Critical Properties of Bose-Einstein Condensates

Konstantin Glaum[1], Parvis Soltan-Panahi[1], Axel Pelster[2], and Hagen Kleinert[1]
[1]Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, 14195 Berlin
[2]Fachbereich Physik, Campus Essen, Universität Duisburg-Essen, Universitätsstrasse 5, 45117 Essen

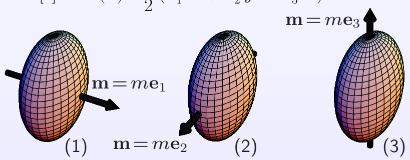
1) Chromium Bose-Einstein Condensate

• Trapped interacting Bose gas:

$$\mathcal{A}[\psi^*,\psi] = \int_0^{\hbar\beta} d\tau \int d^3x \Big\{ \psi^*(\boldsymbol{x},\tau)\Big(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\boldsymbol{\nabla}^2}{2M} - \mu\Big)\psi(\boldsymbol{x},\tau) + U(\boldsymbol{x})\big|\psi(\boldsymbol{x},\tau)\big|^2 + \int d^3x' \big|\psi(\boldsymbol{x}',\tau)\big|^2 V^{(\mathrm{int})}(\boldsymbol{x}-\boldsymbol{x}')\big|\psi(\boldsymbol{x},\tau)\big|^2 \Big\}$$

• Stuttgart experiment [1]: $U(\boldsymbol{x}) = \frac{M}{2}(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2)$

$\omega_1 = 2\pi\cdot 581$ Hz
$\omega_2 = 2\pi\cdot 406$ Hz
$\omega_3 = 2\pi\cdot 138$ Hz



• Interaction potential: $|\boldsymbol{m}| = 6\, m_B,\ a = 105\, a_B$ [2]

$$V^{(\mathrm{int})}(\boldsymbol{x}) = \frac{4\pi\hbar^2 a}{M}\,\delta(\boldsymbol{x}) + \frac{\mu_0}{4\pi}\left\{\frac{\boldsymbol{m}^2}{|\boldsymbol{x}|^3} - \frac{3\,[\boldsymbol{m}\,\boldsymbol{x}]^2}{|\boldsymbol{x}|^5}\right\}$$

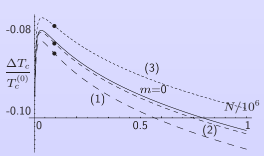
• Critical temperature: sc [3–6] + δ [7,8] + dd [9,10]

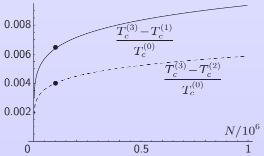
$$\frac{\Delta T_c^{(j)}}{T_c^{(0)}} = -0.727\,\frac{\omega_1+\omega_2+\omega_3}{3\,(\omega_1\omega_2\omega_3)^{1/3}}\,\frac{1}{N^{1/3}} - 3.426\,\frac{a}{\lambda_c^{(0)}} + 1.713\, f^{(j)}\!\left(\frac{\omega_1}{\omega_2},\frac{\omega_1}{\omega_3}\right)\frac{\mu_0 m^2 M}{12\pi\hbar^2\lambda_c^{(0)}}$$

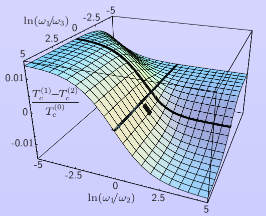
• Geometry factor [11,12]:

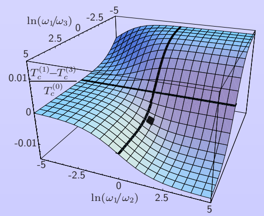
$$f^{(1)}(\eta,\kappa) = 1 + \frac{3\kappa\eta}{\sqrt{1-\kappa^2}\,(1-\eta^2)}\left\{E\left(\arcsin\sqrt{1-\kappa^2},\sqrt{\frac{1-\eta^2}{1-\kappa^2}}\right) - F\left(\arcsin\sqrt{1-\kappa^2},\sqrt{\frac{1-\eta^2}{1-\kappa^2}}\right)\right\}$$

$$f^{(2)}(\eta,\kappa) = f^{(1)}\!\left(\frac{\kappa}{\eta},\frac{1}{\eta}\right)\ ,\quad f^{(3)}(\eta,\kappa) = f^{(1)}\!\left(\frac{1}{\kappa},\frac{\eta}{\kappa}\right)$$









