TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Theory of Condensed Matter and Many Body Systems

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Weakly Interacting Bose Gas on Spheres and Ellipsoids

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CAPES SFB/TR 185 DAAD OPTIMAS

Abstract

- We explore how to describe theoretically a weakly interacting Bose gas on a sphere.
- We determine the critical temperature of a Bose gas on a sphere and its dependence on the particle number.
- In order to derive the corresponding many-body field theory we consider a radial harmonic trap, which confines the three-dimensional Bose gas in the vicinity of the surface of a sphere or of an ellipsoid [1].

Laplace Beltrami operator

- The Laplace Beltrami operator is a generalization of the Laplacian for curved manifolds.
- Laplacian:

$$\Delta u = \text{div}(\text{grad } u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

- Laplace-Beltrami:

$$\Delta_{\text{LB}} u = D_\mu (D^\mu u) = g^{\mu\nu} \partial_\mu (\partial_\nu u) + \Gamma^\mu_{\mu\nu} (\partial^\nu u),$$

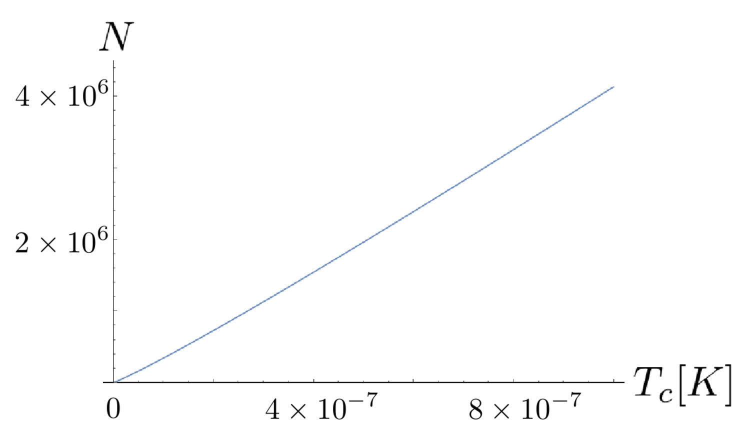
where $g_{\mu\nu}$ is the metric of the manifold and $\Gamma^\mu_{\mu\nu} = \frac{1}{2} g^{\mu\lambda} \partial_\nu g_{\lambda\mu}$ are the Christoffel symbols.

Two dimensional sphere

- Hamiltonian for a free particle on a sphere

$$\hat{H} = \frac{-\hbar^2}{2M} \Delta_{\text{LB}} = \frac{1}{2MR^2} \hat{\mathbf{L}}^2.$$

- Eigenfunctions: Legendre polynomials $Y_{lm}(\theta, \phi)$.
- Energies: $E_{lm} = \frac{\hbar^2 l(l+1)}{2MR^2}$
- Number of particles at critical temperature: $N = \sum_{l=1}^{\infty} \frac{2l+1}{e^{\frac{\tilde{T}}{T_c} l(l+1)} - 1}$ where $\tilde{T} = \frac{\hbar^2}{2k_B M R^2}$ is a temperature scale.
- For atoms of Rubidium ^{87}Rb on a sphere of 30μm radius, we have the following values.



- In the limit of an infinite radius, *i.e.* when the sphere goes over into a plane, we recover the Mermin-Wagner-Hohenberg theorem: "a Bose gas in a uniform infinite system with dimension $d \leq 2$ does not exhibit BEC at finite temperature" [2, 3].

Quasi two dimensional sphere

- Potential and trial function in three dimensions

$$V(\mathbf{r}) = \frac{M\omega^2}{2} (r - R)^2$$

$$\Psi(r, \theta, \varphi, t) = \frac{1}{\sqrt[4]{\pi} R} \frac{e^{-(r-R)^2/2l(\theta,\varphi,t)^2}}{\sqrt{l(\theta,\varphi,t)}} \psi(\theta, \varphi, t)$$

- Action:

$$\int dt d\mathbf{r} \Psi^* \left(i\hbar\partial_t + \frac{\hbar^2}{2m} \Delta - V(\mathbf{r} - U(\theta,\phi) - \frac{1}{2} g N |\Psi|^2) \right) \Psi$$

