Ginzburg-Landau theory for the Jaynes-Cummings Hubbard model

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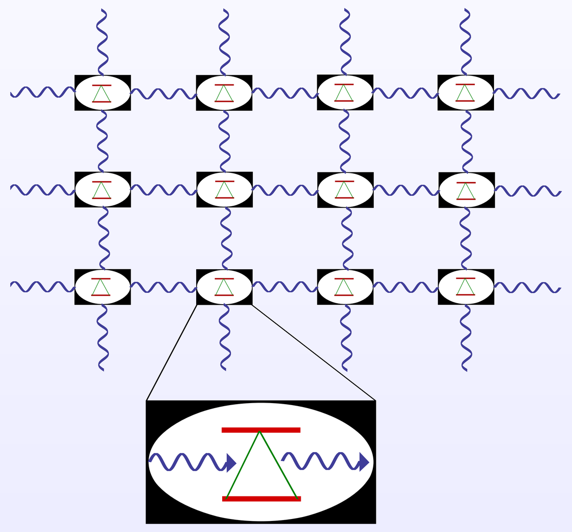
Jaynes-Cummings-Hubbard (JCH) Model [1]

- **Physical assumptions**
  - ⋆ periodical structure built of microcavities
  - ⋆ each cavity contains two-level system and monochromatic photon field
  - ⋆ coupled photonic and atomic excitations (polaritons) are conserved
  - ⋆ Hubbard-like photon hopping between neighboring cavities
  - ⋆ working in the strong-coupling regime $\frac{g^2}{\gamma\lambda} \gg 1$



- **JCH Hamiltonian in grand-canonical ensemble**

$$\hat{\mathcal{H}}^{\rm JCH} = \sum_i \left(\hat{\mathcal{H}}_i^{\rm JC} - \mu \hat{\rm N}_i\right) - \kappa \sum_{\langle i,j\rangle} \hat{\rm a}_i^\dagger \hat{\rm a}_j + \sum_i \left[j_i(\tau)\hat{\rm a}_i^\dagger + j_i^*(\tau)\hat{\rm a}_i\right]$$

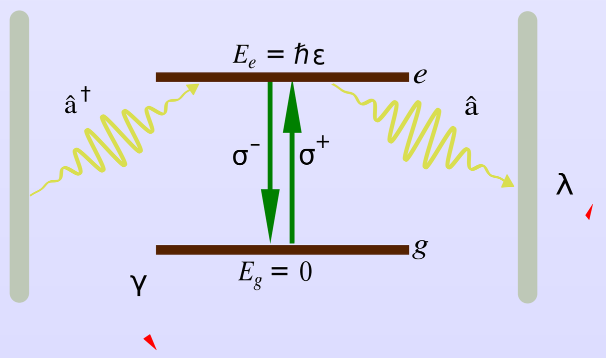
including source currents $j_i(\tau), j_i^*(\tau)$ to break any global symmetry

- **Jaynes-Cummings Hamiltonian [2]**

$$\hat{\mathcal{H}}_i^{\rm JC} = \omega\,\hat{\rm N}_i + \Delta\,\hat{\sigma}_i^+\hat{\sigma}_i^- + g\left(\hat{\rm a}_i^\dagger\hat{\sigma}_i^- + \hat{\rm a}_i\,\hat{\sigma}_i^+\right)$$

polariton number: $\hat{\rm N}_i = \hat{\rm a}_i^\dagger\hat{\rm a}_i + \hat{\sigma}_i^+\hat{\sigma}_i^-$

detuning parameter: $\Delta = \varepsilon - \omega$



- **Partition function in Dirac picture** $(\hbar = 1)$

$$\mathcal{Z}[j,j^*] = {\rm tr}\left\{\hat{\rm U}_{\rm D}(\beta,0)\,e^{-\beta\hat{\mathcal{H}}_0}\right\}, \quad \hat{\rm U}_{\rm D}(\beta,0) = \hat{\rm T}e^{-\int_0^\beta d\tau\,\hat{\mathcal{H}}_{\rm 1D}(\tau)}$$

