Bogoliubov theory of 1D anyons in a lattice

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Abstract

In a one-dimensional lattice anyons can be defined via generalized commutation relations containing a statistical parameter θ, which interpolates between the boson limit $\theta = 0$ and the pseudo-fermion limit $\theta = \pi$ [1]. The corresponding anyon-Hubbard model is mapped to a Bose-Hubbard model via a fractional Jordan-Wigner transformation, yielding a complex hopping term with a density-dependent Peierls phase. Here we work out a corresponding Bogoliubov theory. To this end we start with the underlying mean-field theory, where we allow for the condensate a finite momentum, which is known from previous works [2-4], and determine it from extremizing the mean-field energy. With this we calculate various physical properties and discuss their dependence on the statistical parameter and the lattice size. Among them are both the condensate and the superfluid density as well as the equation of state and the compressibility. Based on the mean-field theory we then analyse the resulting dispersion of the Bogoliubov quasi-particles, which turns out to be in accordance with the Goldstone theorem. In particular, this leads to two different sound velocities for wave propagations to the left and the right, which originates from parity breaking. Furthermore, the quasi-condensate fraction are also investigated.

The Model

- Anyon-Hubbard model in 1D lattice, with periodic boundary condition:

$$\hat{H}_{\text{AHM}} = -J\sum_j \left(\hat{a}_j^\dagger \hat{a}_{j+1} + \text{h.c.}\right) - \mu \sum_j \hat{n}_j + \frac{U}{2}\sum_j \hat{n}_j(\hat{n}_j - 1)$$

- Generalized commutation relations: $\hat{a}_i \hat{a}_j^\dagger - e^{i\theta \text{sign}(i-j)} \hat{a}_i^\dagger \hat{a}_j = \delta_{i,j}, \quad \theta \in [0, \pi]$

- Bosonic fractional Jordan-Wigner transformation: $\hat{a}_j = \hat{b}_j \, e^{i\theta \sum_{l<j} \hat{n}_l}$

- AHM in bosonic representation with density-dependent Peierls phase:

$$\hat{H}_{\text{AHM}}^{(B)} = -J\sum_j \left[\hat{b}_j^\dagger e^{i\theta \hat{n}_j} \hat{b}_{j+1} + \text{h.c.}\right] - \mu\sum_j \hat{n}_j + \frac{U}{2}\sum_j \hat{n}_j(\hat{n}_j - 1)$$

$$\overset{\text{Normal ordered}}{=} -J\sum_{j=1}^{N_s}\sum_{q=0}^{\infty}\sum_{m=0}^{q}\left[\frac{(i\theta)^q}{q!}\underbrace{S(q,m)}_{\substack{\text{Stirling number}\\ \text{of second kind}}}{}^{[5]}(\hat{b}_j^\dagger)^{m+1}(\hat{b}_j)^m \hat{b}_{j+1} + \text{h.c.}\right] - \mu\sum_j \hat{n}_j + \frac{U}{2}\sum_j \hat{n}_j(\hat{n}_j - 1)$$

Bogoliubov Theory

- Modified Bogoliubov ansatz:

$$\hat{b}_j = \frac{1}{\sqrt{N_s}}\left[\left(\sqrt{N_s n_{k_0}} + \delta\hat{b}_{k_0}\right)e^{-ik_0 ja} + \sum_{k\neq k_0}\hat{b}_k \, e^{-ikja}\right]$$

  - Shift of quasi-condensate momentum k_0

- Decomposition of Hamiltonian:

$$\hat{H}_{\text{AHM}}^{(B)} = E_0 + \hat{H}_1 + \hat{H}_2 + \mathcal{O}[(\delta\hat{b}_{k_0}^{(\dagger)})^2] + \mathcal{O}[(\hat{b}_{k\neq k_0}^{(\dagger)})^3]$$

$\hat{H}_2$: Bi-linear operators contributions

$\hat{H}_1$: First-order fluctuations of condensate

E_0: Condensate energy

Ground-state Properties

$$E_0 = N_s\left\{-2Je^{-2n_{k_0}\sin^2\frac{\theta}{2}}\cos\left(n_{k_0}\sin\theta - k_0 a\right)n_{k_0} - \mu n_{k_0} + \frac{U}{2}n_{k_0}^2\right\}$$

