Collective Excitations in Bose-Einstein Condensates

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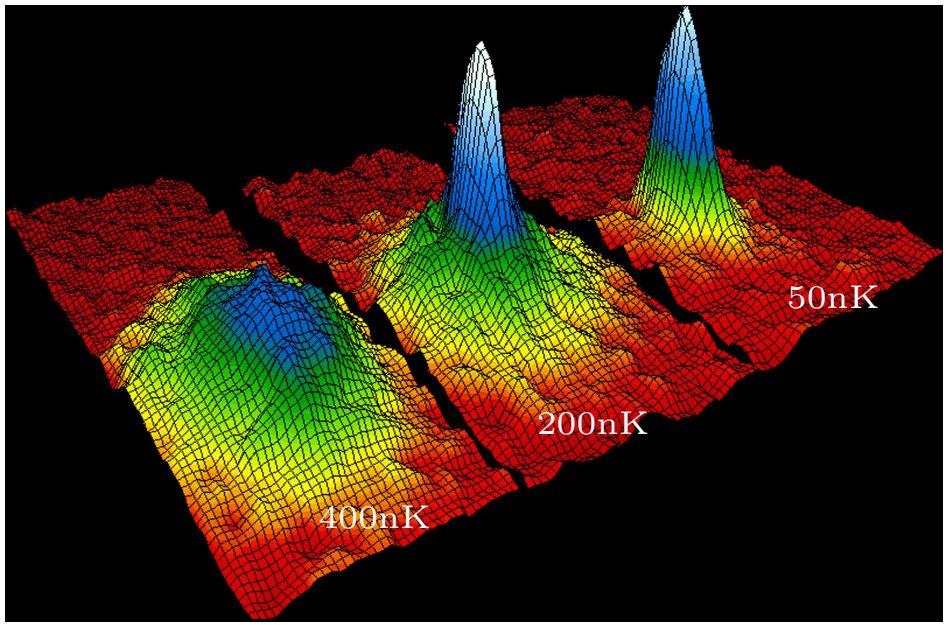
Outline

The Rise of Bose-Einstein Condensation

- Predicted by Einstein (1925) based on work of Bose (1924)
- Experiment with dilute ultracold atomic gases (1995):



⇒ Nobel Prize in Physics (Ketterle, Wieman, Cornell) in 2001

⇒ more than 6000 publications

Gross-Pitaevskii Equation

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \Big[-\frac{\hbar^2}{2M}\Delta + V(\mathbf{r}) + g_2 N\,|\psi(\mathbf{r},t)|^2 + g_3 N^2\,|\psi(\mathbf{r},t)|^4\Big]\psi(\mathbf{r},t)$$

$$V(\mathbf{r}) = \frac{1}{2}M\omega_\rho^2\left(\rho^2 + \lambda^2 z^2\right), \quad \lambda = \omega_z/\omega_\rho$$

- Gaussian variational ansatz: *PRL* **77**, *5320 (1996)*

$$\psi(\rho,z,t) = \frac{1}{\sqrt{\pi^{\frac{3}{2}}u_\rho^2 u_z}}\exp\left\{\sum_{\alpha=\rho,z}\left[\left(-\frac{1}{2u_\alpha^2} - i\varphi_\alpha\right)\alpha^2\right]\right\}$$

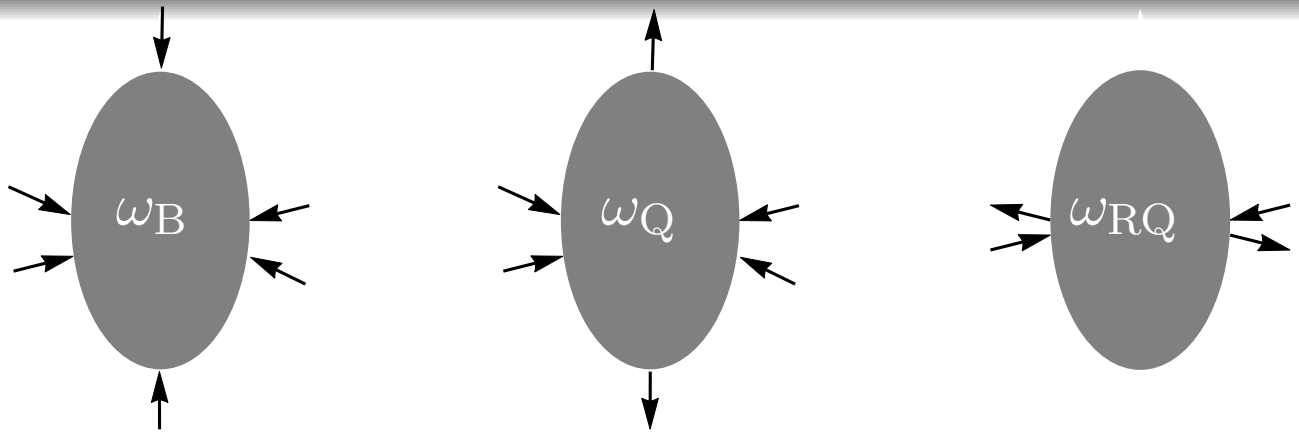
- Equations of motion: $u_\alpha \to u_\alpha l$, $l = \sqrt{\hbar/M\omega_\rho}$

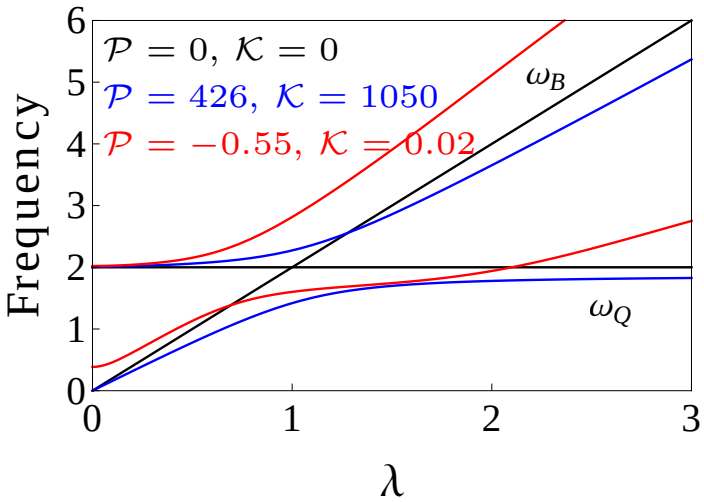
$$\ddot{u}_\rho + u_\rho - \frac{1}{u_\rho^3} - \frac{\mathcal{P}}{u_\rho^3 u_z} - \frac{\mathcal{K}}{u_\rho^5 u_z^2} = 0\,, \quad \mathcal{P} = 426$$

