Critical Properties of Bose-Einstein Condensates

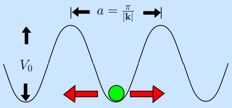
H. Kleinert[1] and A. Pelster[2]
[1]Institut für Theoretische Physik, Freie Universität Berlin, 14195 Berlin, Germany
[2]Fachbereich Physik, Universität Duisburg-Essen, 45117 Essen, Germany

1. Reentrant Effect in Optical Lattice

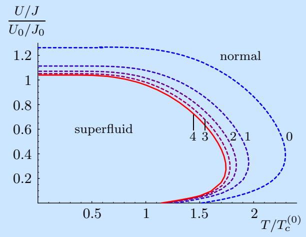
- Periodic potential [1,2]: $V(\boldsymbol{x}) = V_0 \sum_{i=1}^{3} \sin^2(k_i x_i) \quad , \quad E_r = \frac{\hbar^2 \boldsymbol{k}^2}{2M}$
- Tight-binding approximation [3]:

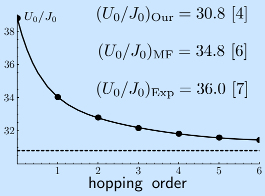
$$J = \frac{4}{\sqrt{\pi}} E_r \left(\frac{V_0}{E_r}\right)^{3/4} \exp\left\{-2\left(\frac{V_0}{E_r}\right)^{1/2}\right\}$$

$$U = \frac{2\sqrt{2}}{\sqrt{\pi}} \, |\boldsymbol{k}| a_s E_r \left(\frac{V_0}{E_r}\right)^{3/4}$$



- Quantum phase diagram from Bose-Hubbard model [4,5]:





2. Reentrant Effect in Power Potentials

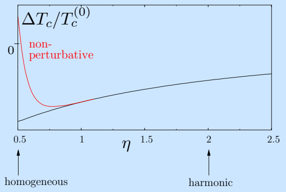
- Trapped Bose gas with δ-interaction:

$$\mathcal{A}[\psi^*, \psi] = \int_0^{\hbar\beta} d\tau \int d^3x \left\{ \psi^*(\boldsymbol{x},\tau) \left(\hbar\,\partial_\tau - \frac{\hbar^2 \boldsymbol{\nabla}^2}{2M} - \mu\right) \psi(\boldsymbol{x},\tau) + V(\boldsymbol{x}) \left|\psi(\boldsymbol{x},\tau)\right|^2 + \frac{2\pi\hbar^2 a_s}{M} \left|\psi(\boldsymbol{x},\tau)\right|^4 \right\}$$

- Trapping potential [8–10]:

$$V(\boldsymbol{x}) = \sum_{i=1}^{3} q_i |x_i|^{p_i}$$

$$\eta = \frac{1}{2} + \sum_{i=1}^{3} \frac{1}{p_i}$$



- T_c-shift: $\frac{\Delta T_c}{T_c^{(0)}} = c_1 \hat{a} + (c_2' \ln \hat{a} + c_2)\,\hat{a}^2 + \mathcal{O}\left(\hat{a}^3\right)$

	$\hat{a}$	c_1	c_2'	c_2
harmonic	$a_s/\lambda_c^{(0)}$	-3.43 [11,12]	-45.86 [13]	-155 [13,14]
uniform	$a_s n^{1/3}$	1.3 [14–21]	19.75 [22]	75 [14,22]

- **Methods:** perturbative and nonperturbative [23,24]

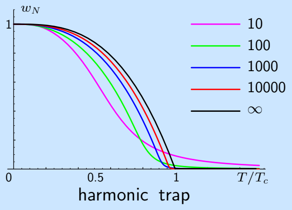
3. Canonical Approach to BEC

- Recursion for partition function [25,26]:

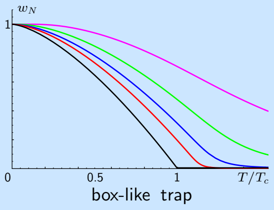
$$Z_N(\beta) = \frac{1}{N} \sum_{n=1}^{N} \left(\sum_{\boldsymbol{k}} e^{-n\beta E_{\boldsymbol{k}}}\right) Z_{N-n}(\beta) \quad \text{with} \quad Z_0(\beta) = 1$$

- Ground-state occupancy [27]:

$$w_N(\beta) = \frac{1}{N} \sum_{n=1}^{N} e^{-n\beta E_{\boldsymbol{0}}} \, \frac{Z_{N-n}(\beta)}{Z_N(\beta)}$$



harmonic trap



box-like trap

- **Goal 1:** Treat two-particle interaction
- **Goal 2:** Determine depletion and superfluid density
- **Method:** Variational perturbation theory [23,24]

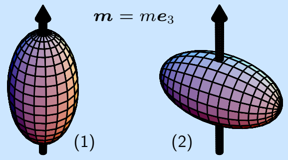
4. Chrom BEC

- Stuttgart experiment in the group of T. Pfau [28]:

$$V(\boldsymbol{x}) = \frac{M}{2} \sum_{j=1}^{3} \omega_j^2 x_j^2$$

$$\omega_1^{(1)} = \omega_2^{(1)} = \omega_\perp \;,\; \omega_3^{(1)} = \omega_\parallel$$

$$\omega_1^{(2)} = \omega_\parallel \;,\; \omega_2^{(2)} = \omega_3^{(2)} = \omega_\perp$$

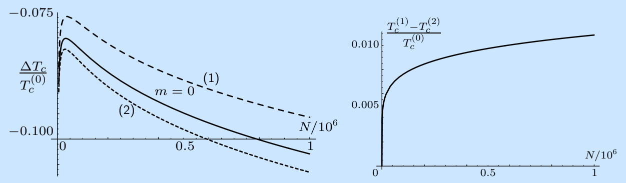


- Interaction potential:

$$V^{(\rm int)}(\boldsymbol{x}) = \frac{\mu_0 m^2}{4\pi} \left(\frac{1}{|\boldsymbol{x}|^3} - \frac{3x_3^2}{|\boldsymbol{x}|^5}\right) + \frac{4\pi\hbar^2 a_s}{M} \delta(\boldsymbol{x})$$

magnetic moment of ^{52}Cr: $m = 6\ m_B$
s-wave scattering length [29]: $a_s = 105\ a_B$

- Critical temperature versus particle number:

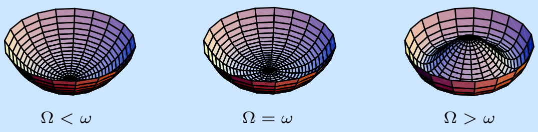


- **Goal:** Temperature dependence of condensate density and heat capacity
- **Method 1:** Mean-field theory
- **Method 2:** Exact canonical calculations

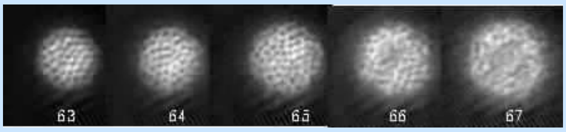
5. Rotating BEC

- Potential in co-rotating frame [30,31]:

$$V(\boldsymbol{x}) = \frac{M}{2}(\omega^2 - \Omega^2)(x^2 + y^2) + \frac{M}{2}\omega_z^2 z^2 + \frac{K}{4}(x^2 + y^2)^2$$



- **Goal 1:** Collective modes at $T = 0$
- **Goal 2:** Melting of vortex lattice [32] due to central depletion [30]



- **Method:** Variational perturbation theory [23,24]

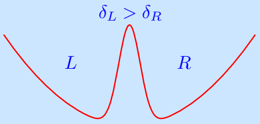
6. Josephson Effect near Feshbach Resonance