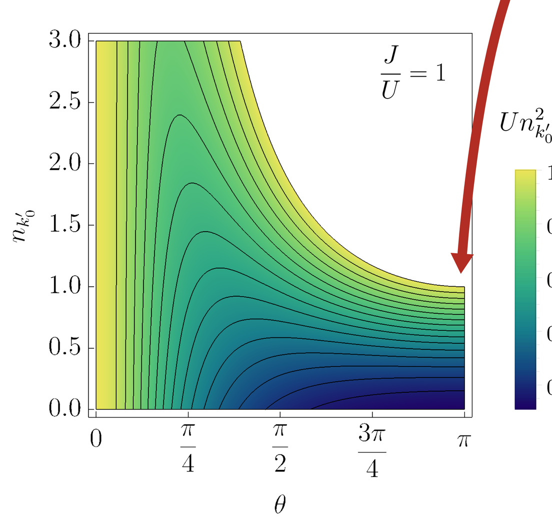
- Physical "ground state" $\Rightarrow$ Minimum w.r.t. order parameters k_0 & n_{k_0}

- Results from Hessian matrix: $\Rightarrow$ Condition for k_0: $k_0' \cdot a = n_{k_0'}\sin\theta + 2m\pi, \; m \in \mathbb{Z}$

  $\Rightarrow$ Equation of state: $\mu = Un_{k_0'} - 2Je^{-2n_{k_0'}\sin^2\frac{\theta}{2}}\left(1 - 2n_{k_0'}\sin^2\frac{\theta}{2}\right) \;\Rightarrow\; \hat{H}_1 \overset{\mu(n_{k_0},k_0')}{=} 0$

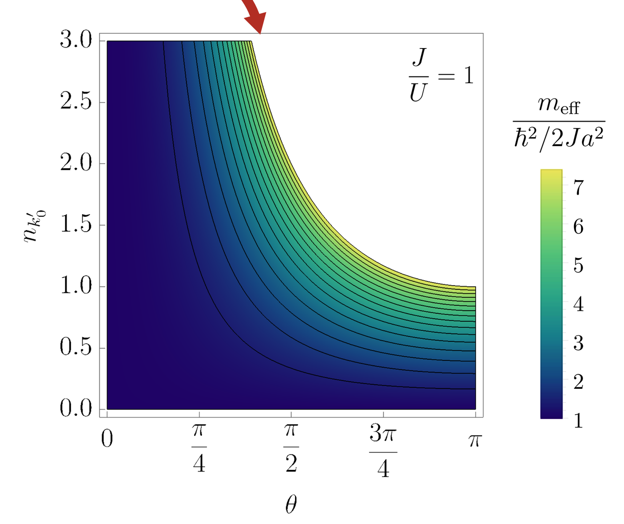
  $\Rightarrow$ Constraint of condensate average density from statistic parameter!

$n_{k_0'}(\theta \to \pi) < 1$ $\quad \boxed{n_{k_0'} < \sin^{-2}\frac{\theta}{2}} \quad$ $n_{k_0'}(\theta \to 0) < \infty$



- **Fig.** Isothermal compressibility:

$$\kappa_T = \left[Un_{k_0'}^2 + 8n_{k_0'}^2 Je^{-2n_{k_0'}\sin^2\frac{\theta}{2}}\sin^2\frac{\theta}{2}\left(1 - n_{k_0'}\sin^2\frac{\theta}{2}\right)\right]^{-1}$$



- **Fig.** Effective mass:

$$m_{\text{eff}} = \frac{\hbar^2}{2Ja^2 e^{-2n_{k_0'}\sin^2\frac{\theta}{2}}}$$

Elementary Excitations

- H_2 under translation $q = k - k_0'$:
- quasi-particle with Bogoliubov angle α_q: $\begin{bmatrix}\hat{d}_{k_0'+q}\\ \hat{d}_{k_0'-q}^\dagger\end{bmatrix} = \begin{bmatrix}\cosh\alpha_q & \sinh\alpha_q\\ \sinh\alpha_q & \cosh\alpha_q\end{bmatrix}\begin{bmatrix}\hat{b}_{k_0'+q}\\ \hat{b}_{k_0'-q}^\dagger\end{bmatrix}$

  - Diagonalized Hamiltonian:

$$\hat{H}_{\text{diag}} \approx E_0 + \frac{1}{2}\sum_{q\neq 0}[E_{-q} - A_{-q}] + \sum_{q\neq 0} E_q \, \hat{d}_{k_0'+q}^\dagger \hat{d}_{k_0'-q}$$