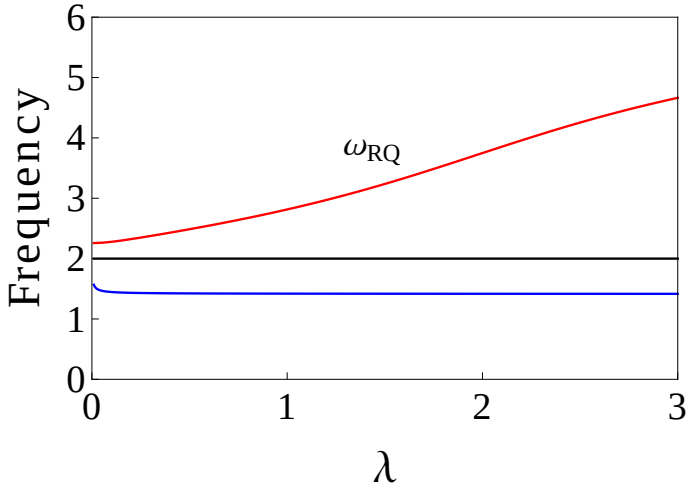
$$\ddot{u}_z + \lambda^2 u_z - \frac{1}{u_z^3} - \frac{\mathcal{P}}{u_\rho^2 u_z^2} - \frac{\mathcal{K}}{u_\rho^4 u_z^3} = 0\,, \quad \mathcal{K} = 1050$$

- Experiment: $N = 10^5$ ^{87}Rb, $\omega_\rho = 2\pi \times 112$ Hz, *PRL* **86**, *2196 (2001)*

