Geometric Resonances in Bose-Einstein Condensates (BECs) with Two- and Three-Body Interactions*

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July 16, 2012

Outline

- We study geometric resonances in a BEC as for instance,

$$\omega_{\mathrm{B}} = 2\omega_{\mathrm{Q}}$$

- We use a Poincaré-Lindstedt analysis of a Gaussian variational approach.
  I. Vidanović, A. Balaž, H. Al-Jibbouri, and A. Pelster, PRA 84, 013618 (2011)
- By changing the anisotropy of the confining potential λ, we obtain
  - ⋆ Geometric resonances and shifts in the frequencies of collective modes
  - ⋆ Mode coupling of collective modes
- We discuss the stability of a BEC for:
  - ⋆ Repulsive and attractive two-body interaction.
  - ⋆ Attractive two-body and repulsive three-body interactions.

- ⋆ The dynamics of a BEc in a trap at zero temperature is described:

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \Big[-\frac{\hbar^2}{2M}\Delta + V(\mathbf{r}) + g_2 N\,|\psi(\mathbf{r},t)|^2 + g_3 N^2\,|\psi(\mathbf{r},t)|^4\Big]\psi(\mathbf{r},t)$$

- ⋆ where $\psi(\mathbf{r},t)$ is a condensate wave function
- ⋆ N is the total number of atoms in the condensate
- ⋆ $V(\mathbf{r}) = \frac{1}{2}M\omega_\rho^2\left(\rho^2 + \lambda^2 z^2\right)$, trap anisotropy $\lambda = \omega_z/\omega_\rho$
- ⋆ $g_2 = 4\pi\hbar^2 a/M$.
- ⋆ Experimental value, for instance, $g_3/\hbar$ of the order 10^{-27} to $10^{-26}\mathrm{cm}^6\mathrm{s}^{-1}$ Phys. Rev. Lett. **89**, 050402 (2002)

- ⋆ Lagrangian

$$L(t) = \int \mathcal{L}(\mathbf{r}, t)\, d\mathbf{r}\,,$$

- ⋆ Lagrange density

$$\mathcal{L}(\mathbf{r}, t) = \frac{i\hbar}{2}\left(\psi\frac{\partial\psi^*}{\partial t} - \psi^*\frac{\partial\psi}{\partial t}\right) - \frac{\hbar^2}{2M}|\nabla\psi|^2 - V(\mathbf{r})|\psi|^2 - \frac{g_2 N}{2}|\psi|^4 - \frac{g_3 N^2}{3}|\psi|^6$$

- ⋆ Gaussian variational ansatz
  Phys. Rev. Lett. 77, 5320 (1996)
  Phys. Rev. A 56, 1424 (1997)

$$\psi^{\mathrm{G}}(\rho, z, t) = \mathcal{N}(t)\exp\left[-\frac{1}{2}\left(\frac{\rho^2}{u_\rho(t)^2} + \frac{z^2}{u_z(t)^2}\right) + i\left(\rho^2\phi_\rho(t) + z^2\phi_z(t)\right)\right]$$

- ⋆ where $\mathcal{N}(t) = 1/\sqrt{\pi^{\frac{3}{2}} u_\rho^2 u_z}$
- ⋆ Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0, \qquad q \in \{\phi_i, u_i\}$$

- ⋆ Phases ϕ_ρ and ϕ_z

$$\phi_\rho = \frac{M\dot{u}_\rho}{2\hbar u_\rho}, \qquad \phi_z = \frac{M\dot{u}_z}{2\hbar u_z}$$

- ⋆ Dimensionless parameters

$$\tilde{\omega}_i = \omega_i/\omega_\rho\,, \quad \tilde{u}_i = u_i/\ell\,, \quad \ell = \sqrt{\frac{\hbar}{M\omega_\rho}}\,, \quad \tilde{t} = t/\omega_\rho$$

- ⋆ Equations of motion in the dimensionless form

$$\ddot{u}_\rho + u_\rho - \frac{1}{u_\rho^3} - \frac{p}{u_\rho^3 u_z} - \frac{k}{u_\rho^5 u_z^2} = 0$$

$$\ddot{u}_z + \lambda^2 u_z - \frac{1}{u_z^3} - \frac{p}{u_\rho^2 u_z^2} - \frac{k}{u_\rho^4 u_z^3} = 0$$

