From Maxwell equations to Bose-Einstein condensation of photons

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About photons in mircrocavities

Light confined in a microcavity [3] is described by Maxwell's equations with appropriate boundary conditions. A careful analysis of the corresponding boundary value problem in oblate spheroidal coordinates provides a systematic approach to determine the underlying mode functions. In the paraxial approximation, this three-dimensional microcavity problem can be reduced to an effective two-dimensional trapped massive Bose gas. This result supports the heuristic derivation of Ref. [2], where even the Bose-Einstein condensation of these massive photons was observed.

Experimental setup in Weitz group

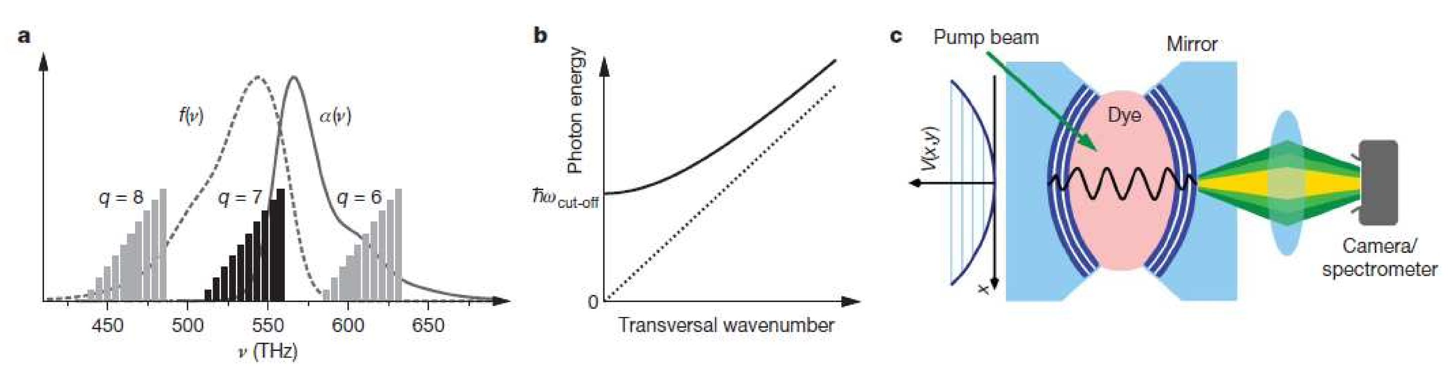

FIGURE 1: a. Schematic spectrum of cavity modes with absorption coefficient $\alpha(\nu)$ and fluorescence strength $f(\nu)$ b. Dispersion relation of photons in the cavity (solid line) with fixed longitudinal mode ($q = 7$) and the free photon dispersion (dashed line) c. Schematic experimental setup with trapping potential imposed by the curved mirrors.

- photons cannot be cooled without loosing photons (Stefan-Boltzmann law, no chemical potential)
- **How to thermalize photons?** → modified resonator environment
- dye-filled curved micro resonator with mirror distance $L = 1.46\,\mu$m, see Fig.1.c
- axial cut-off $k_{\text{cut}} \propto \frac{2\pi}{L} \,\widehat{=}\, 550$ THz
  → thermal excitation suppressed by $\exp\left[-\frac{\hbar c k_{\text{cut}}}{k_B T}\right] = \exp(-80)$
- k_z fixed, transversal modes thermalize with dye solution as heat bath (rovibronic levels)
- system effective 2D, k_z yields effective mass term

A careful look at photon modes

- treating this microcavity system involves 3 main problems
  - ◇ **coordinate system matching to the boundary**
  - ◇ while **scalar Helmholtz operator** should remain **separable**
  - ◇ construction of **solenoidal vector field**
- boundary geometry parametrized ($\rho \ll R$)

$$z|_{\partial\Omega_\pm}(\rho) = \pm\left(\frac{L}{2} - (R - \sqrt{R^2-\rho^2}\right) \approx \pm\left(\frac{L}{2} - \frac{\rho^2}{2R}\right)$$

- mirrors ideal conductors

$$\mathbf{n}\times\mathbf{E}|_{\partial\Omega_\pm} = -\mathbf{n}\times\dot{\mathbf{A}}|_{\partial\Omega_\pm} = \mathbf{0} \qquad (1)$$
$$\mathbf{n}\cdot\mathbf{B}|_{\partial\Omega_\pm} = \mathbf{n}\cdot\nabla\times\mathbf{A}|_{\partial\Omega_\pm} = \mathbf{0} \qquad (2)$$

