Anisotropic Two-Fluid Hydrodynamics

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Isotropic Landau-Khalatnikov Two-Fluid Model

General Properties

- **Two Ideal Fluids**
  normal fluid and superfluid
- **Continuity Equation**
  $\dot{\rho} + \mathrm{div}\,\boldsymbol{j} = 0$
  density $\rho_s + \rho_n = \rho$
  current $\boldsymbol{j} = \rho_s \boldsymbol{v}_s + \rho_n \boldsymbol{v}_n$
- **Conservation of Momentum**

$$\partial_t j_i + \partial_k \Pi_{ik} = 0$$

$$\Pi_{ik} = \rho_s v_{s_i} v_{s_k} + \rho_n v_{n_i} v_{n_k} + p\delta_{ik}$$

- **Pressure**

$$p = -u + Ts + \mu\rho + \tfrac{1}{2}\rho_n (\boldsymbol{v}_n - \boldsymbol{v}_s)^2$$

with μ: chemical potential per unit mass,
s: entropy density, u: internal energy density

- **Entropy Conservation**

$$\dot{s} + \mathrm{div}\,(s\boldsymbol{v}_n) = 0;\ s = s_n, s_s = 0$$

- **Superfluid Euler Equation**

$$\dot{\boldsymbol{v}}_s + \nabla\left(\tfrac{1}{2}\boldsymbol{v}_s^2 + \bar{\mu}\right) = \boldsymbol{0},\ \mathrm{rot}\,\boldsymbol{v}_s = \boldsymbol{0}$$

Propagation of Sound

- **Linearization, Ansatz** $\delta s, \delta\rho \sim e^{i(\boldsymbol{q}\cdot\boldsymbol{r} - \omega t)}$
- **Approximation** $\left(\frac{\partial p}{\partial T}\right)_\rho \simeq 0$
- **First Sound**
  density and pressure oscillations

$$u_1^2 = \left(\frac{\partial p}{\partial \rho}\right)_{\bar{s}}, \qquad \delta\boldsymbol{v}_n = \delta\boldsymbol{v}_s$$

- **Second Sound**
  entropy and temperature oscillations

$$u_2^2 = \bar{s}_0^2 \frac{\rho_{s_0}}{\rho_{n_0}} \left(\frac{\partial T}{\partial \bar{s}}\right)_\rho, \qquad \delta\boldsymbol{v}_n = -\delta\boldsymbol{v}_s$$

Anisotropic Extension

Motivation

- **Solution of the GP Eq. for a dipolar BEC with disorder [1] yields**

$$n_{s_{ik}} = n\delta_{ik} - \int \frac{\mathrm{d}^3k}{(2\pi)^2} \frac{4nR(\boldsymbol{k})k_i k_k}{\boldsymbol{k}^2\left[\hbar^2\boldsymbol{k}^2/2m + 2nV_{\mathrm{int}}(\boldsymbol{k})\right]^2} + \ldots$$

- **Super- and normalfluid densities are tensors of second rank**

$$\rho_{n_{ij}} + \rho_{s_{ij}} = \rho_{ij} = \tilde{\rho}\delta_{ij}$$

- **What are the consequences for two-fluid hydrodynamic?**

Action Principle

Action

$$\mathcal{A} = \iint \mathrm{d}^3r\,\mathrm{d}t\ \Big\{\tfrac{1}{2}v_{s_i}(\tilde{\rho}\delta_{ij} - \rho_{n_{ij}})v_{s_j} + \tfrac{1}{2}v_{n_i}\rho_{n_{ij}}v_{n_j} - u(\tilde{\rho}, \rho_{n_{ij}}, s) + \lambda\left[\frac{\partial\tilde{\rho}}{\partial t} + \partial_i((\tilde{\rho}\delta_{ij} - \rho_{n_{ij}})v_{s_j} + \rho_{n_{ij}}v_{n_j})\right] + \kappa\left[\frac{\partial s}{\partial t} + \mathrm{div}(s\boldsymbol{v}_n)\right]\Big\}$$

Differential equation of state

$$\mathrm{d}u = T\mathrm{d}s + \bar{\mu}\mathrm{d}\tilde{\rho} + \frac{\partial u}{\partial\rho_{n_{ij}}}\mathrm{d}\rho_{n_{ij}}$$

Extremize action to obtain 7 sets of equations

$$\frac{\delta\mathcal{A}}{\delta\rho} = 0,\ \frac{\delta\mathcal{A}}{\delta\rho_{n_{ik}}} = 0,\ \frac{\delta\mathcal{A}}{\delta s} = 0,\ \frac{\delta\mathcal{A}}{\delta v_{n_i}} = 0,\ \frac{\delta\mathcal{A}}{\delta v_{s_i}} = 0,\ \frac{\delta\mathcal{A}}{\delta\lambda} = 0,\ \frac{\delta\mathcal{A}}{\delta\kappa} = 0$$

Classical Approach

Classical approach analog to [2]

Conservation laws without explicit expressions for Π_{ik}, $\boldsymbol{j}$, etc.