- ⋆ $p = g_2 N/(2\pi)^{3/2}\hbar\omega_\rho\ell^3 = \sqrt{2/\pi}\,Na/\ell$
- ⋆ $k = 2g_3 N^2/9\sqrt{3}\pi^3\omega_\rho\hbar\ell^6 = \frac{16 g_3\hbar\omega_\rho}{9\sqrt{3}g_2^2}p^2$

- ⋆ Time-independent solutions

$$u_{\rho 0} = \frac{1}{u_{\rho 0}^3} + \frac{p}{u_{\rho 0}^3 u_{z0}} + \frac{k}{u_{\rho 0}^5 u_{z0}^2} \,,$$

$$\lambda^2 u_{z0} = \frac{1}{u_{z0}^3} + \frac{p}{u_{\rho 0}^2 u_{z0}^2} + \frac{k}{u_{\rho 0}^4 u_{z0}^3}$$

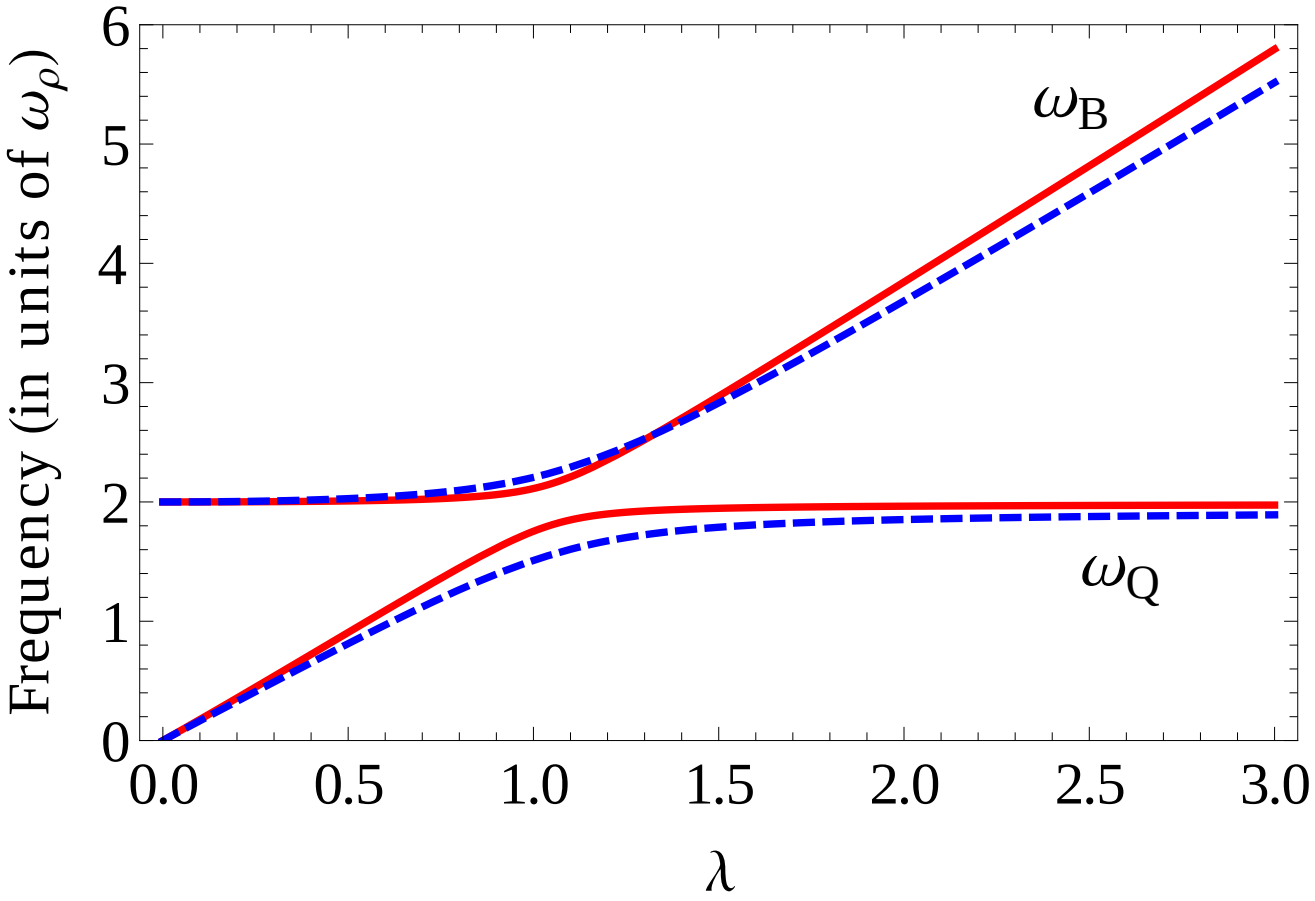
- ⋆ In order to estimate the frequencies of collective modes, we insert