unperturbed system: $\hat{\mathcal{H}}_0 = \sum_i \left[(\omega-\mu)\,\hat{\rm N}_i + \Delta\,\hat{\sigma}_i^+\hat{\sigma}_i^- + g\left(\hat{\rm a}_i^\dagger\hat{\sigma}_i^- + \hat{\rm a}_i\,\hat{\sigma}_i^+\right)\right]$

perturbation: $\hat{\mathcal{H}}_{\rm 1D}(\tau) = -\kappa\sum_{\langle i,j\rangle}\hat{\rm a}_{i\rm D}^\dagger(\tau)\hat{\rm a}_{j\rm D}(\tau) + \sum_i\left[j_i^*(\tau)\hat{\rm a}_{i\rm D}(\tau) + j_i(\tau)\hat{\rm a}_{i\rm D}^\dagger(\tau)\right]$

- **Free energy functional**

$$\mathcal{F}[j,j^*] = -\frac{1}{\beta}\ln\mathcal{Z}[j,j^*], \quad \mathcal{F}_0 = -\frac{1}{\beta}\ln\mathcal{Z}_0, \quad \mathcal{Z}_0 = {\rm tr}\left\{e^{-\beta\hat{\mathcal{H}}_0}\right\}$$

perform transformation to Matsubara space

$$g(\omega_m) = \frac{1}{\sqrt{\beta}}\int_0^\beta d\tau\, g(\tau)\,e^{i\omega_m\tau}, \qquad \omega_m = \frac{2\pi m}{\beta}$$

Ginzburg-Landau Theory [3, 4]

- **Ginzburg-Landau order parameter**

$$\psi_i(\omega_m) = \langle\hat{a}_i(\omega_m)\rangle_0 = \beta\,\frac{\delta\mathcal{F}}{\delta j_i^*(\omega_m)}, \quad \psi_i^*(\omega_m) = \langle\hat{a}_i^\dagger(\omega_m)\rangle_0 = \beta\,\frac{\delta\mathcal{F}}{\delta j_i(\omega_m)}$$

- **Legendre transformation to effective action**

$$\Gamma\left[\psi_i(\omega_m),\psi_i^*(\omega_m)\right] = \mathcal{F} - \frac{1}{\beta}\sum_i\sum_{\omega_m}\left[\psi_i(\omega_m)\,j_i^*(\omega_m) + \psi_i^*(\omega_m)\,j_i(\omega_m)\right]$$

- **Physical equations of motion**

$$j_i(\omega_m) = -\beta\,\frac{\delta\Gamma}{\delta\psi_i^*(\omega_m)} \overset{!}{=} 0, \quad j_i^*(\omega_m) = -\beta\,\frac{\delta\Gamma}{\delta\psi_i(\omega_m)} \overset{!}{=} 0$$

- **Since $\hat{\mathcal{H}}_0$ is local use cummulant expansion [5]**

$$\mathcal{F}[j,j^*] \approx \mathcal{F}_0 - \frac{1}{\beta}\int_0^\beta d\tau_1 d\tau_2\left[\sum_{i,j} a_2^{(0)}(\tau_1,\tau_2)\,\delta_{ij} + \kappa\sum_{\langle i,j\rangle} a_2^{(1)}(\tau_1,\tau_2)\right] j_i(\tau_1)j_j^*(\tau_2)$$

$$\Gamma\left[\psi_i(\omega_m),\psi_i^*(\omega_m)\right] \approx \mathcal{F}_0 + \frac{1}{\beta}\sum_{\omega_m}\left[\sum_{i,j}\frac{\delta_{ij}}{a_2^{(0)}(\omega_m)} - \kappa\sum_{\langle i,j\rangle}\right]\psi_i(\omega_m)\,\psi_j^*(\omega_m)$$