Galilean transformations impose structure of flux tensors

velocity	$\boldsymbol{v}_n = \boldsymbol{v}_s + \boldsymbol{v}_{n_0}$
current	$\boldsymbol{j} = \tilde{\rho}\boldsymbol{v}_s + \boldsymbol{j}_0$
momentum density	$\Pi_{ik} = \Pi_{0_{ik}} + \tilde{\rho}v_{s_i}v_{s_k} + v_{s_i}j_{0_k} + j_{0_i}v_{s_k}$
energy density	$e = \frac{1}{2}\tilde{\rho}\boldsymbol{v}_s^2 + \boldsymbol{v}_s\cdot\boldsymbol{j}_0 + e_0$
energy flux	$\boldsymbol{q} = \left(\frac{1}{2}\tilde{\rho}\boldsymbol{v}_s^2 + \boldsymbol{v}_s\cdot\boldsymbol{j}_0 + e_0\right)\boldsymbol{v}_s + \frac{1}{2}\boldsymbol{v}_s^2\boldsymbol{j}_0 + \Pi_0\boldsymbol{v}_s + \boldsymbol{q}_0$
entropy flux	$\boldsymbol{f} = s\boldsymbol{v}_s + \boldsymbol{f}_0$

Notice: energy flux does not depend on space and time derivatives

Eliminate those dependencies to obtain expressions for flux quantities

Result & Outlook

Hydrodynamic Equations

- **Equations inherit their structure from isotropic case**
- **Current** $j_i = \rho_{n_{ij}}v_{n_j} + \rho_{s_{ij}}v_{s_j}$
- **Pressure gradient** $\partial_i p = \rho_{n_{jk}}(v_n - v_s)_k\,\partial_i(v_n - v_s)_j + \tilde{\rho}\partial_i\bar{\mu} + s\partial_i T$
- **Asymmetric momentum density tensor**

$$\Pi_{ik} = \rho_{s_{kj}}v_{s_j}v_{s_i} + \rho_{n_{ij}}v_{n_j}v_{n_k} - \rho_{n_{ij}}v_{n_k}v_{s_j} - \rho_{n_{kj}}v_{n_j}v_{s_i} + p\delta ik$$

consequence of broken rotational invariance
indicates the possibility of intrinsic angular momentum

Sound Modes

- **In a fully polarized quantum gas, the tensoric densities simplify to**

$$\rho_{n,s_{x,y}} = \rho_{n,s_\perp},\ \rho_{n,s_z} = \rho_{n,s_\parallel}$$

- **Linearization yields coupled wave-equations**

$$\begin{pmatrix} (\frac{\partial p}{\partial\tilde{\rho}})_{\bar{s}} - u_{\parallel,\perp}^2 & (\frac{\partial p}{\partial\bar{s}})_{\tilde{\rho}} \\ \bar{s}_0^2\frac{\rho_{s_{\parallel,\perp}}}{\rho_{n_{\parallel,\perp}}}(\frac{\partial T}{\partial\tilde{\rho}})_{\bar{s}} & \bar{s}_0^2\frac{\rho_{s_{\parallel,\perp}}}{\rho_{n_{\parallel,\perp}}}(\frac{\partial T}{\partial\bar{s}})_{\tilde{\rho}} - u_{\parallel,\perp}^2 \end{pmatrix} \begin{pmatrix} \delta\tilde{\rho} \\ \delta\bar{s} \end{pmatrix} = \boldsymbol{0}$$

- **Sound velocities depend on direction $\parallel$, $\perp$**

$$u_{1,2_{\parallel,\perp}}^2 = \frac{1}{2}\left[\left(\frac{\partial p}{\partial\tilde{\rho}}\right)_{\bar{s}} + \frac{\rho_{s_{0\parallel,\perp}}}{\rho_{n_{0\parallel,\perp}}}\frac{T\bar{s}_0^2}{c_\nu}\right] \pm \sqrt{\frac{1}{4}\left[\left(\frac{\partial p}{\partial\tilde{\rho}}\right)_{\bar{s}} + \frac{\rho_{s_{0\parallel,\perp}}}{\rho_{n_{0\parallel,\perp}}}\frac{T\bar{s}_0^2}{c_\nu}\right]^2 - \frac{\rho_{s_{0\parallel,\perp}}}{\rho_{n_{0\parallel,\perp}}}\left(\frac{T\bar{s}_0^2}{c_\nu}\right)\left(\frac{\partial p}{\partial\tilde{\rho}}\right)_T}$$

- **First and second sound modes are coupled**
- **Amplitude ratio** $\frac{\delta\tilde{\rho}}{\delta\bar{s}} = \frac{\left(\frac{\partial p}{\partial\bar{s}}\right)_{\tilde{\rho}}}{u_{\parallel,\perp}^2 - \left(\frac{\partial p}{\partial\tilde{\rho}}\right)_{\bar{s}}}$
- **Normal fluid and superfluid oscillate in-phase or out-of-phase**

$$\delta\boldsymbol{v}_s = \nu\delta\boldsymbol{v}_n,\ \nu = \frac{\gamma\rho_{n_{0\parallel,\perp}}}{\alpha - \rho_{s_{0\parallel,\perp}}\gamma}$$

Outlook

- **Calculate thermodynamic relations from a microscopic model**
- **Obtain sound velocities in dependence of temperature, dipole interaction strength and disorder**

References

[1] C. Krumnow and A. Pelster. Dipolar Bose-Einstein condensates with weak disorder. *Phys. Rev. A*, 84:021608, 2011.

[2] I. M. Khalatnikov. *An Introduction to the Theory of Superfluidity*. Westview Press, New York, 2000.