2) Canonical Approach to BEC

• N-particle partition function [13,14]:

$$Z_N^B(\beta) = \frac{1}{N!}\sum_P \prod_{n=1}^N \left(\int d^3x_n \int_{\boldsymbol{x}_n(0)=\boldsymbol{x}_n}^{\boldsymbol{x}_n(\hbar\beta)=\boldsymbol{x}_{P(n)}} \mathcal{D}^3 x_n\right) e^{-\mathcal{A}[\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N]/\hbar}$$

• Action:

$$\mathcal{A}[\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N] = \sum_{n=1}^N \int_0^{\hbar\beta} d\tau \left[\frac{M}{2}\dot{\boldsymbol{x}}_n^2 + U(\boldsymbol{x}_n) + {\sum_{m=1}^N}' V^{(\mathrm{int})}(\boldsymbol{x}_n - \boldsymbol{x}_m)\right]$$

• Recursion for interaction-free partition function [15–18]:

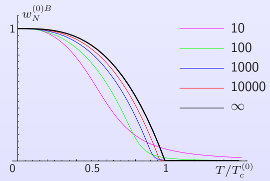
$$Z_N^{(0)B}(\beta) = \frac{1}{N}\sum_{n=1}^N Z_1(n\beta)\, Z_{N-n}^{(0)B}(\beta) \quad \text{with} \quad Z_0^{(0)B}(\beta) = 1$$

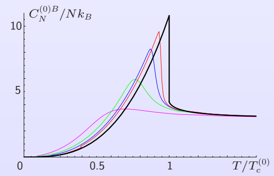
$$Z_1(\beta) = \sum_{\boldsymbol{k}} e^{-\beta E_{\boldsymbol{k}}}$$

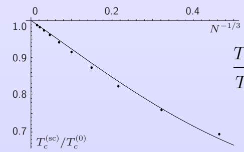
• Ground-state occupancy without interaction [16]:

$$w_N^{(0)B}(\beta) = \frac{1}{N}\sum_{n=1}^N e^{-n\beta E_{\boldsymbol{0}}}\, Z_{N-n}^{(0)B}(\beta)\,/\,Z_N^{(0)B}(\beta)$$

• Interaction-free results for harmonic case:



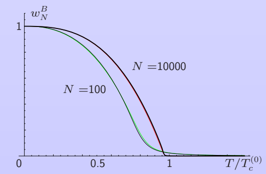


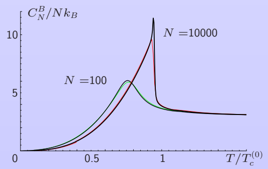


$$\frac{T_c^{(\mathrm{sc})}}{T_c^{(0)}} = 1 - \frac{\zeta(2)}{2\,\zeta^{2/3}(3)}\frac{1}{N^{1/3}} + \frac{\zeta^2(2)}{4\,\zeta^{4/3}(3)}\frac{1}{N^{2/3}} - \frac{\ln\left[8N/\zeta(3)\right]}{9\,\zeta^{1/3}(3)}\frac{1}{N^{2/3}} + \ldots \quad [19]$$

• Interacting Bose gas [20]: $V^{(\mathrm{int})}(\boldsymbol{x}) = g\,\delta(\boldsymbol{x})$

$$Z_N^B(\beta) = \frac{1}{N}\sum_{n=1}^N Z_1(n\beta)\, Z_{N-n}^B(\beta) \times \left\{1 + ng\frac{(M\omega/2\pi\hbar)^{3/2}}{\hbar Z_1(n\beta)}\sum_{l=1}^{n-1}\Big[Z_1(l\beta)\,Z_1([n-l]\beta)\,Z_1(n\beta)\Big]^{1/2} + \ldots\right\}^{-1}$$