$$u_\rho(t) = u_{\rho 0} + \delta u_\rho(t) \qquad u_z(t) = u_{z0} + \delta u_z(t)$$

$$\omega_{\mathrm{B,Q}}^2 = \frac{m_1 + m_3 \pm \sqrt{(m_1 - m_3)^2 + 8m_2^2}}{2}$$

- ⋆ where

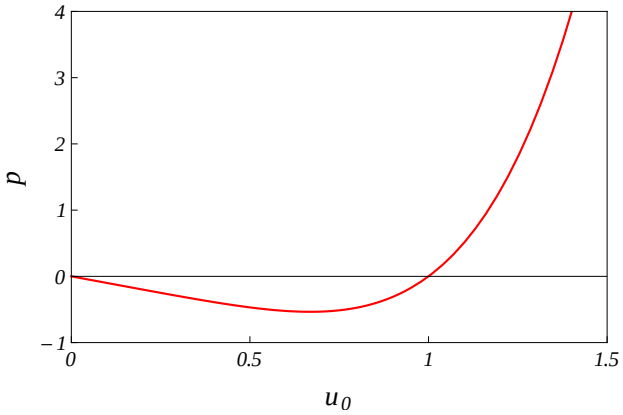
$$m_1 = 4 + \frac{2k}{u_{\rho 0}^6 u_{z0}^2}, \quad m_2 = \frac{p}{u_{\rho 0}^3 u_{z0}^2} + \frac{2k}{u_{\rho 0}^5 u_{z0}^3}, \quad m_3 = 4\lambda^2 - \frac{p}{u_{\rho 0}^2 u_{z0}^3}$$



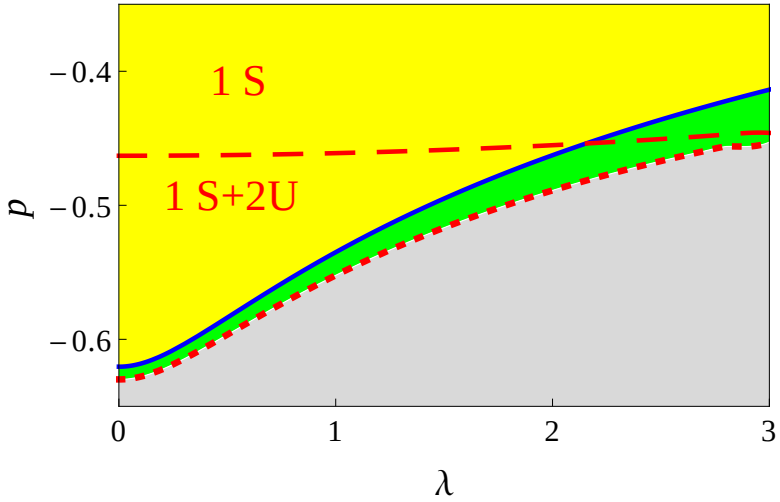
- ⋆ $p = 1$, $k = 0.001$ (solid lines)
- ⋆ $p = 10$, $k = 0.1$ (dashed lines)

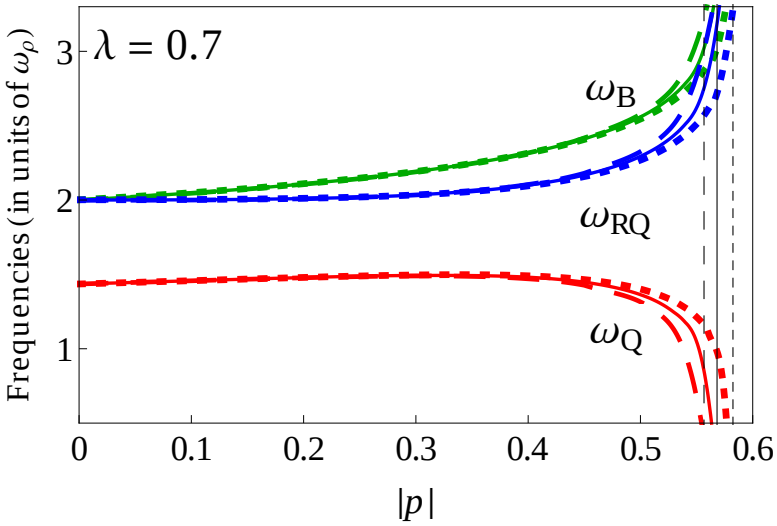
- First we consider isotropic case $\lambda = 1$