Frequencies of Collective Modes





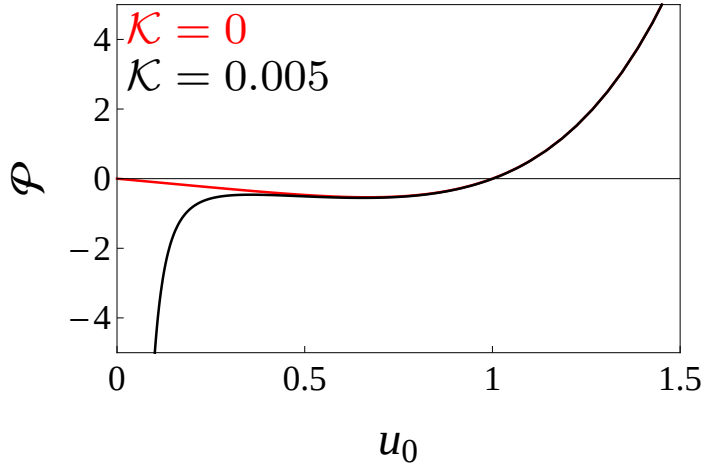


H. Al-Jibbouri, I. Vidanović, A. Balaž, and A. Pelster, J. Phys B **46**, *065303 (2013)*

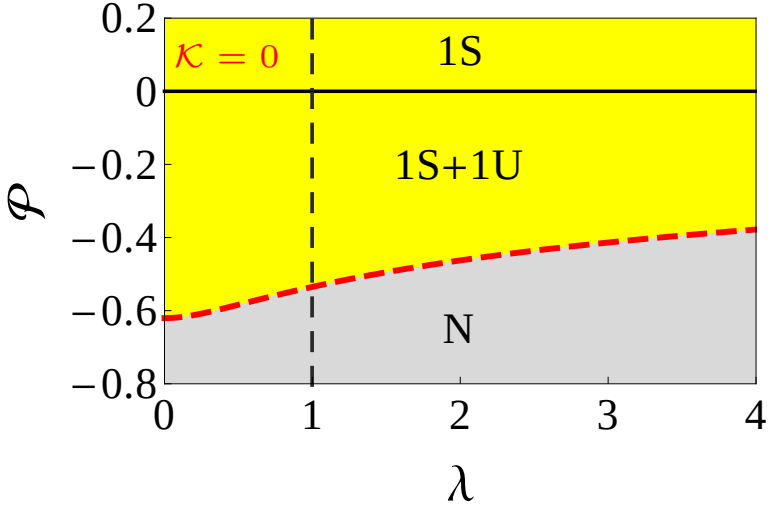
Stability Diagram

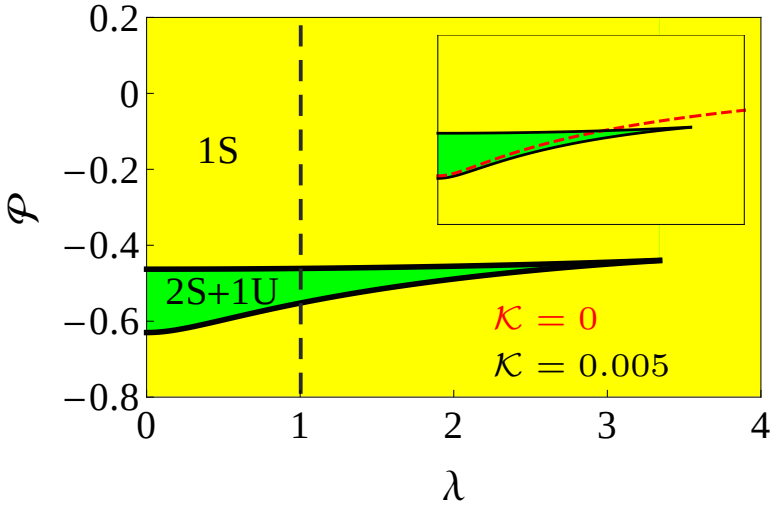
- Isotropic case $\lambda = 1$

$$\mathcal{P} = u_0^5 - u_0 - \frac{\mathcal{K}}{u_0^3}$$



- Axially-symmetric case





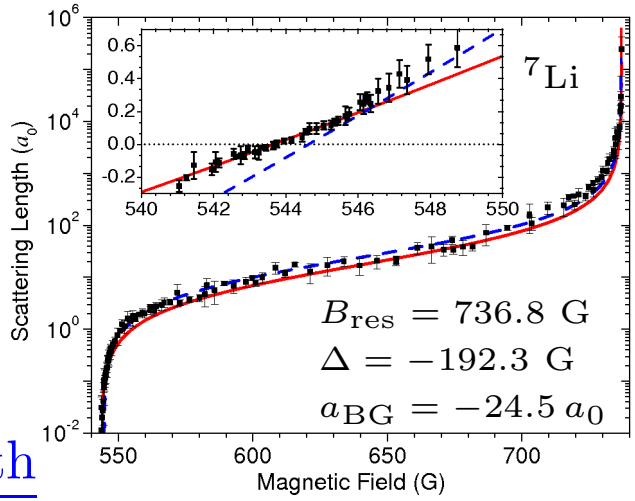
H. Al-Jibbouri, I. Vidanović, A. Balaž, and A. Pelster, J. Phys B **46**, *065303 (2013)*

Outline

Feshbach Resonance

$$g_2 = \frac{4\pi\hbar^2 a_s}{M}\,,\ a_s = a_{BG}\left[1 - \frac{\Delta}{B - B_{res}}\right]$$

S. E. Pollack, D. Dries, M. Junker, Y. P. Chen, T. A. Corcovilos, and R. G. Hulet, PRL **102**, *090402 (2009)*



Periodic Modulation of Interaction Strength

$$B(t) = B_0 + \delta B \cos\Omega t\,, \quad B_0 = 565\,\mathrm{G},\ \delta B = 10\,\mathrm{G}$$

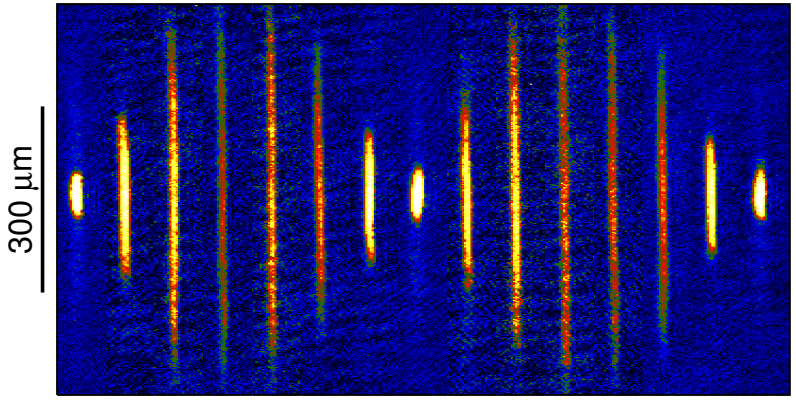
$$\Rightarrow \quad a_s(t) = a_{av} + \delta a \cos\Omega t, \qquad a_{av} = 3a_0,\ \delta a = 2a_0$$

$$\Rightarrow \quad \mathcal{P}(t) = \mathcal{P} + \mathcal{Q}\cos\Omega t, \qquad \mathcal{P} = 15,\ \mathcal{Q} = 10$$

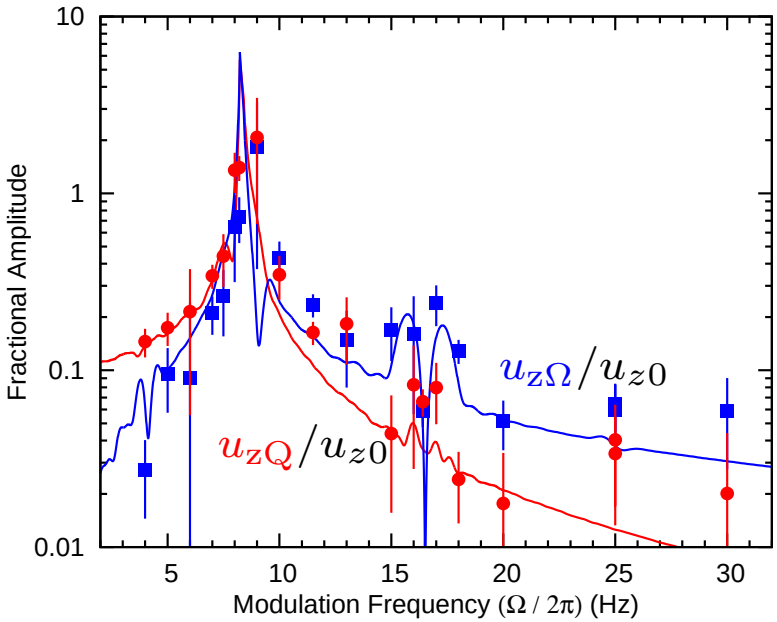
S. E. Pollack, D. Dries, R. G. Hulet, K. M. F. Magalhães, E. A. L. Henn, E. R. F. Ramos, M. A. Caracanhas, and V. S. Bagnato, Phys. Rev. A **81**, *053627 (2010)*

Experimental Result

Stroboscopic pictures:



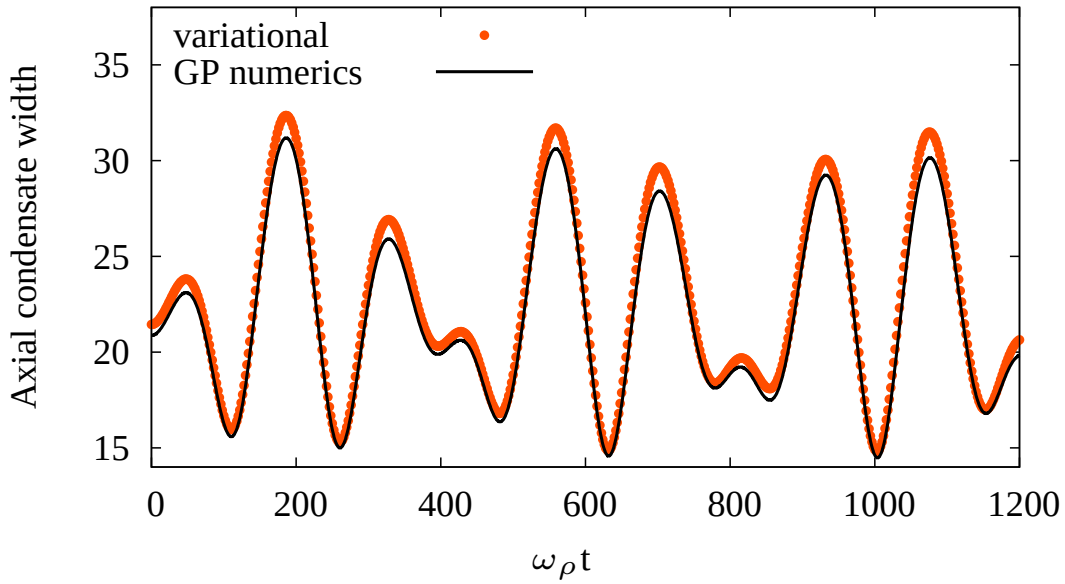
$\longleftarrow t = 100\,\mathrm{ms} \longrightarrow$



Fit: $u_z(t) = u_{z0} + u_{z\Omega}\sin(\Omega t + \Phi) + u_{zQ}\sin(\omega_Q t + \phi)$

S. E. Pollack, D. Dries, R. G. Hulet, K. M. F. Magalhães, E. A. L. Henn, E. R. F. Ramos, M. A. Caracanhas, and V. S. Bagnato, Phys. Rev. A **81**, *053627 (2010)*

Condensate Dynamics

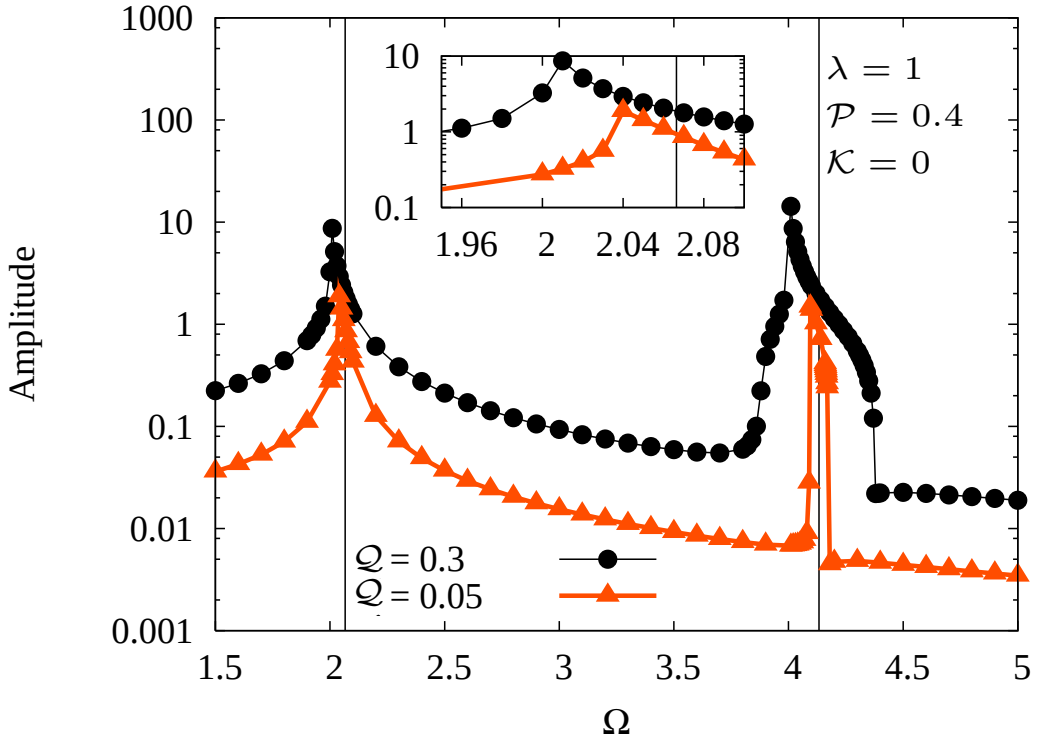


$\omega_\rho = 2\pi \times 4.85$ Hz, $\lambda = 0.021$, $\mathcal{P} = 15$, $\mathcal{Q} = 10$, $\Omega = 0.05\omega_\rho$, and $\mathcal{K} = 0$

I. Vidanović, A. Balaž, H. Al-Jibbouri, and A. Pelster, PRA **84**, *013618 (2011)*
D. Vudragović, I. Vidanović, A. Balaž, P. Muruganandam, and S. K. Adhikari, Comput. Phys. Commun. **183**, *2021 (2012)*