3) Spinor Bose-Einstein Condensate

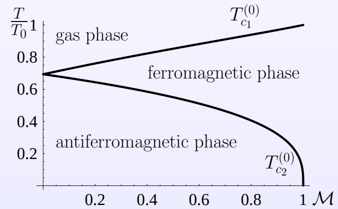
• Non-interacting spinor gas [21,22]: $U(\boldsymbol{x}) = \frac{M}{2}\omega_i^2 x_i^2$, $E_{\boldsymbol{0}} = \frac{\hbar}{2}(\omega_1+\omega_2+\omega_3)$

$$\mathcal{A}^{(0)} = \int_0^{\hbar\beta} d\tau \int d^3x\ \psi_i^*(\boldsymbol{x},\tau)\left[\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\boldsymbol{\nabla}^2}{2M} + U(\boldsymbol{x}) - \mu\right)\delta_{ij} - \eta F_{ij}^{(z)}\right]\psi_j(\boldsymbol{x},\tau)$$

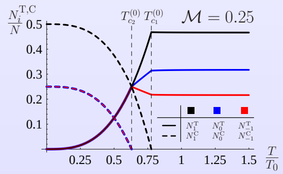
• Dependence of critical temperatures on magnetization $\mathcal{M}$ [23]:

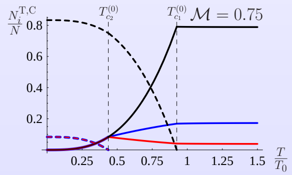
Gross-Pitaevskii equations:

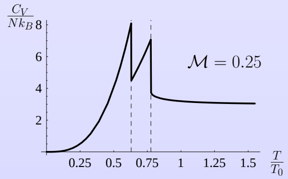
$$(E_{\boldsymbol{0}} - \mu - \eta)\Psi_1(\boldsymbol{x}) = 0$$
$$(E_{\boldsymbol{0}} - \mu)\Psi_0(\boldsymbol{x}) = 0$$
$$(E_{\boldsymbol{0}} - \mu + \eta)\Psi_{-1}(\boldsymbol{x}) = 0$$

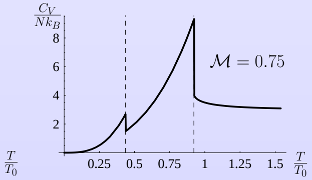


• Particle number and heat capacity [24]:









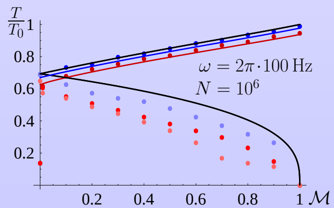
• Interacting spinor gas: $\mathcal{A} = \mathcal{A}^{(0)} + \mathcal{A}^{(\mathrm{int})}$, $c_0 \propto a_0 + 2a_2$, $c_2 \propto a_2 - a_0$

$$\mathcal{A}^{(\mathrm{int})} = \frac{1}{2}\int_0^{\hbar\beta} d\tau \int d^3x \left\{c_0\Big[\psi_i^*(\boldsymbol{x},\tau)\psi_i(\boldsymbol{x},\tau)\Big]^2 + c_2\sum_{a=x,y,z}\Big[\psi_i^*(\boldsymbol{x},\tau)F_{ij}^a\,\psi_j(\boldsymbol{x},\tau)\Big]^2\right\}$$

• First-order T_{c_1}-shift [25]:

$$\frac{\Delta T_{c_1}}{T_{c_1}^{(0)}} = \alpha_0(\mathcal{M})\frac{a_0}{\lambda_0} + \alpha_2(\mathcal{M})\frac{a_2}{\lambda_0}$$

[26]	T_{c_1}	T_{c_2}	T_{c_3}	$T_{c_1}^{\mathrm{analytic}}$
^{87}Rb	●	●	●	—
^{23}Na	●	●		—



Critical Properties of Bose-Einstein Condensates

Konstantin Glaum[1], Parvis Soltan-Panahi[1], Axel Pelster[2], and Hagen Kleinert[1]
[1]Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, 14195 Berlin
[2]Fachbereich Physik, Campus Essen, Universität Duisburg-Essen, Universitätsstrasse 5, 45117 Essen