$$u_0^2 + u_0 - p = 0$$



- Axially-symmetric case





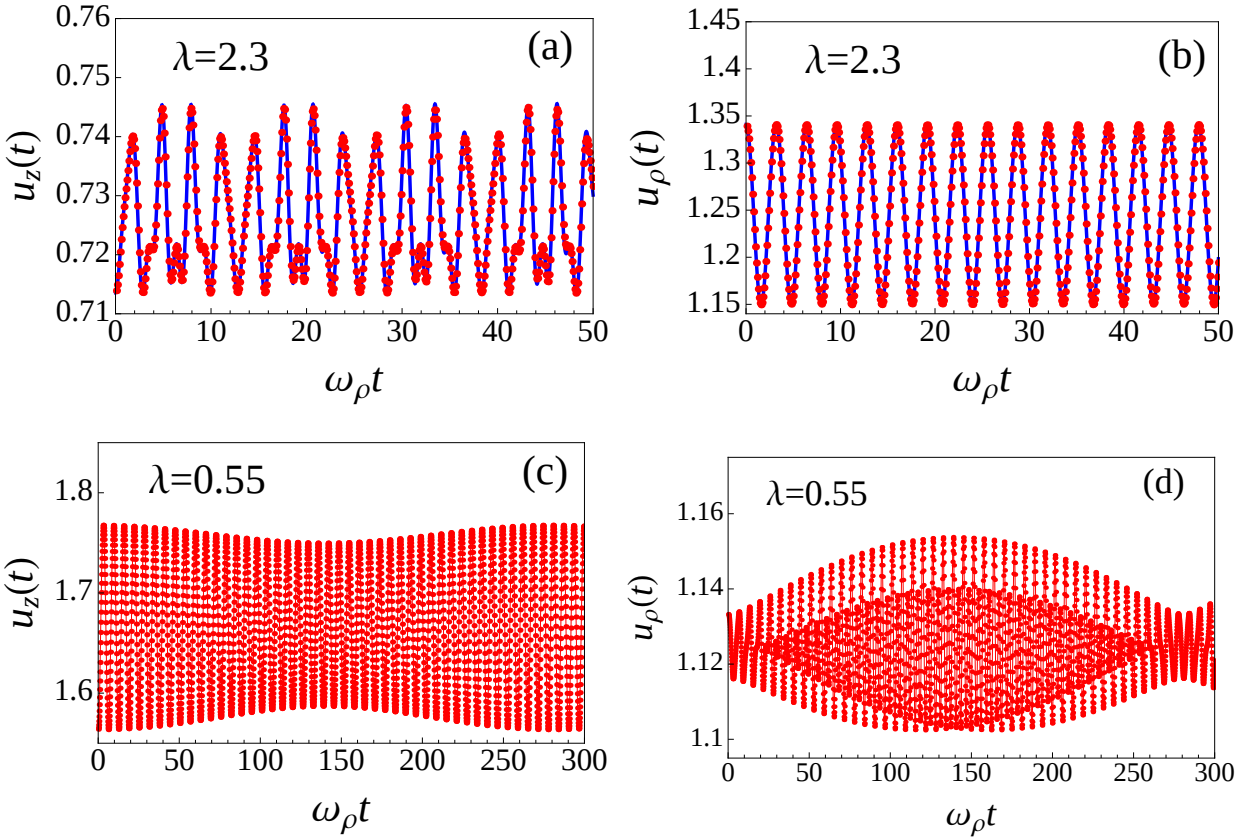
- $k = 0$ (solid lines)
- $k = 0.005$ (dashed lines)

$k = 0$ (sold lines)