  - Coefficients:

$$A_q = -2Je^{-2n_{k_0'}\sin^2\frac{\theta}{2}}\Big[-4n_{k_0'}^2\cos\theta\sin^2\frac{\theta}{2} - 2n_{k_0'}\sin^2\frac{\theta}{2} + n_{k_0'}\left(\cos(\theta - qa) - \cos qa\right) + \cos qa - 1\Big] + Un_{k_0'}$$

$$B_q = -Je^{-2n_{k_0'}\sin^2\frac{\theta}{2}}\Big[-4n_{k_0'}^2\cos\theta\sin^2\frac{\theta}{2} - 2n_{k_0'}\sin^2\frac{\theta}{2} + i\cdot n_{k_0'}\sin\theta + n_{k_0'}(e^{-i\theta} - 1)\,e^{-iqa}\Big] + \frac{U}{2}n_{k_0'}$$

- Bogoliubov spectrum:

$$E_q = \underbrace{\frac{1}{2}\sqrt{(A_q + A_{-q})^2 - \left(B_q + B_{-q} + B_q^* + B_{-q}^*\right)^2}}_{\text{Even}} + \underbrace{\frac{1}{2}(A_q - A_{-q})}_{\text{Odd}}$$

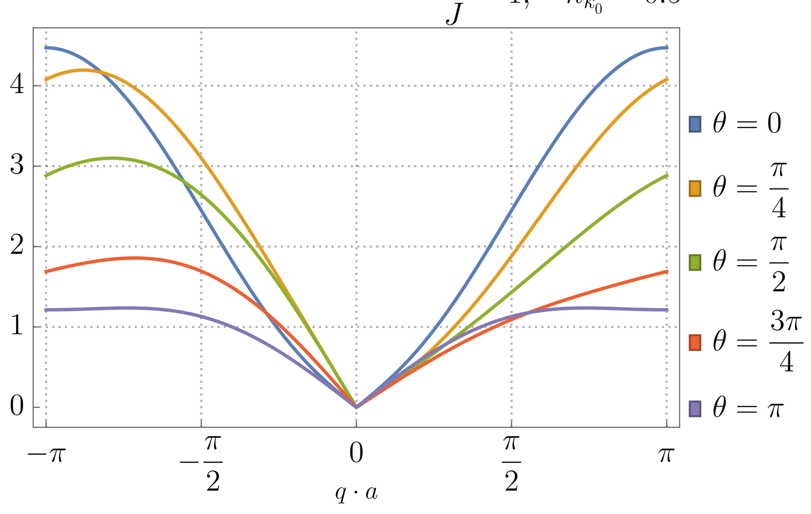
  - Asymmetric dispersion around the condensate momentum

  $\Rightarrow$ Unbalanced transport [6]

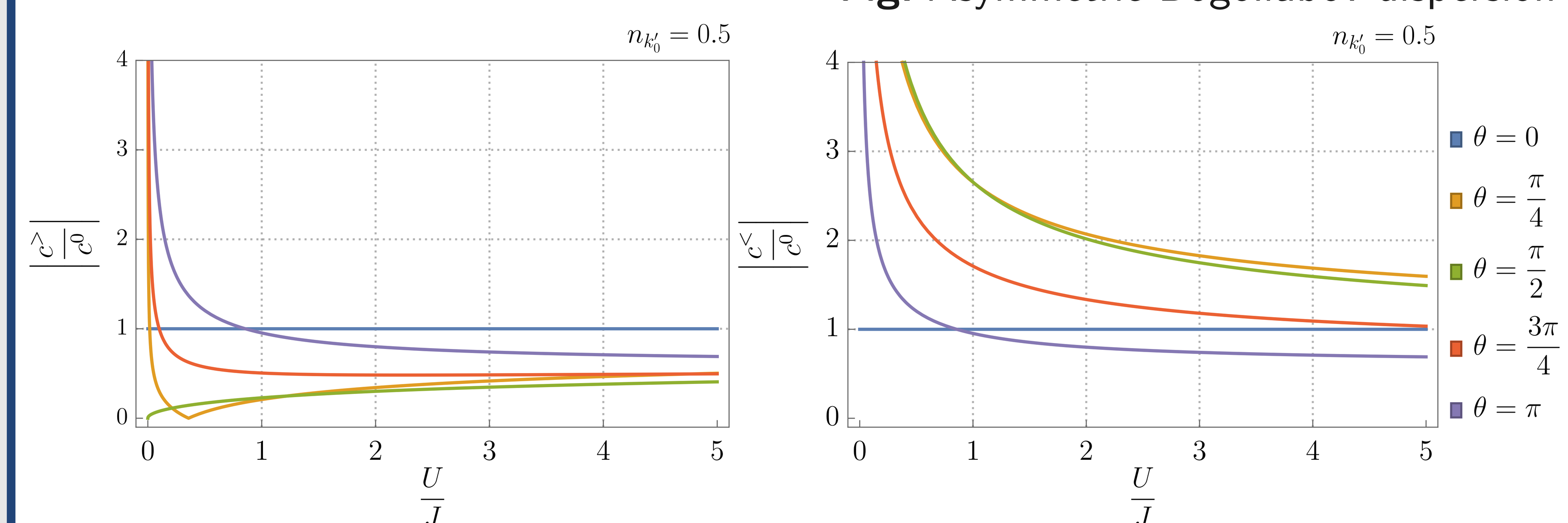
  $\Rightarrow$ Broken $\mathcal{PT}$ symmetry