- rewrite Maxwell equations with $\mathbf{A}$ given in radiation gauge $\phi = 0$ and

$$\text{div}\mathbf{A} = 0 \qquad \left[\triangle + |\mathbf{k}|^2\right]\mathbf{A} = \mathbf{0} \qquad (3)$$

- due to rotational symmetry choose oblate spheroidal coordinates

$$x = a\ \cosh\mu\sin\theta\cos\phi$$
$$y = a\ \cosh\mu\sin\theta\sin\phi$$
$$z = a\ \sinh\mu\cos\theta$$

- μ = constant $\widehat{=}$ ellipsoid and for $\mu \ll 1$

$$z \approx a\mu\left(1 - \frac{\rho^2}{2a^2}\right)$$

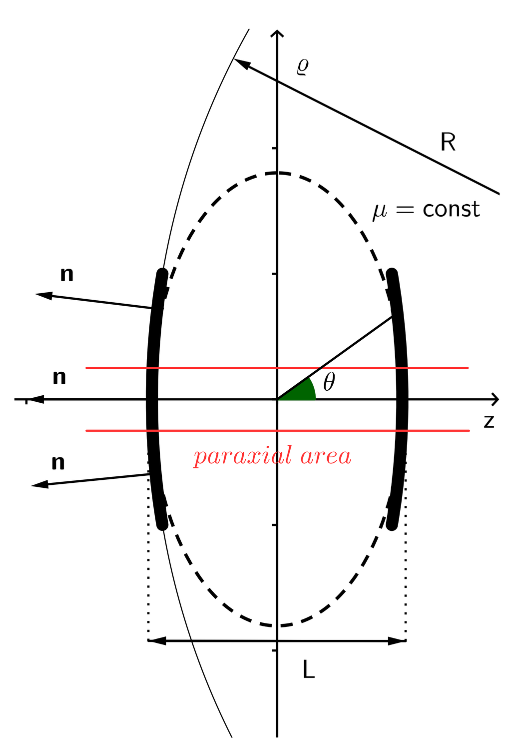

FIGURE 2: Schematic of geometric situation.

- scale factor $a = \sqrt{\frac{RL}{2}}$ contains geometry
- boundary parameter is simply $\mu_\pm = \pm\sqrt{\frac{L}{2R}}$

Scalar Helmholtz equation

- 3D scalar Helmholtz equation separable [4] with ansatz $u = R(\theta)P(\mu)e^{im\phi}$
- rescaled eigenvalue decomposition $\gamma^2 + (ak_\perp)^2 = (ak)^2$ with γ separation constant
- large trapping potential $(ak)^2\sin^2\theta \propto 1/L$

$$-\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta\frac{\mathrm{d}R}{\mathrm{d}\theta} + \frac{m^2R}{\sin^2\theta} + (ak)^2\sin^2\theta R = (ak_\perp)^2 R$$

- small θ-expansion $\widehat{=}$ **modes strongly confined** to optical axis
- paraxial approximation yields $\theta = \rho/a$ in leading order **2D harmonic oscillator**
- normalizability of solution $R(\theta)$ gives

$$k_\perp^2 = 2k\left(\frac{2l+m+1}{a}\right)$$

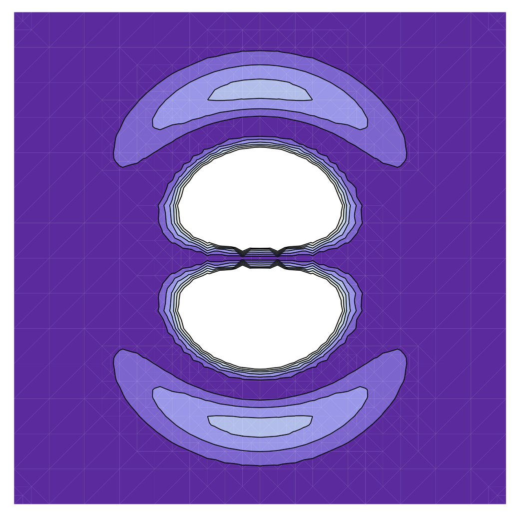

FIGURE 3: Intensity in xy-plane for u_{nlm} with ($n = 2, l = 2, m = 1$).