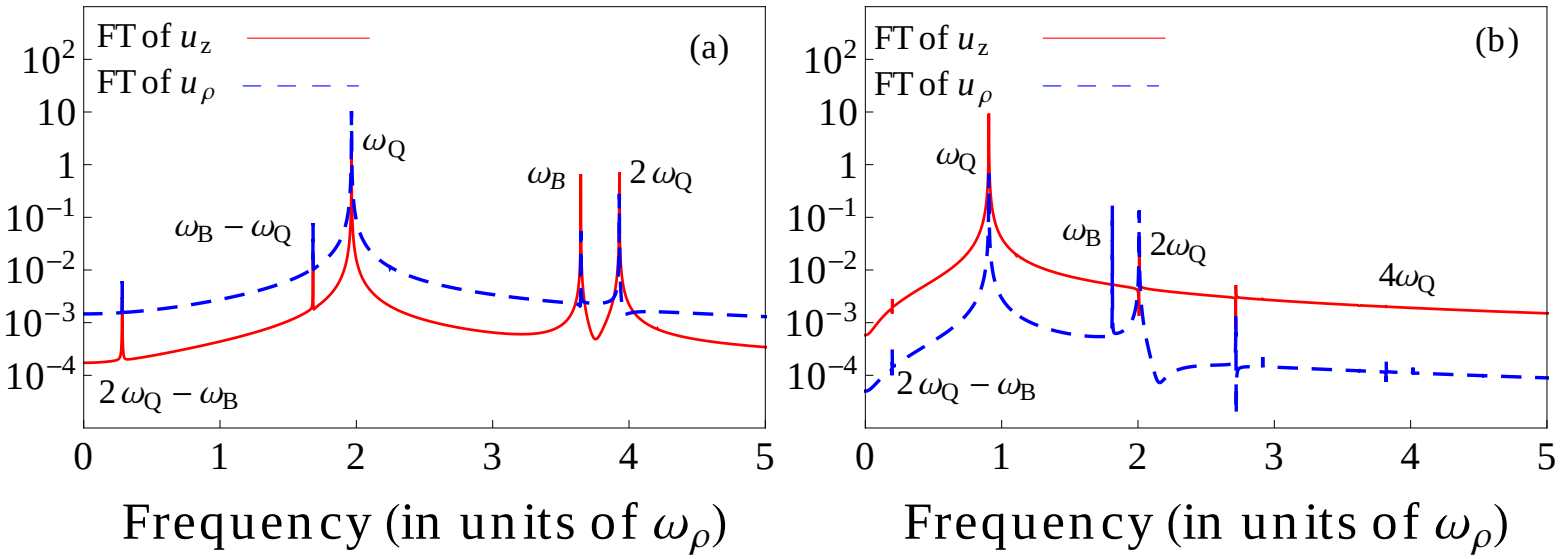
$k = 0.005$ (dotted lines)

$k = -0.005$ (dashed lines)

$$\mathbf{u}(0) = \mathbf{u}_{\rm eq} + \varepsilon\, \mathbf{u}_Q, \quad \dot{\mathbf{u}}(0) = \mathbf{0}$$



- ⋆ $p = 1$, $k = 0$, $\varepsilon = 0.1$
- ⋆ analytic ($2^{\rm nd}$ order perturbation theory): solid lines
- ⋆ numeric: dotted lines



- ⋆ (a) $\lambda = 1.9$ and (b) $\lambda = 0.5$
- ⋆ $p = 1$, $k = 0.001$, and $\varepsilon = 0.1$.

- We use perturbation theory

$$u_\rho(t) = u_{\rho 0} + \varepsilon u_{\rho 1}(t) + \varepsilon^2 u_{\rho 2}(t) + \varepsilon^3 u_{\rho 3}(t) + \dots ,$$

$$u_z(t) = u_{z0} + \varepsilon u_{z1}(t) + \varepsilon^2 u_{z2}(t) + \varepsilon^3 u_{z3}(t) + \dots .$$

- We obtain system of linear differential equations

$$\ddot{u}_{\rho n}(t) + m_1 u_{\rho n}(t) + m_2 u_{zn}(t) + \chi_{\rho n}(t) = 0 ,$$

$$\ddot{u}_{zn}(t) + 2m_2 u_{\rho n}(t) + m_3 u_{zn}(t) + \chi_{zn}(t) = 0$$

- where $n = 1, 2, 3, ...$
- $\chi_{\rho n}(t)$ and $\chi_{zn}(t)$ depend only on the solutions $u_{\rho i}(t)$ and $u_{zi}(t)$ of the lower order i, such that $i < n$.
- For $n = 1$ we have $\chi_{\rho 1}(t) = 0$ and $\chi_{z1}(t) = 0$
- Linear transformation