References

[1] M. Griesmaier, J. Werner, S. Hensler, J. Stuhler, and T. Pfau, *Bose-Einstein Condensation of Chromium*, Phys. Rev. Lett. **94**, 160401 (2005)
[2] J. Werner, A. Griesmaier, S. Hensler, J. Stuhler, T. Pfau, A. Simoni, and E. Tiesinga, *Observation of Feshbach Resonances in an Ultracold Gas of* ^{52}*Cr*, Phys. Rev. Lett. **94**, 183201 (2005)
[3] S. Grossmann and M. Holthaus, *On Bose-Einstein Condensation in Harmonic Traps*, Phys. Lett. A **208**, 188 (1995)
[4] T. Haugset, H. Haugerud, and J. O. Andersen, *Bose-Einstein Condensation in Anisotropic Harmonic Traps*, Phys. Rev. A **55**, 2922 (1997)
[5] F. Dalfovo, S. Giorgini, L.P. Pitaevskii, and S. Stringari, *Theory of Bose-Einstein Condensation in Trapped Gases*, Rev. Mod. Phys. **71**, 463 (1999)
[6] Ph. W. Courteille, V. S. Bagnato, and V. I. Yukalov, *Bose-Einstein Condensation of Trapped Atomic Gases*, Laser Phys. **11**, 659 (2001)
[7] S. Giorgini, L. P. Pitaevskii, and S. Stringari, *Condensate Fraction and Critical Temperature of a Trapped Interacting Bose Gas*, Phys. Rev. A **54**, R4633 (1996)
[8] F. Gerbier, J. H. Thywissen, S. Richard, M. Hugbart, P. Bouyer, and A. Aspect, *Critical Temperature of a Trapped, Weakly Interacting Bose Gas*, Phys. Rev. Lett. **92**, 030405 (2004)
[9] K. Glaum, A. Pelster, H. Kleinert, and T. Pfau, *Critical Temperature of a Chromium Condensate*, eprint: `cond-mat/0606569`
[10] K. Glaum, A. Pelster, H. Kleinert, and T. Pfau, *Bose-Einstein Condensation Temperature of Chromium in Anisotropic Harmonic Trap* (in preparation)
[11] S. Giovanazzi, P. Pedri, L. Santos, A. Griesmaier, M. Fattori, T. Koch, J. Stuhler, and T. Pfau, *Expansion Dynamics of a Dipolar Bose-Einstein Condensate*, eprint: `cond-mat/0605708`
[12] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, 1965)
[13] R. P. Feynman, *Statistical Mechanics* (Benjamin, Reading, 1972)
[14] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics*, Third Extended Edition (World Scientific, Singapore, 2004)
[15] F. Brosens, J.T. Devreese, and L.F. Lemmens, *Thermodynamics of Coupled Identical Oscillators within the Path-Integral Formalism*, Phys. Rev. E **55**, 227 (1997)
[16] P. T. Landsberg, *Thermodynamics - with Quantum Statistical Illustrations* (Interscience Publishers, New York, 1961)
[17] M. Wilkens and C. Weiss, *Particle Number Fluctuations in an Ideal Bose Gas* , J. Mod. Opt. **44**, 1801 (1997)
[18] C. Weiss and M. Wilkens, *Particle Number Counting Statistics in Ideal Bose Gases*, Optics Express **1**, 272 (1997)
[19] V. I. Yukalov, *Modified Semiclassical Approximation for Trapped Bose Gases*, Phys. Rev. A **72**, 033608 (2005)
[20] K. Glaum, H. Kleinert, and A. Pelster, *Bose-Einstein Condensation in Canonical Ensembles* (in preparation)
[21] T.-L. Ho, *Spinor Bose Condensates in Optical Traps*, Phys. Rev. Lett. **81**, 741 (1998)
[22] T. Ohmi and K. Machida, *Bose-Einstein Condensation with Internal Degrees of Freedom in Alkali Atom Gases*, J. Phys. Soc. Jap. **67**, 1822 (1998)
[23] T. Isoshima, T. Ohmi, and K. Machida, *Double Phase Transition in Magnetized Spinor Bose-Einstein Condensation*, J. Phys. Soc. Jap. **69**, 3864 (2000)
[24] P. Soltan-Panahi and A. Pelster, *Thermodynamic Properties of* $F = 1$ *Spinor Gases* (in preparation)
[25] P. Soltan-Panahi, *Spinor Bose–Einstein Condensates*, Diploma-Thesis, Freie Universiät Berlin (in preparation)
[26] W. Zhang, S. Yi, and L. You, *Bose-Einstein Condensation of Trapped Interacting Spin-1 Atoms*, Phys. Rev. A **70**, 046311 (2004)