Excitation Spectra: Numerical Result

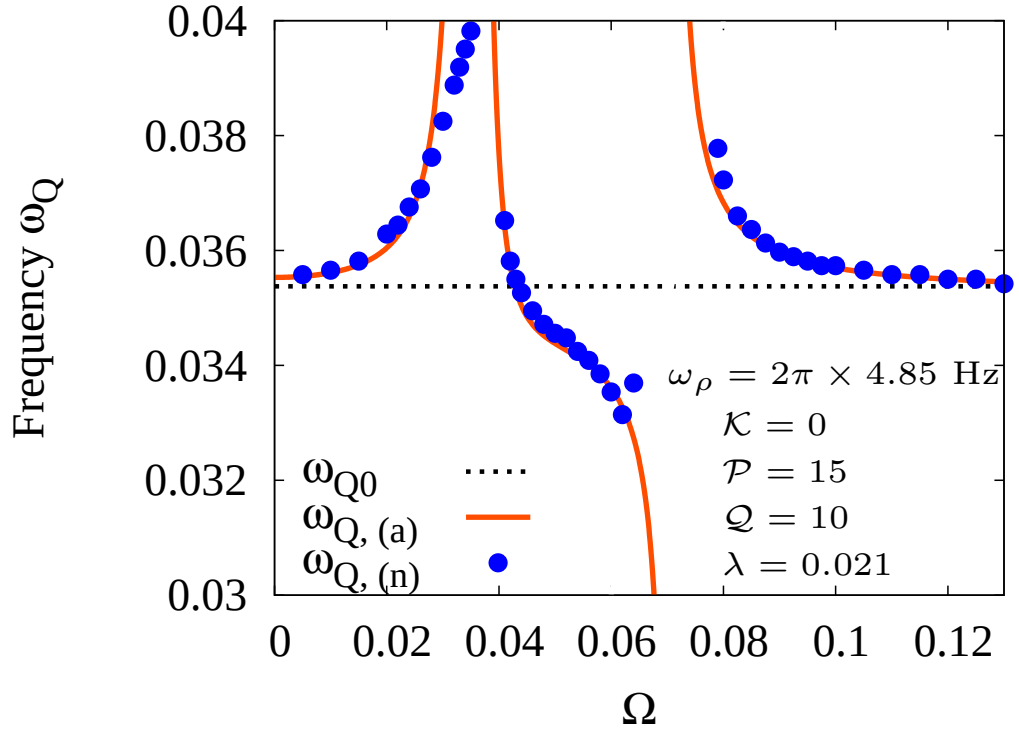
Parametric resonance:
$\Omega = 2\omega_B/n$



I. Vidanović, A. Balaž, H. Al-Jibbouri, and A. Pelster, PRA **84**, *013618 (2011)*

Result From Poincaré-Lindstedt Method

$$\omega_Q(\Omega) = \omega_{Q_0} + \mathcal{Q}^2 \frac{\mathcal{F}_Q(\omega_{Q_0}, \omega_{B_0}, \Omega, u_{\rho 0}, u_{z0}, \mathcal{P}, \mathcal{K} = 0)}{(\Omega^2 - \omega_{Q_0}^2)^2(\Omega^2 - 4\omega_{Q_0}^2)} + \dots$$



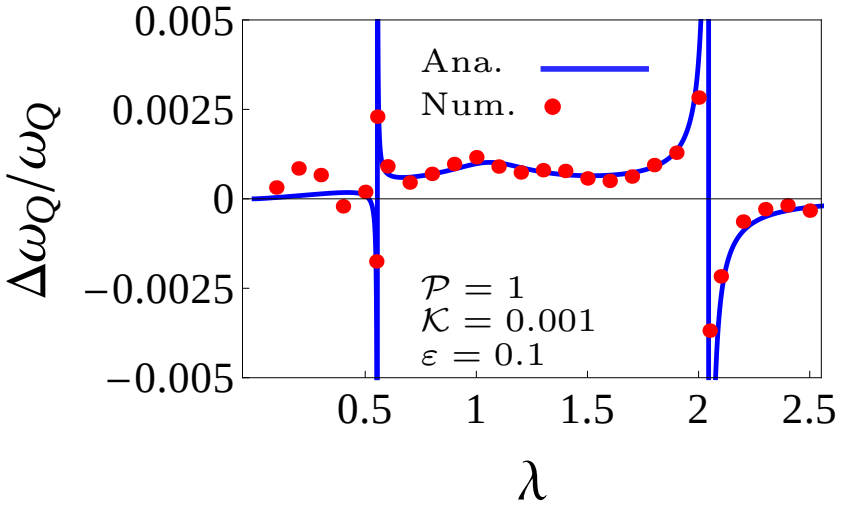
I. Vidanović, A. Balaž, H. Al-Jibbouri, and A. Pelster, PRA **84**, *013618 (2011)*

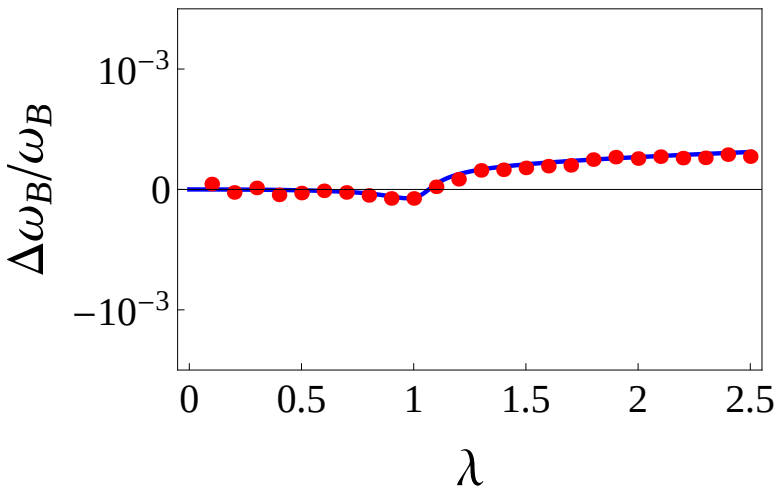
Outline

Quadrupole and Breathing Mode Frequency Shifts

Initial condition: $\mathbf{u}(0) = \mathbf{u}_0 + \varepsilon\, \mathbf{u}_{Q_0,B_0}, \quad \dot{\mathbf{u}}(0) = \mathbf{0}$

$$\Rightarrow \quad \omega_{Q,B}(\varepsilon) = \omega_{Q_0,B_0} + \varepsilon^2 \frac{\mathcal{G}_{Q,B}(\omega_{Q0}, \omega_{B0}, u_{\rho 0}, u_{z0}, \mathcal{P}, \mathcal{K}, \lambda)}{(\omega_{B_0,Q_0} - 2\omega_{Q_0,B_0})(\omega_{B_0,Q_0} + 2\omega_{Q_0,B_0})} + \ldots$$