- **Explicit form of Ginzburg-Landau coefficient $a_2^{(0)}$**

$$a_2^{(0)}(\omega_m) = \sum_{\alpha,\alpha'=\pm1}\sum_{n=0}^{\infty}\frac{e^{-\beta E_{n,\alpha}}}{\beta\,\mathcal{Z}_0}\left\{\left[\frac{e^{\beta\left(E_{n,\alpha}-E_{n+1,\alpha'}+i\omega\right)}-1}{\left(E_{n,\alpha}-E_{n+1,\alpha'}+i\omega\right)^2} - \frac{\beta}{E_{n,\alpha}-E_{n+1,\alpha'}+i\omega}\right]\left(t_{(n+1)\alpha'\alpha}\right)^2 + (1-\delta_{n,0})\left[\frac{e^{\beta\left(E_{n,\alpha}-E_{n-1,\alpha'}-i\omega\right)}-1}{\left(E_{n,\alpha}-E_{n-1,\alpha'}-i\omega\right)^2} - \frac{\beta}{E_{n,\alpha}-E_{n-1,\alpha'}-i\omega}\right]\left(t_{n\alpha\alpha'}\right)^2\right\}$$

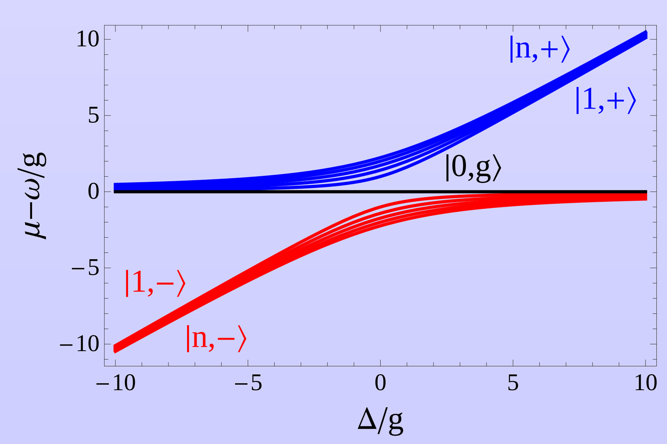
- **Energies of unperturbed system**

$$E_{n,\alpha} = (\omega-\mu)\,n + \frac{1}{2}\left(\Delta + \alpha\sqrt{\Delta^2 + 4\,g^2\,n}\right)$$

transition factors $t_{n\alpha\alpha'}$

$$t_{n\pm-} = \sqrt{n}\,a_{n\pm}\,b_{n-1+} + \sqrt{n-1}\,b_{n\pm}\,b_{n-1-}$$
$$t_{n\pm+} = \sqrt{n}\,a_{n\pm}\,a_{n-1+} + \sqrt{n-1}\,b_{n\pm}\,a_{n-1-}$$
$$\begin{pmatrix} a_{n+} & b_{n+} \\ a_{n-} & b_{n-}\end{pmatrix} = \begin{pmatrix}\sin\theta_n & \cos\theta_n \\ \cos\theta_n & -\sin\theta_n\end{pmatrix}$$

mixing angle: $\theta_n = \frac{1}{2}\arctan\left(\frac{2g\sqrt{n}}{\Delta}\right)$



Results

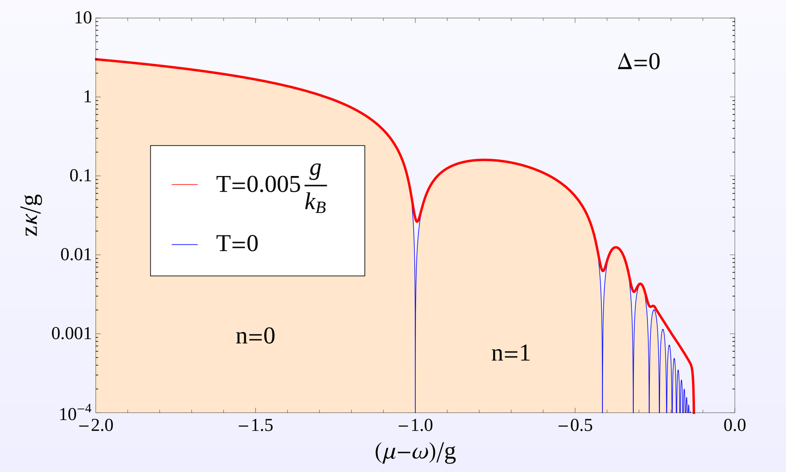
- **Phase boundary for quantum phase transition [6, 7]**