- Hamiltonian: $H_{\rm tot} = \sum_{\alpha = L,R} H_\alpha + H_T$

$$H_\alpha = \sum_\sigma f_{\alpha,\sigma}^+ \left(-\frac{\hbar^2 \boldsymbol{\nabla}^2}{2M} + V_\alpha\right) f_{\alpha,\sigma} + b_\alpha^+ \left(-\frac{\hbar^2 \boldsymbol{\nabla}^2}{4M} + V_\alpha + \delta_\alpha\right) b_\alpha + g\left(b_\alpha^+ f_{\alpha,\downarrow} f_{\alpha,\uparrow} + f_{\alpha,\uparrow}^+ f_{\alpha,\downarrow}^+ b_\alpha\right) - \mu_\alpha N_\alpha$$

$$N_\alpha = 2 b_\alpha^+ b_\alpha + \sum_\sigma f_{\alpha,\sigma}^+ f_{\alpha,\sigma}$$

$$H_T = \sum_\sigma \left(T f_{L\sigma}^+ f_{R\sigma} + T^+ f_{R\sigma}^+ f_{L\sigma}\right)$$



- **Goal 1:** Study phase coherence between Cooper pairs and molecules on different sides
- **Goal 2:** Control Josephson oscillations via detuning $\delta_{L,R}$

Critical Properties of Bose-Einstein Condensates

H. Kleinert[1] and A. Pelster[2]

[1]Institut für Theoretische Physik, Freie Universität Berlin, 14195 Berlin, Germany

[2]Fachbereich Physik, Universität Duisburg-Essen, 45117 Essen, Germany

References

[1] M.P.A. Fisher, P.B. Weichman, G. Grinstein, and D.S. Fisher, *Boson Localization and the Superfluid-Insulator Transition*, Phys. Rev. B **40**, 546 (1989)

[2] D. Jaksch, C. Bruder, J.I. Cirac, C.W. Gardiner, and P. Zoller, *Cold Bosonic Atoms in Optical Lattices*, Phys. Rev. Lett. **81**, 3108 (1998)

[3] W. Zwerger, *Mott-Hubbard Transition of Cold Atoms in Optical Lattices*, Journal Optics B **5**, S9 (2003)

[4] H. Kleinert, S. Schmidt, and A. Pelster, *Reentrant Phenomenon in the Quantum Phase Transitions of a Gas of Bosons Trapped in an Optical Lattice*, Phys. Rev. Lett. **93**, 160402 (2004)

[5] H. Kleinert, S. Schmidt, and A. Pelster, *Quantum Phase Diagram for Homogeneous Bose-Einstein Condensate*, Ann. Phys. (Leipzig) **14**, 214 (2005)

[6] D. van Oosten, P. van der Straten, and H. T. Stoof, *Quantum Phases in an Optical Lattice*, Phys. Rev. **A 63**, 053601 (2001)

[7] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Quantum Phase Transition From a Superfluid to a Mott Insulator in a Gas of Ultracold Atoms*, Nature **415**, 39 (2002)

[8] V. Bagnato, D.E. Pritchard, and D. Kleppner, *Bose-Einstein Condensation in an External Potential*, Phys. Rev. A **35**, 4354 (1987)

[9] Ph.W. Courteille, V.S. Bagnato, and V.I. Yukalov, *Bose-Einstein Condensation of Trapped Atomic Gases*, Laser Phys. **11**, 659 (2001)

[10] O. Zobay, G. Metikas, and H. Kleinert, *Nonperturbative Effects on T_c of Interacting Bose Gases in Power-Law Traps*, `cond-mat/0411133`

[11] S. Giorgini, L.P. Pitaevskii, and S. Stringari, *Condensate Fraction and Critical Temperature of a Trapped Interacting Bose Gas*, Phys. Rev. A **54**, R4633 (1996)

[12] F. Gerbier, J.H. Thywissen, S. Richard, M. Hugbard, P. Bouyer, and A. Aspect, *Critical Temperature of a Trapped, Weakly Interacting Bose Gas*, Phys. Rev. Lett. **92**, 030405 (2004)