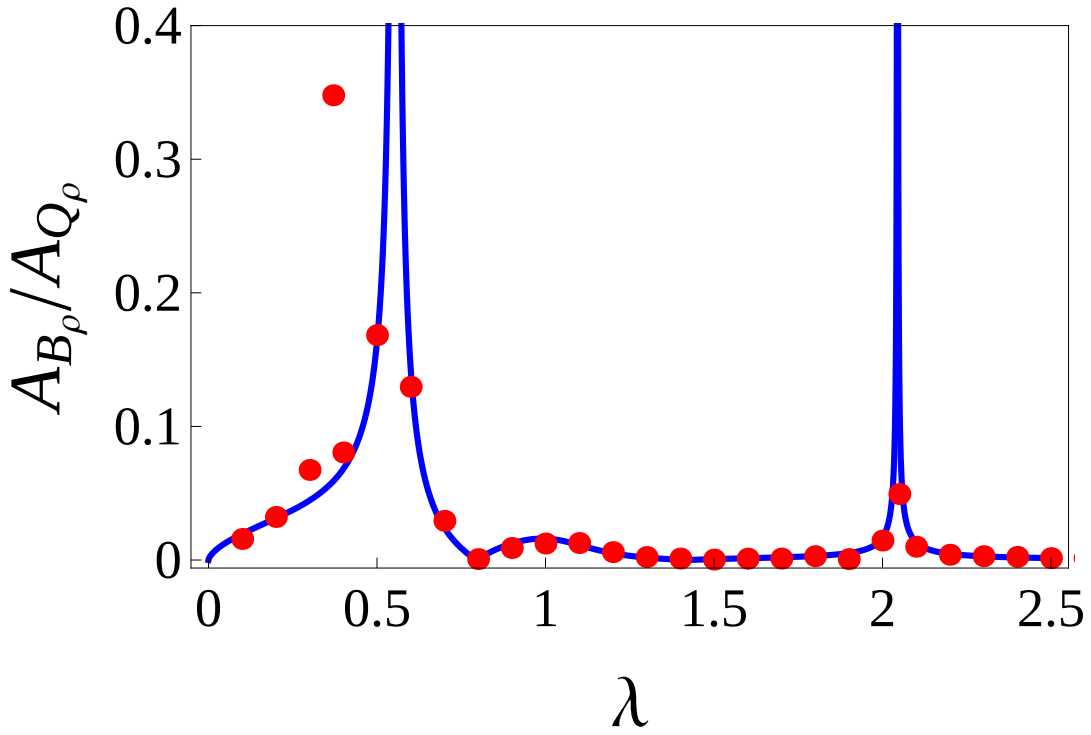
Geometric resonances at $\omega_{B0} = 2\omega_{Q0}$:
$\lambda_1 = 0.55$ and $\lambda_2 = 2.056$

No geometric resonances: $\omega_{Q0} \neq 2\omega_{B0}$

H. Al-Jibbouri, I. Vidanović, A. Balaž, and A. Pelster, J. Phys B **46**, *065303 (2013)*

Resonant Mode Coupling for Quadrupole Mode

$$\mathbf{u}(t) = \mathbf{u}_0 + \mathbf{A}_Q \cos\omega_{Q_0} t + \mathbf{A}_B \cos\omega_{B_0} t + \ldots$$



$\mathcal{P} = 1$
$\mathcal{K} = 0.001$
$\varepsilon = 0.1$

Special case: $\Rightarrow$ $A_{B_\rho} = 0$ at $\lambda = 0.85$, $\lambda = 1.42$

H. Al-Jibbouri, I. Vidanović, A. Balaž, and A. Pelster, J. Phys B **46**, *065303 (2013)*

Outline

Kohn Theorem Near Feshbach Resonance

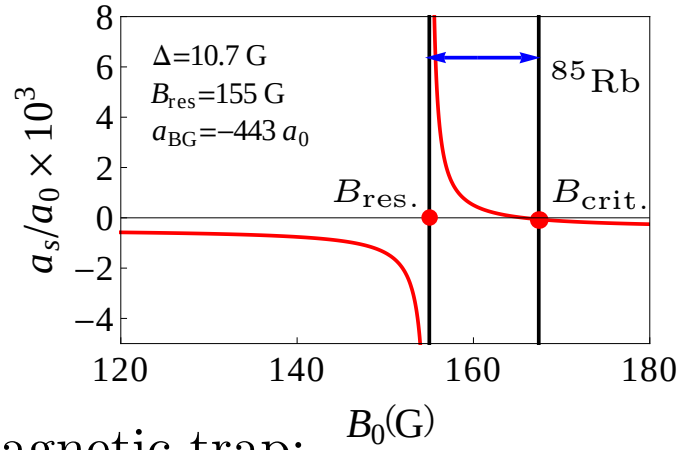
- Dipole mode (Kohn Theorem): $\omega_{\mathrm{D}} = \omega_z = \lambda\omega_\rho$

  W. Kohn, PR **123**, *1242 (1961)*

- Feshbach resonance:

$$g_2 = \frac{4\pi\hbar^2 a_{\mathrm{s}}}{M}\,,\quad a_{\mathrm{s}} = a_{\mathrm{BG}}\left[1 - \frac{\Delta}{B - B_{\mathrm{res}}}\right]$$

C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Rev. Mod. Phys. **82**, *1225 (2010)*



- Kohn theorem is not applicable in magnetic trap:

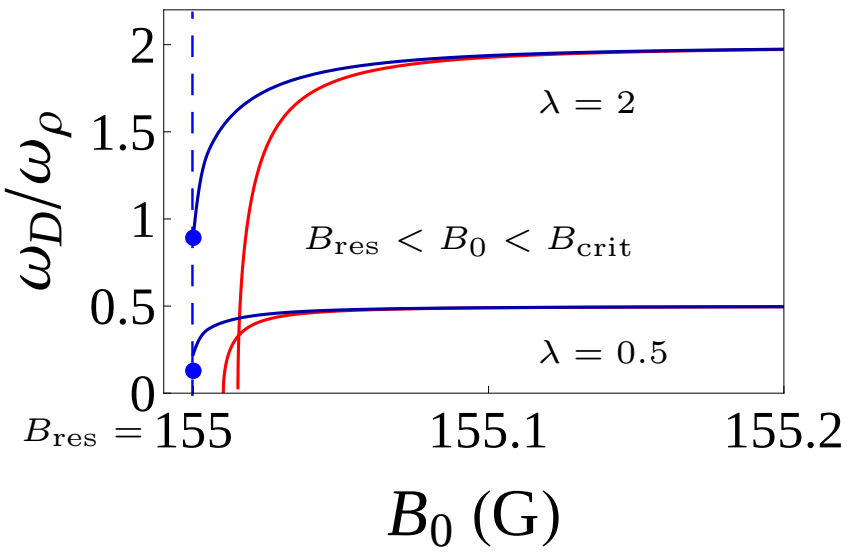
$$V(\mathbf{r}) = V_0 + \frac{M\omega_\rho^2}{2}\left(\rho^2 + \lambda^2 z^2\right)$$
$$B(\mathbf{r}) = B_0 + \frac{M\omega_\rho^2}{2\mu_{\mathrm{B}}}\left(\rho^2 + \lambda^2 z^2\right)\,,\quad B_0 = V_0/\mu_{\mathrm{B}}$$