- Minimizing the action $\Rightarrow$ Gross-Pitaevskii equation.
- We will consider that $\hat{\mathbf{L}}^2 \psi$ is much larger than $\hat{\mathbf{L}}^2 l$.
- Integrating the action on r and then minimizing it:

$$i\hbar\partial_t \psi = \frac{\hat{\mathbf{L}}^2 \psi}{2mR^2} + \frac{\hbar^2 \psi}{4ml^2} + \frac{m\omega^2 l^2 \psi}{4} + U\psi + \frac{gN|\psi|^2}{\sqrt{2\pi} l R^2},$$

$$\frac{\hbar}{2ml^3} - \frac{m\omega^2 l}{2} + \frac{gN|\psi|^2}{2\sqrt{2\pi} l^2 R^2} = 0.$$

Quasi two dimensional sphere without interactions

- Gaussian length coincides with harmonic oscillator length $l = l_{\text{osc}} = \sqrt{\frac{\hbar}{m\omega}}$ and

$$i\hbar\partial_t \psi = \frac{\hat{\mathbf{L}}^2 \psi}{2mR^2} + \frac{\hbar\omega\psi}{2} + U\psi.$$

Quasi two dimensional ellipsoid

- Let us consider ellipsoids with surface equation given by $\frac{x^2}{a^2} + \frac{y^2}{a^2} + z^2 = R^2$.
- Following the same arguments as for a sphere, we get the following equations for ellipsoids.

$$i\hbar\partial_t \psi = -\frac{\hbar^2 \Delta_{\text{ell}} \psi}{2mR^2} + \frac{\hbar^2 \psi}{4ml^2} (\sin^2\theta + a^2 \cos^2\theta) + \frac{m\omega^2 l^2 \psi}{4} + U\psi + \frac{gN|\psi|^2}{\sqrt{2\pi} l R^2},$$

$$\frac{\hbar}{2ml^3} (\sin^2\theta + a^2 \cos^2\theta) - \frac{m\omega^2 l}{2} + \frac{gN|\psi|^2}{2\sqrt{2\pi} l^2 R^2} = 0.$$

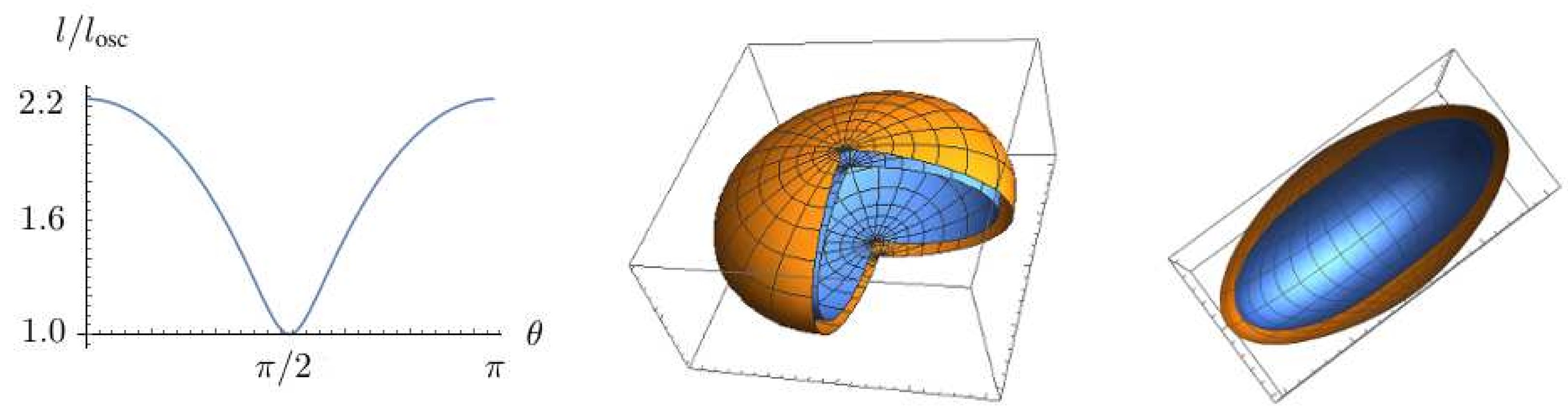
Quasi two dimensional ellipsoid without interactions