obtained from 2nd order coefficient

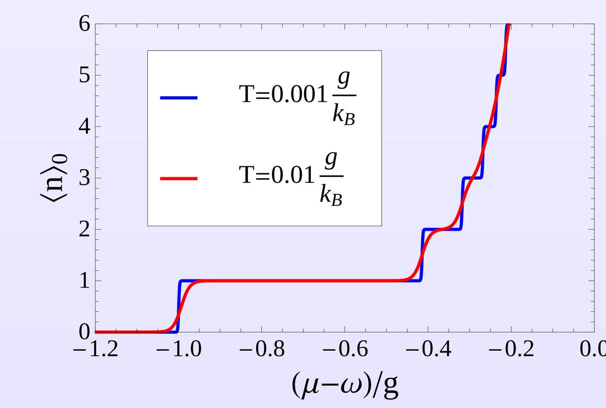
$$\sum_{\omega_m}\left[\sum_{i,j}\frac{\delta_{i,j}}{a_2^{(0)}(\omega_m)} - \kappa\sum_{\langle i,j\rangle}\right]\psi_i(\omega_m) \overset{!}{=} 0$$

equilibrium: $\psi_i(\omega_m) = \frac{1}{\sqrt{\beta}}\,\psi\,\delta_{m,0}$

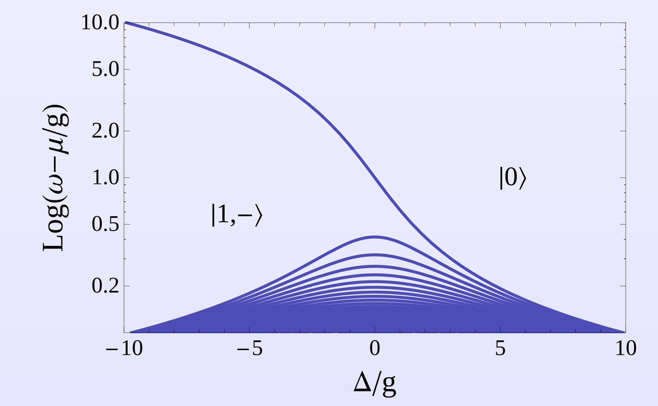
yields: $\left[\frac{1}{a_2^{(0)}(0)} - \kappa z\right]\psi \overset{!}{=} 0$