E. R. F. Ramos, F. E. A. dos Santos, M. A. Caracanhas, and V. S. Bagnato, PRA **85**, *033608 (2012)*

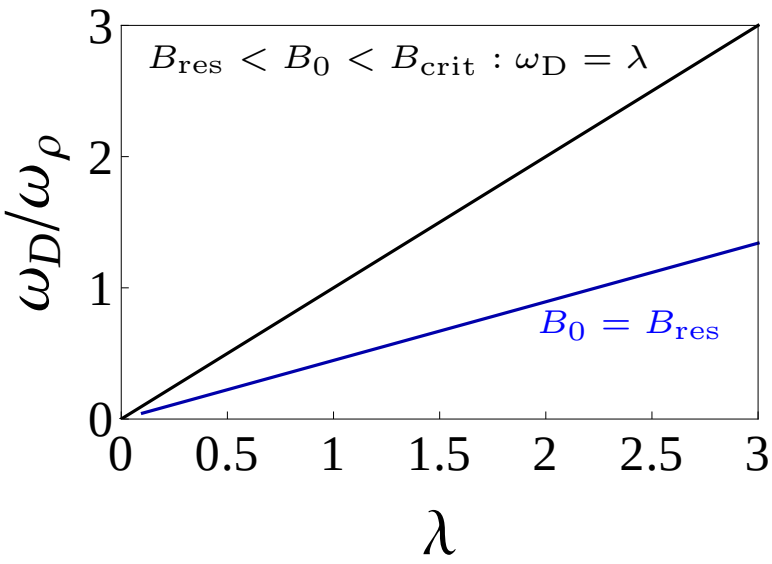
H. Al-Jibbouri and A. Pelster, PRA **88**, *033621 (2013)*

Dipole Mode Frequency in Vicinity of Feshbach Resonance

On right-hand side



On top



E. R. F. Ramos, F. E. A. dos Santos, M. A. Caracanhas, and V. S. Bagnato, PRA **85**, *033608 (2012)*

H. Al-Jibbouri and A. Pelster, PRA **88**, *033621 (2013)*

$\Rightarrow$ Experimentally: B_0 controllable down to 1 mG, *PRL* **106**, *255303 (2011)*

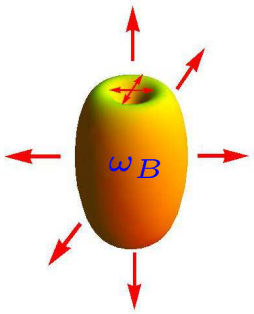
Outline

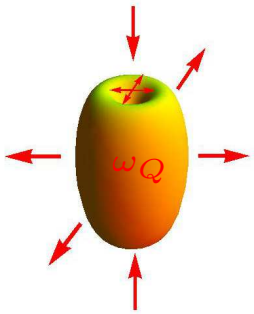
Frequencies with a Single Vortex

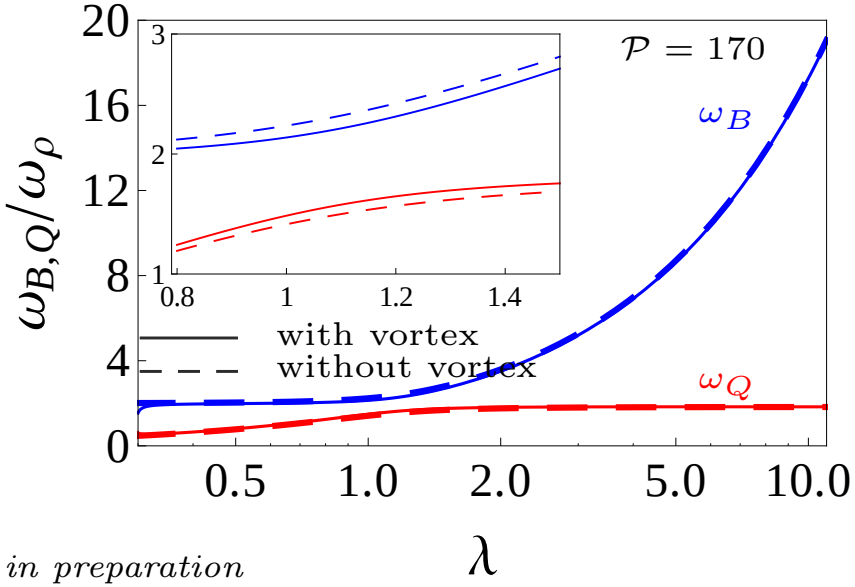
- Thomas-Fermi variational ansatz with vorticity $\gamma = 1$

$$\psi(\rho, z, \phi) = C\sqrt{\frac{\rho^2}{\rho^2+R_\rho^2\beta^2}\left[1-\left(\frac{\rho}{R_\rho}\right)^2-\left(\frac{z}{R_z}\right)^2\right]}\,e^{i\phi+i\varphi_\rho\rho^2+i\varphi_z z^2}$$

D. H. J. O' Dell and C. Eberlein, PRA **75**, *013604 (2007)*







- *H. Al-Jibbouri, A. Balaž, and A. Pelster, in preparation*
- *R. P. Teles, V. S. Bagnato, and F. E. A. dos Santos, ArXiv:1306.2534*