- simultaneously separated axial differential equation also depends on $k_\perp$ and m
- Schrödinger like form with $\zeta = \sinh\mu$

$$-\frac{\mathrm{d}^2P}{\mathrm{d}\zeta^2} - \frac{(m^2-1)P}{(1+\zeta^2)^2} + \frac{(k_\perp a)^2P}{(1+\zeta^2)} = (ka)^2P$$

- since $\mu \ll 1 \Leftrightarrow \zeta \ll 1$ potential terms essentially constant
- oscillating solutions for

$$(ka)^2 - (k_\perp a)^2 + m^2 - 1 = \gamma^2 + m^2 - 1 > 0$$

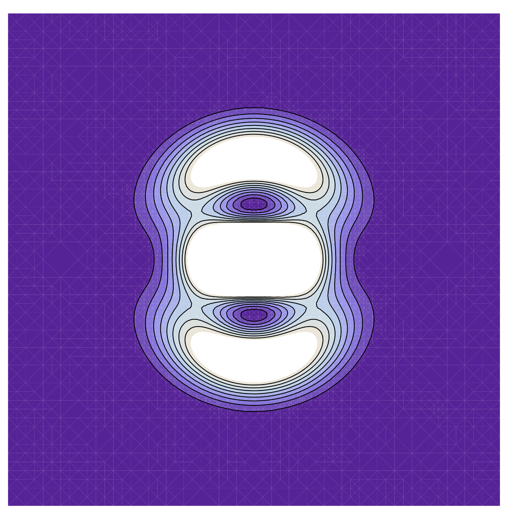

FIGURE 4: Intensity in xy-plane from vector solution ($n = 2, l = 2, m = 1$) and $\mathbf{n} = \mathbf{e_z}$.

- full scalar solution with L_l^m generalised Laguerre polynomials is

$$u_{nlm}(\rho,\mu,\phi) = \exp\left[-\frac{k_\perp^2\rho^2}{4}\right]\rho^m L_l^m\left(\frac{k_\perp^2\rho^2}{2}\right)\left[C\exp\left[i\sqrt{\gamma^2+m^2-1}\mu\right] + D\exp\left[-i\sqrt{\gamma^2+m^2-1}\mu\right]\right]$$

Construction of vector solution

- if u solution to scalar Helmholtz equation and $\mathbf{n}$ constant unit vector (theorem [1])

$$\mathbf{f} = \nabla\times(\mathbf{n}u) \qquad \mathbf{g} = \frac{1}{|\mathbf{k}|}\nabla\times\mathbf{f}$$

  then $\mathbf{A} = \mathbf{f} + \mathbf{g}$ is a vector solution of (3)
- $\mathbf{f}$ and $\mathbf{g}$ are TE- and TM-modes
- imposing (1) and (2) to find u Dirichlet type for n even and von Neumann type for n odd
- "axial" $\gamma = a\sqrt{\frac{n^2\pi^2}{L^2} + \frac{1-m^2}{a^2}}$ mixed with angular momentum quantum number m

Characteristic numbers extracted

- expanding linear dispersion relation

$$\omega = c|\mathbf{k}| \approx \gamma a + \frac{c(ak_\perp)^2}{\gamma} \approx \frac{cn\pi}{L} + c\frac{(2l+m+1)}{a}$$

- free spectral range $\frac{c\triangle\gamma}{2\pi} = 77$ THz and $\frac{c\triangle k_\perp}{2\pi} = 42$ GHz
- tranverse modes "sitting" on top of axial modes
- fixed quantum number leads to quadratic dispersion → massive Boson gas
- obtained frequency c/a corresponds to Ref. [2]

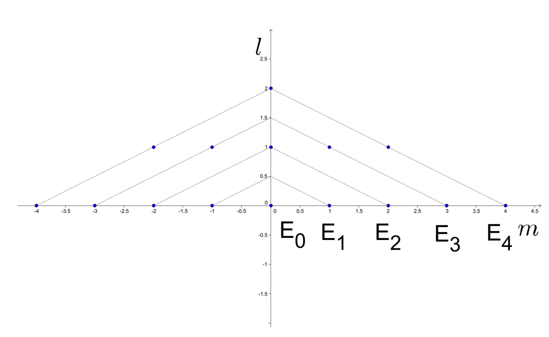

FIGURE 5: Degeneracy of modes may be modified by vector construction.

Outlook

- non constant normal vector $\mathbf{n}$
- curl-construction implements extra contributions to ground state u_{n00}
- vector ground state $\mathbf{A}$ with $J_z = L_z + S_z = \pm 1$ (Doughnut vs Gauss)
- spatial correlation function

References

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