_0 m^2}{\hbar^2 a_x^{\text{osc}}}$$

- **Gas Geometry**

Optical Density — Fermi Surface — Aspect Ratio



- **Outlook**
  - ⋆ Magnetostriction
  - ⋆ Collective Oscillations [9]
  - ⋆ Anisotropy [10,11]
  - ⋆ Release Energy
  - ⋆ Expansion [12]

Polarized Rotating Fermi Gas [13]

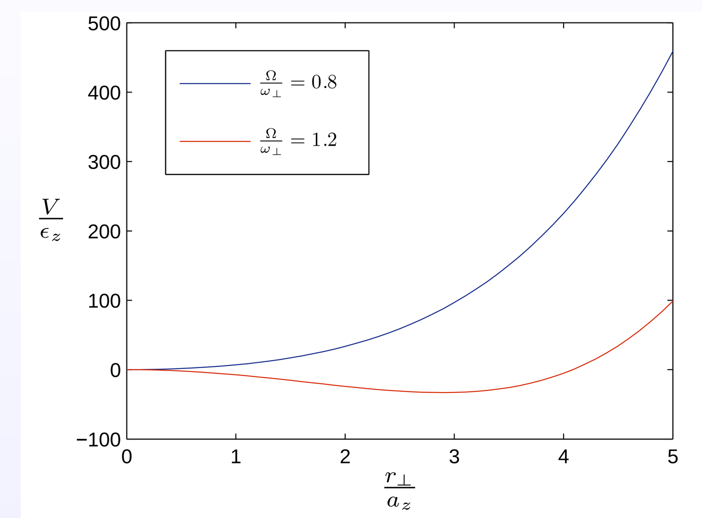
- **Anharmonic Trap [14–16]**

Potential in Rotating Frame

$$V(\mathbf{x}) = \frac{\epsilon_z}{2}\left(\eta\frac{r_\perp^2}{\gamma^2 a_z^2} + \frac{z^2}{a_z^2} + \frac{\kappa r_\perp^4}{2\,a_z^4}\right)$$

$$\epsilon_z = \hbar\omega_z \quad a_z = \sqrt{\hbar/M\omega_z}$$
$$\gamma = \omega_z/\omega_\perp \quad \kappa = ka_z^4/\epsilon_z$$
$$\eta = 1 - \Omega^2/\omega_\perp^2$$

$\Omega \rightarrow$ Rotation Frequency



- **Semiclassical Density of States**

$$g(E) = \frac{1}{6}\begin{cases}\frac{4(E+\Delta E)^{3/2}}{\epsilon_z^{5/2}\kappa^{1/2}} - \frac{3\eta E}{\gamma^2\epsilon_z^2\kappa} - \frac{2\eta\Delta E}{\gamma^2\epsilon_z^2\kappa}; & E \geq 0\\ \frac{8(E+\Delta E)^{3/2}}{\epsilon_z^{5/2}\kappa^{1/2}}; & -\Delta E \leq E \leq 0\end{cases} \qquad \Delta E = \frac{\eta^2\epsilon_z}{4\gamma^4\kappa}$$

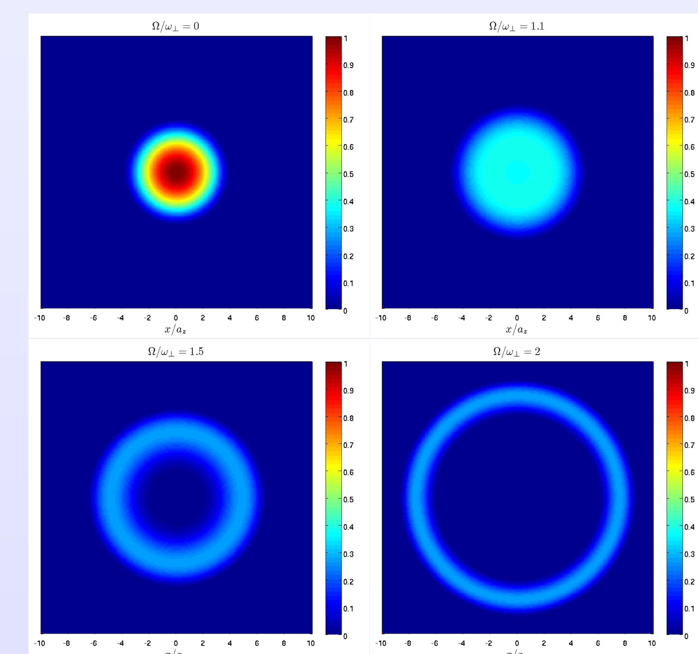
- **Particle Density** ($T = 0$)

Rotating-frame Density

$$n(\mathbf{x}) = \frac{\sqrt{2}\left[E_F - V(\mathbf{x})\right]^{3/2}}{3\pi^2 a_z^3\epsilon_z^{3/2}}$$

Donut Regime [17]

$$\Omega_{\text{DO}} = \omega_\perp\sqrt{1 + \gamma^2(60\kappa^3 N)^{1/5}}$$

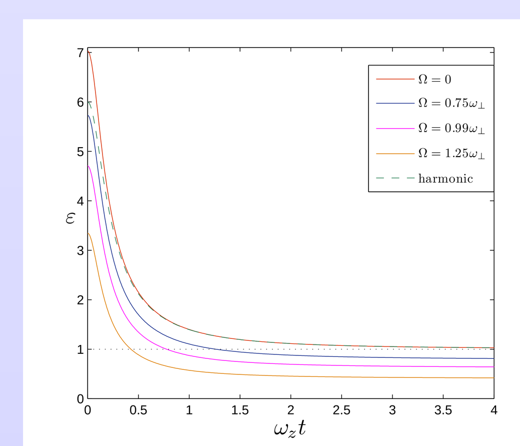


- **TOF Expansion** ($T = 0$)

Ballistic Substitution: $(\mathbf{x},\mathbf{p}) \rightarrow (\mathbf{x} - \frac{\mathbf{p}t}{M}, \mathbf{p})$

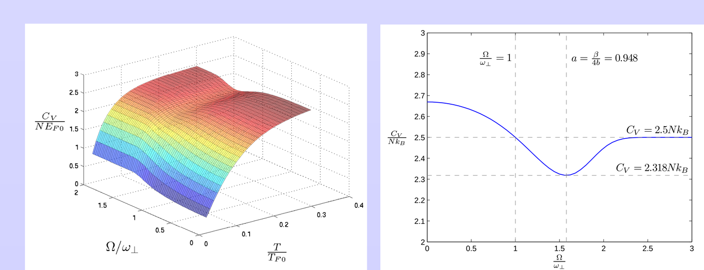
Aspect Ratio $\varepsilon(t) = \frac{W_z(t)}{W_\perp(t)}$

$$W_i(t) = \left[\int d^3x x_i^2 n(\mathbf{x},t)\right]^{1/2}$$



- **Heat Capacity**

$$\Omega_{\text{dip}} = \omega_\perp\sqrt{1 + \gamma^2\left(\frac{3.792\kappa k_B T}{\epsilon_z}\right)^{1/2}}$$



- **Outlook**
  - ⋆ Contact Interaction
  - ⋆ Dipole-Dipole Interaction
  - ⋆ Quantum Hall Regime [18]
  - ⋆ Fractional Quantum Hall Regime [19]

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