Outline

Summary and Outlook

- We presented prominent nonlinear features of collective excitations due to parametric and geometric resonances.
- The Kohn theorem can not be directly applied in the vicinity of a Feshbach resonance in a magnetic trap.
- A vortex changes the collective excitation frequencies.
- Outlook
  1. Anharmonic trap
  2. Quantum fluctuations
  3. Finite temperature

Thank you for your attention

Poincaré-Lindstedt Method: Isotropic Case $\lambda = 1$

- To calculate collective modes to higher orders, we rescale time as $s = \omega t$ and use expansions:

$$\omega^2\,\ddot{u}(s) + u(s) - \frac{1}{u(s)^3} - \frac{\mathcal{P}}{u(s)^4} - \frac{\mathcal{Q}}{u(s)^4}\,\cos\frac{\Omega s}{\omega} = 0$$

- Perturbative expansions

$$\begin{aligned} u(s) &= u_0 + \mathcal{Q}\,u_1(s) + \mathcal{Q}^2\,u_2(s) + \mathcal{Q}^3\,u_3(s) + \ldots \\ \omega &= \omega_0 + \mathcal{Q}\,\omega_1 + \mathcal{Q}^2\,\omega_2 + \mathcal{Q}^3\,\omega_3 + \ldots \end{aligned}$$

$$\begin{aligned} \omega_0^2\ddot{u}_1(s) + \omega_0^2 u_1(s) &= \frac{1}{u_0^4}\cos\frac{\Omega s}{\omega} \\ \omega_0^2\ddot{u}_2(s) + \omega_0^2 u_2(s) &= -2\omega_0\omega_1\ddot{u}_1(s) - \frac{4}{u_0^5}u_1(s)\cos\frac{\Omega s}{\omega} + \alpha u_1(s)^2 \end{aligned}$$

Poincaré-Lindstedt Method: Isotropic Case $\lambda = 1$

- $n = 3$

$$\omega_0^2 \ddot{u}_3(s) + \omega_0^2 u_3(s) + C \cos s + \ldots = 0,$$

- Particular solution

$$u_3(s) = -Cs \sin s + \ldots$$

- Last term can be absorbed into first-order solution

$$\begin{aligned} u(s) &= u_0 + \mathcal{Q}A \cos s - C\mathcal{Q}^3 s \sin s + \ldots \\ &\approx u_0 + \mathcal{Q}A \cos(s + \delta s), \quad \delta = \mathcal{Q}^2 C/A \end{aligned}$$

Physical Motivation of Kohn Theorem Near Feshbach Resonance

- Thomas-Fermi approximation: $\mu = V(\mathbf{r}) + g_2 n(\mathbf{r})$
- Far away from Feshbach resonance: $\mathcal{H} = B_0 - B_{\rm res} > 0$

$$\mu = V(\mathbf{r}) + \frac{4\pi\hbar^2 a_{\rm BG} n(\mathbf{0})}{M}\left[1 - \frac{\Delta}{\mathcal{H}} + \frac{\Delta M \omega_\rho^2}{2\mathcal{H}^2 \mu_{\rm B}}\left(\rho^2 + \lambda^2 z^2\right) + ...\right]$$

$$\Rightarrow \qquad a_{\rm eff} = a_{\rm BG}\left(1 - \frac{\Delta}{\mathcal{H}}\right)$$

$$\Rightarrow \qquad \omega_D = \lambda\omega_\rho\sqrt{1 - \frac{4\pi\hbar^2 |a_{\rm BG}| n(\mathbf{0})\mu\Delta}{M\mathcal{H}^2\mu_{\rm B}}} \quad < \lambda\omega_\rho$$

⋆ This is initial qualitative finding result for the dipole mode frequency: *H. Al-Jibbouri and A. Pelster, PRA* **88**, *033621 (2013)*

Quantized Vortices

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \Big[-\frac{\hbar^2}{2M}\Delta + V(\mathbf{r}) + g_2 N\left|\psi(\mathbf{r},t)\right|^2\Big]\psi(\mathbf{r},t)$$

- Madelung transformation: $\psi(\mathbf{r},t) = \sqrt{n(\mathbf{r},t)}e^{iS(\mathbf{r},t)}$
- Continuity equation: $\frac{\partial n}{\partial t} + \nabla(n\mathbf{v}) = \mathbf{0}$
- Euler equation for irrotational superfluid:
  $M\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{1}{2}Mv^2 + V + g_2 n - \frac{\hbar^2\nabla^2\sqrt{n}}{2M\sqrt{n}}\right) = \mathbf{0}$
- Current density $\mathbf{j} = n\mathbf{v} \quad \Rightarrow \quad \mathbf{v} = \frac{\hbar}{M}\nabla S$
- Condensate wave function must be single-valued, so its phase can only change by multiples of 2π

  $k = \oint_C \mathbf{v}d\mathbf{r} = 2\pi\gamma\frac{\hbar}{M} \quad , \gamma = 1, 2, ...$

Structure of Single Vortex

- Condensate wave function

$$\psi(\rho, z, \phi) = f(\rho, z) e^{i\gamma\phi}\,, \qquad f(\rho, z) = \sqrt{n(\rho, z)}$$

- Time-independent GP equation

$$-\frac{\hbar^2}{2M}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{\partial^2 f}{\partial z^2}\right] + \frac{\hbar^2}{2M\rho^2}\gamma^2 f + \frac{M}{2}\omega_\rho^2\left(\rho^2 + \lambda^2 z^2\right) f + g_2 f^3 = \mu f$$

- Healing length: $\xi = \sqrt{\frac{\hbar^2}{2Mg_2 f(\xi,0)^2}} = \frac{1}{\sqrt{8\pi a f(\xi,0)^2}} = \frac{l^2}{R_\rho}$
- $\rho \ll \xi$: dominant term arises from kinetic energy

$$\Rightarrow \quad f = c\rho^\gamma, \qquad n = c^2\rho^{2\gamma}$$

- $\xi \ll \rho \ll R_\rho$: Thomas-Fermi solution

$$\Rightarrow \quad n_{\mathrm{TF}}(\rho, z) = \frac{\mu}{g_2}\left[1 - \left(\frac{\rho}{R_\rho}\right)^2 - \left(\frac{z}{R_z}\right)^2\right]$$