- Gaussian length proportional to the harmonic oscillator length $l = \sqrt[4]{1 + (a^2 - 1)\cos^2\theta}\, l_{\text{osc}}$ and

$$i\hbar\partial_t \psi = -\frac{\hbar^2 \Delta_{\text{ell}} \psi}{2mR^2} + \frac{\hbar\omega}{2} \sqrt{1 + (a^2 - 1)\cos^2\theta}\, \psi + U(\theta, \varphi)\psi.$$

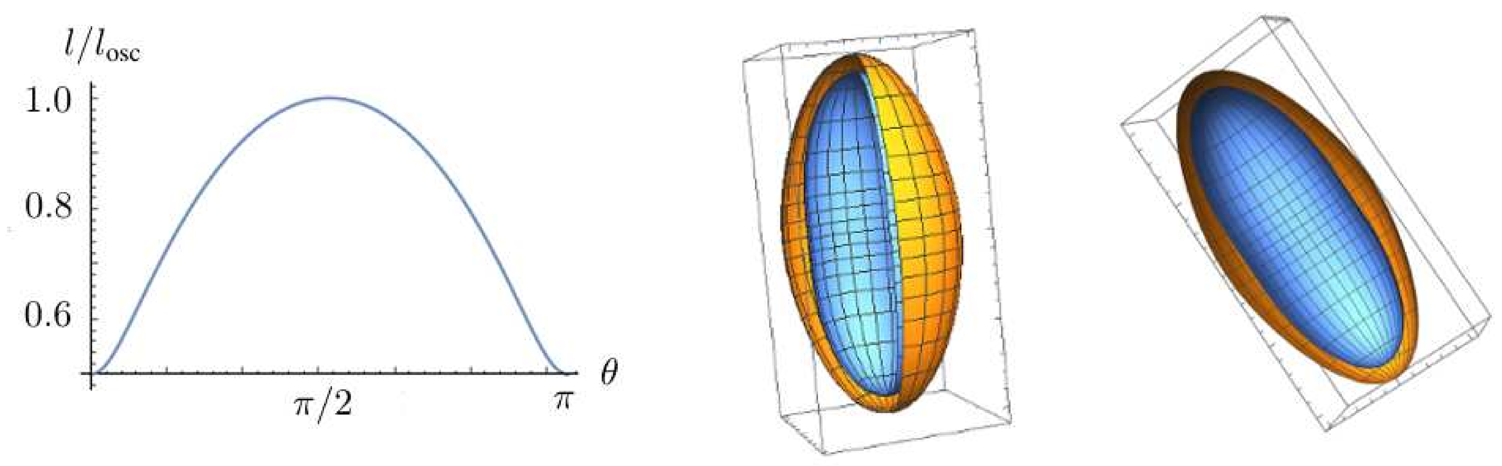
Oblate ellipsoid

- If $a = 5$ the equation for the Gaussian length becomes $l = \sqrt[4]{1 + 24\cos^2\theta}\, l_{\text{osc}}$.



Prolate ellipsoid

- If $a = 0.2$ the equation for the Gaussian length becomes $l = \sqrt[4]{1 - 0.96\cos^2\theta}\, l_{\text{osc}}$.



Conclusions and perspectives

Conclusions

- We derived equations for the wave functions for particles confined on the surface of a sphere or of an ellipsoid.
- The behaviour on the sphere is symmetric, but on the ellipsoid we can obtain an angular dependence.

Perspectives

- Apply the derived equations to obtain properties of the quantum gases on curved manifolds.
- Derive the full consistency equations by including angular derivatives of l.

References

[1] L. Salasnich, A. Parola, and L. Reatto, Phys. Rev. A **65**, 043614 (2002)

[2] N. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966).

[3] P. Hohenberg, Phys. Rev. **158**, 383 (1967).