[13] P. Arnold and B. Tomášik, *T_c For Trapped Dilute Bose Gases: A Second-Order Result*, Phys. Rev. A **64**, 053609 (2001)

[14] B. Kastening, *Non-Universal Critical Quantities from Variational Perturbation Theory and Their Application to the BEC Temperature Shift*, Phys. Rev. A **70**, 043621 (2004)

[15] H. Kleinert, *Five-Loop Critical Temperature Shift in Weakly Interacting Homogeneous Bose-Einstein Condensate*, Mod. Phys. Lett. B **17**, 1011 (2003)

[16] B. Kastening, *Bose-Einstein Condensation Temperature of a Homogeneous Weakly Interacting Bose Gas in Variational Perturbation Theory Through Six Loops*, Phys. Rev. A **68**, 061601(R) (2003)

[17] B. Kastening, *Bose-Einstein Condensation Temperature of a Homogenous Weakly Interacting Bose Gas in Variational Perturbation Theory Through Seven Loops*, Phys. Rev. A **69**, 043613 (2004)

[18] B. Kastening, *Shift of BEC Temperature of Homogenous Weakly Interacting Bose Gas*, Laser Phys. **14**, 586 (2004)

[19] V.A. Kashurnikov, N.V. Prokof´ev, and B.V. Svistunov, *Critical Temperature Shift in Weakly Interacting Bose Gas*, Phys. Rev. Lett. **87** 120402, (2001)

[20] P. Arnold and G.D. Moore, *Transition Temperature of a Dilute Homogeneous Imperfect Bose Gas*, Phys. Rev. Lett. **87**, 120401 (2001)

[21] P. Arnold and G. Moore, *Monte Carlo Simulation of O(2) Φ^4 Field Theory in Three Dimensions*, Phys. Rev. E **64**, 066113 (2001)

[22] P. Arnold, G. Moore, and B. Tomášik, *T_c For Homogeneous Dilute Bose Gases: A Second Order Result*, Phys. Rev. A **65**, 013606 (2002)

[23] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics*, Third Extended Edition (World Scientific, Singapore, 2004)

[24] H. Kleinert and V. Schulte-Frohlinde, *Critical Phenomena in Φ^4-Theory* (World Scientific, Singapore, 2001)

[25] R.P. Feynman, *Statistical Mechanics* (Bejamin, Reading, 1972)

[26] F. Brosens, J.T. Devreese, and L.F. Lemmens, *Thermodynamics of Coupled Identical Oscillators Within the Path-Integral Formalism*, Phys. Rev. E **55**, 227 (1997)

[27] K. Glaum, H. Kleinert, and A. Pelster, *Bose-Einstein Condensation in Canonical Ensembles*, in preparation

[28] `http://www.physik.uni-stuttgart.de/institute/pi/5/news/041126/ChromBEC.html`

[29] P.O. Schmidt, S. Hensler, J.Werner, A. Griesmaier, A. Görlitz, T. Pfau, and A. Simoni, *Determination of the s-Wave Scattering Length of Chromium*, Phys. Rev. Lett. **91**, 193201 (2003)

[30] V. Bretin, S. Stock, Y. Seurin, and J. Dalibard, *Fast Rotation of a Bose-Einstein Condensate*, Phys. Rev. Lett. **92**, 050403 (2004)

[31] S. Stock, V. Bretin, F. Chevy, and J. Dalibard, *Shape Oscillation of a Rotating Bose-Einstein Condensate*, Europhys. Lett. **65**, 594 (2004)

[32] H. Kleinert, *Gauge Fields in Condensed Matter*, Vol. I: *Superflow and Vortex Lines – Vortex Fields, Phase Transitions*; Vol. II: *Stresses and Defects – Differential Geometry,Crystal Melting* (World Scientific, Singapore, 1989)