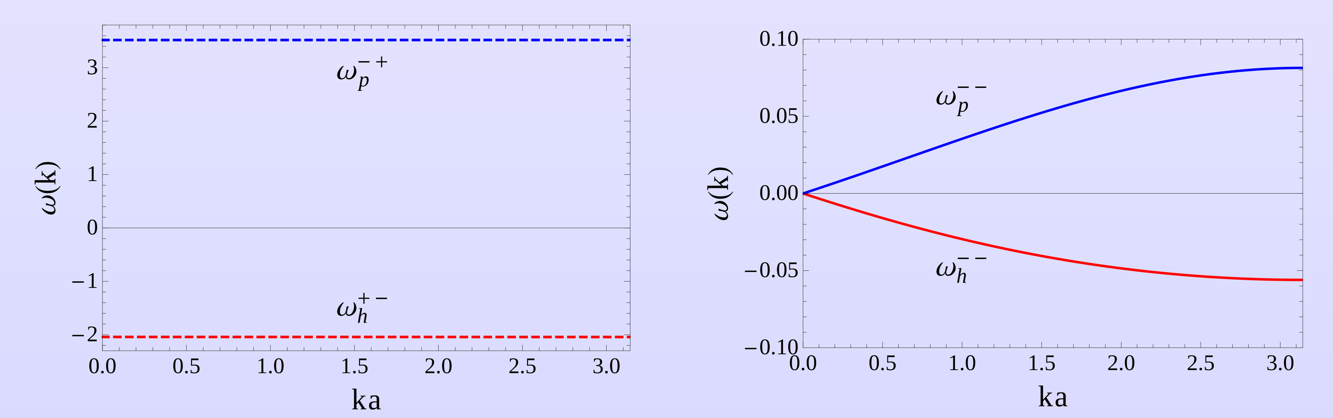
- **Polariton number** $(\Delta = 0)$



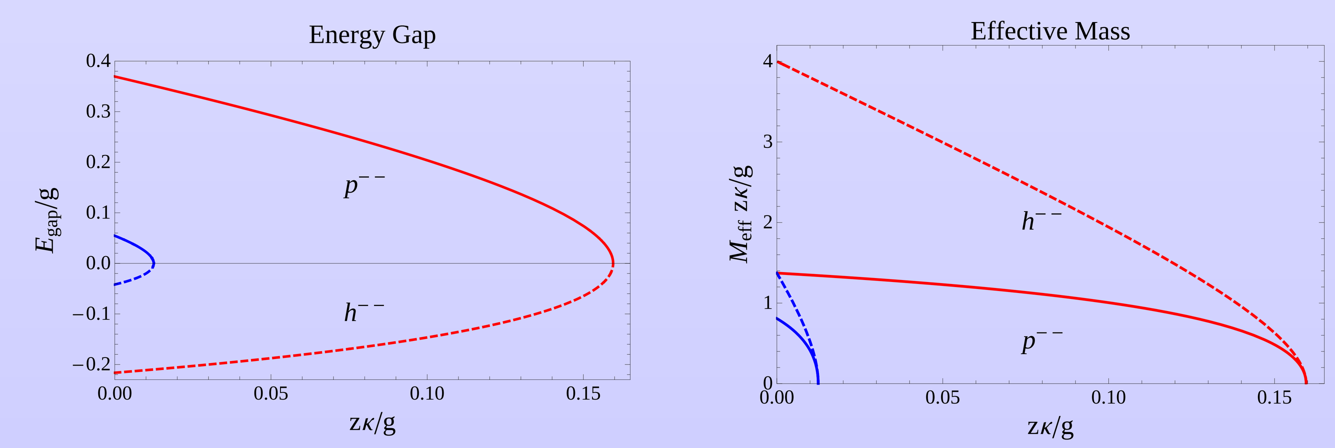
critical chemical potential



- **Excitation spectra in Mott phase:** $\Delta = 0,\ T = 0,\ n = 2$ (tip of the lobe)



- **Effective mass and energy gap [8]:** $\Delta = 0,\ T = 0$ (tip of the lobe)



red line for $n = 1$, blue line for $n = 2$

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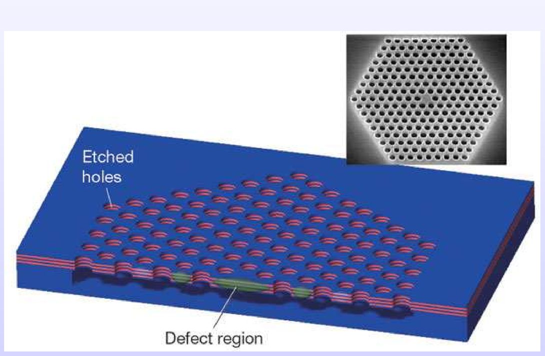
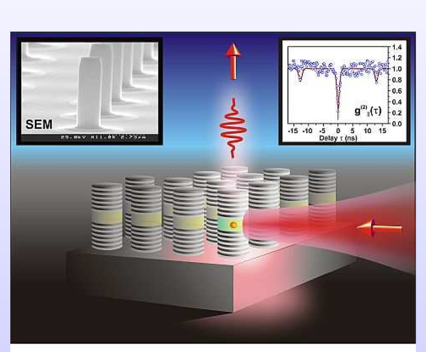
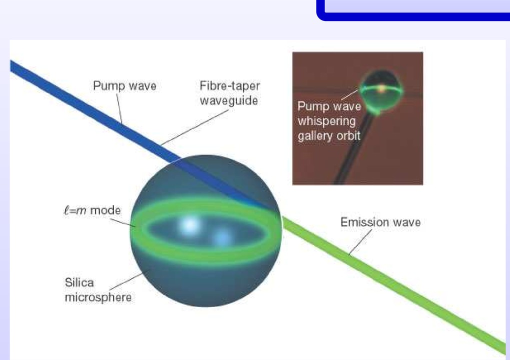
Possible Experimental Realizations