  $\Rightarrow$ Two sound modes with different velocities $c^>$ & $c^<$

  - Bosonic velocity: $c^0 = \sqrt{\frac{2Ja^2}{\hbar^2}Un_{k_0}} = \sqrt{\frac{Un_{k_0}}{m_{\text{eff},\theta=0}}}$



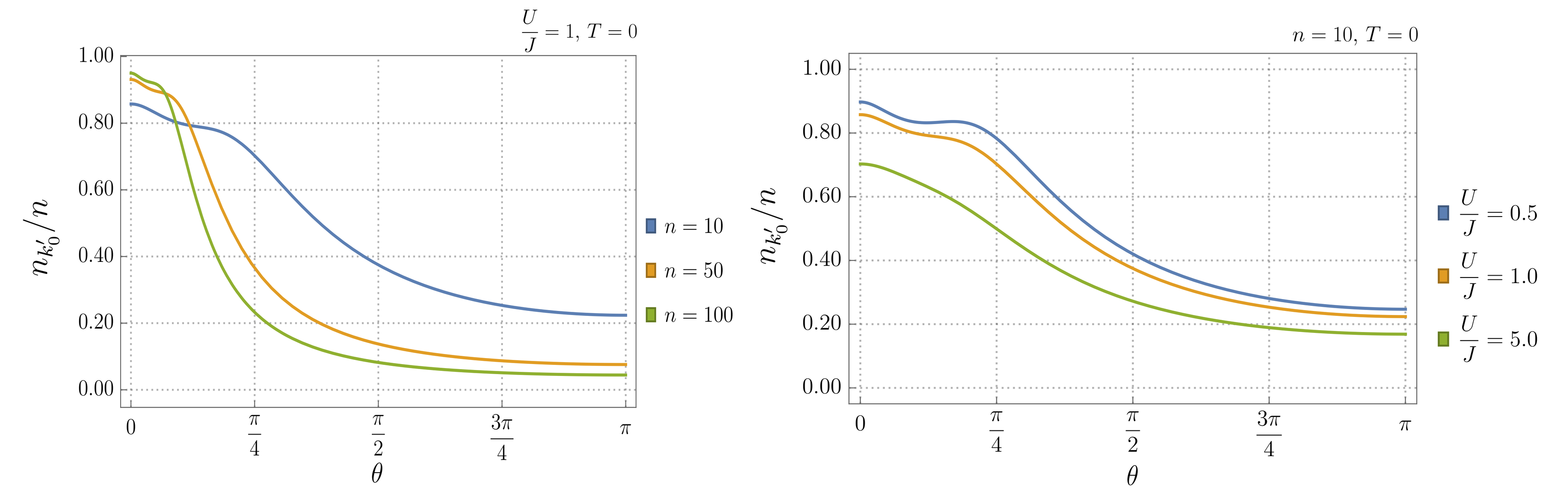
- **Fig.** Asymmetric Bogoliubov dispersion



- **Fig.** Comparsion between right- and left-propagating sound velocities

- Condensate density (Bogoliubov approximation):

$$n_{k_0'} = n - \frac{1}{N_s}\sum_{q\neq 0}\left\{\frac{A_q - E_q}{2E_q - (A_q - A_{-q})} + \frac{A_q + A_{-q}}{2E_q - (A_q - A_{-q})}\frac{1}{e^{\beta E_q} - 1}\right\}$$



- **Fig.** Condensate fractions. Left: various total density; Right: various U/J

Conclusion & Outlook

- Analytical calculation for anyonic Bogoliubov mean-field theory
- Interpolation between BE & FD statistics recovered, resemble to Haldane exclusion principle on mean-field level
- Asymmetric Bogoliubov dispersion, sound modes, quasi-condensate fraction
- Further analysis, instabilities, dynamical properties, open system...

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