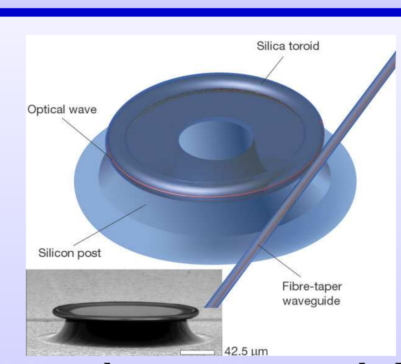
Quantum dots coupled to photonic band gap defect microcavities [9, 10]



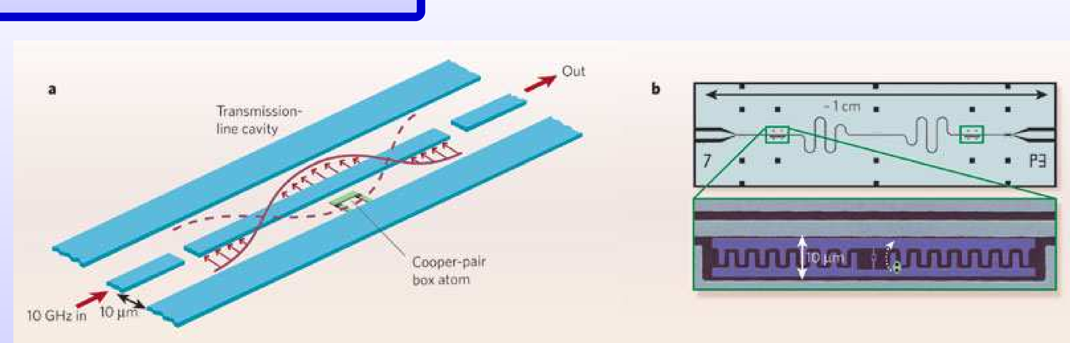
Array of pillar microcavities coupled to quantum dots [11, 12]



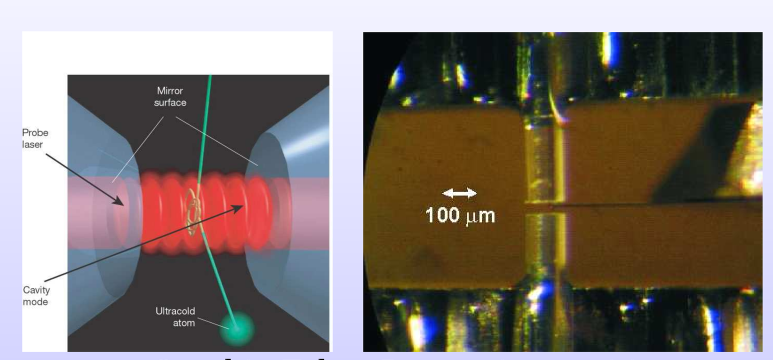
Microsphere cavities doped with nanocrystals [13, 14]



Microdisc cavities coupled to quantum dots [15, 16]



Stripline resonators coupled to Cooper-pair boxes [17, 18]



On-chip Fabry-Perot microcavities coupled with trapped ions [19, 20]

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References

[1] M. J. Hartmann, F. G. S. L. Brandão, and M. B. Plenio. Strongly interacting polaritons in coupled arrays of cavities. *Nature Physics*, 2:849–855, 2006.
[2] F. W. Cummings and E. T. Jaynes. Comparison of quantum and semiclassical radiation theories with application to the beam maser. *Proceedings of the IEEE*, 51:89 – 109, 1963.
[3] F. E. A. dos Santos and A. Pelster. Quantum phase diagram of bosons in optical lattices. *Physical Review A*, 79:013614, 2009.
[4] B. Bradlyn, F. E. A. dos Santos, and A. Pelster. Effective Action Approach for Quantum Phase Transitions in Bosonic Lattices. *Physical Review A*, 79:13615, 2008.
[5] W. Metzner. Linked-cluster expansion around the atomic limit of the Hubbard model. *Physical Review B*, 43:8549–8563, 1991.
[6] J. Koch and K. Le Hur. Superfluid-Mott-insulator transition of light in the Jaynes-Cummings lattice. *Physical Review A*, 80:023811, 2009.
[7] S. Schmidt and G. Blatter. Excitations of Strongly Correlated Lattice Polaritons. *Physical Review Letters*, 104:216402, 2010.
[8] S. Schmidt and G. Blatter. Strong Coupling Theory for the Jaynes-Cummings-Hubbard Model. *Physical Review Letters*, 103:086403, 2009.
[9] M. Bayindir, B. Temelkuran, and E. Ozbay. Propagation of photons by hopping: A waveguiding mechanism through localized coupled cavities in three-dimensional photonic crystals. *Physical Review B*, 61:R11855–R11858, 2000.
[10] B.-S. Song, S. Noda, T. Asano, and Y. Akahane. Ultra-high-Q photonic double-heterostructure nanocavity. *Nature Materials*, 4:207–210, 2005.
[11] J. M. Gerard, D. Barrier, J. Y. Marzin, R. Kuszelewicz, L. Manin, E. Costard, V. Thierry-Mieg, and T. Rivera. Quantum boxes as active probes for photonic microstructures: The pillar microcavity case. *Applied Physics Letters*, 69:449–451, 1996.
[12] M. Kahl, T. Thomay, V. Kohnle, K. Beha, J. Merlein, M. Hagner, A. Halm, J. Ziegler, T. Nann, Y. Fedutik, U. Woggon, M. Artemyev, F. Pérez-Willard, A. Leitenstorfer, and R. Bratschitsch. Colloidal quantum dots in all-dielectric high-Q pillar microcavities. *Nano letters*, 7:2897–2900, 2007.
[13] M. Cai, O. Painter, K. J. Vahala, and P. C. Sercel. Fiber-coupled microsphere laser. *Optics letters*, 25:1430–1432, 2000.
[14] B. Möller, U. Woggon, and M. V. Artemyev. Photons in coupled microsphere resonators. *Journal of Optics A: Pure and Applied Optics*, 8:S113–S121, 2006.
[15] P. E. Barclay, K. Srinivasan, O. Painter, B. Lev, and H. Mabuchi. Integration of fiber-coupled high-Q SiN[sub x] microdisks with atom chips. *Applied Physics Letters*, 89:131108, 2006.
[16] S. M. Spillane, T. J. Kippenberg, and K. J. Vahala. Ultrahigh- Q toroidal microresonators for cavity quantum electrodynamics. *Physical Review A*, 71:013817, 2005.
[17] J. M. Fink, M. Göppl, M. Baur, R. Bianchetti, P. J. Leek, A. Blais, and A. Wallraff. Climbing the Jaynes-Cummings ladder and observing its nonlinearity in a cavity QED system. *Nature*, 454:315–318, 2008.
[18] A. Wallraff, D. I. Schuster, A. Blais, L. Frunzio, R. S. Huang, J. Majer, S. Kumar, S. M. Girvin, and R. J. Schoelkopf. Strong coupling of a single photon to a superconducting qubit using circuit quantum electrodynamics. *Nature*, 431:162–167, 2004.
[19] Y. Colombe, T. Steinmetz, G. Dubois, F. Linke, D. Hunger, and J. Reichel. Strong atom-field coupling for Bose-Einstein condensates in an optical cavity on a chip. *Nature*, 450:272–276, 2007.
[20] M. Trupke, E. A. Hinds, S. Eriksson, E. A. Curtis, Z. Moktadir, E. Kukharenka, and M. Kraft. Microfabricated high-finesse optical cavity with open access and small volume. *Applied Physics Letters*, 87:211106, 2005.