$$u_{\rho n}(t) = x_n(t) + y_n(t), \quad u_{zn}(t) = c_1 x_n(t) + c_2 y_n(t)$$

- ⋆ with coefficients

$$c_{1,2} = \frac{m_3 - m_1 \mp \sqrt{(m_3 - m_1)^2 + 8m_2^2}}{2m_2}$$

- ⋆ Decouples the system at the n-th order and leads to

$$\ddot{x}_n(t) + \omega_Q^2 x_n(t) + \frac{c_2 \chi_{\rho n}(t) - \chi_{zn}(t)}{c_2 - c_1} = 0\,,$$

$$\ddot{y}_n(t) + \omega_B^2 y_n(t) + \frac{c_1 \chi_{\rho n}(t) - \chi_{zn}(t)}{c_1 - c_2} = 0$$

- ⋆ This happens for the first time at level $n = 3$

$$\ddot{\mathbf{u}}_3(t) + M\mathbf{u}_3(t) + \mathbf{I}_{Q,3} \cos\omega_Q t + \ldots = 0,$$

- where $M = \begin{pmatrix} m_1 & m_2 \\ 2m_2 & m_3 \end{pmatrix}$
- ⋆ The particular solution has the form

$$\mathbf{u}_{3,P}(t) = -\varepsilon^2 \frac{(\mathbf{u}_Q^L)^T \mathbf{I}_{Q,3}}{2\omega_Q} \mathbf{u}_Q t \sin\omega_Q t + \ldots$$

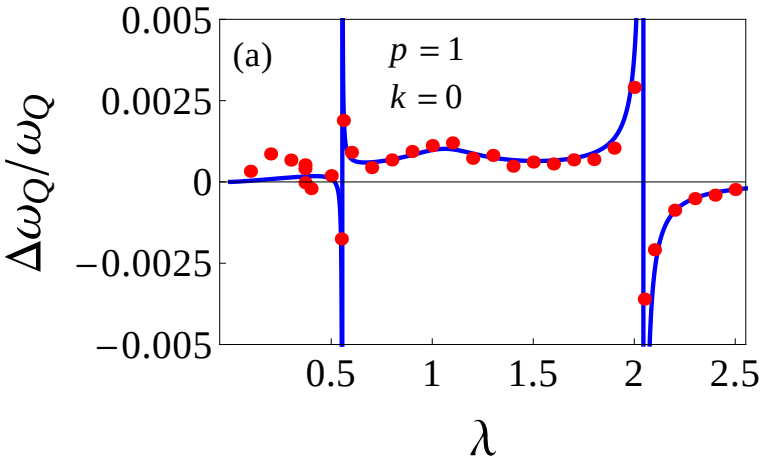
- The secular term can be now absorbed by a shift in the quadrupole mode frequency,

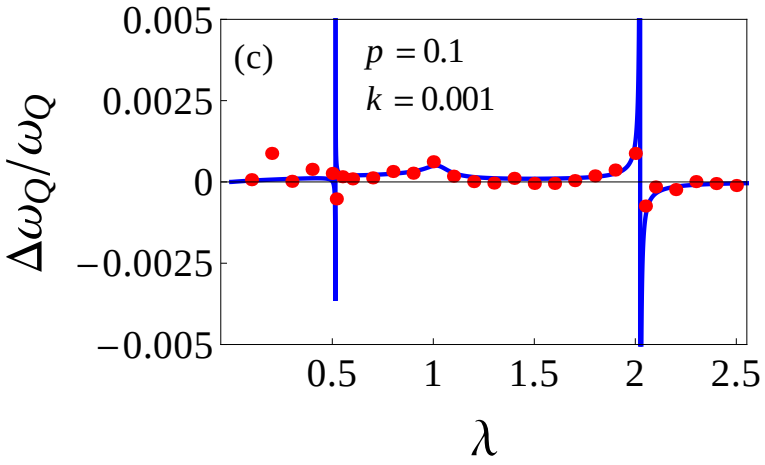
$$\mathbf{u}_3(t) = \mathbf{u}_Q \cos\omega_Q t - \varepsilon^2 \frac{(\mathbf{u}_Q^L)^T \mathbf{I}_{Q,3}}{2\omega_Q} \mathbf{u}_Q t \sin\omega_Q t + \dots$$

$$\approx \quad \mathbf{u}_Q \cos(\omega_Q + \Delta\omega_Q)t + \dots ,$$

- Quadrupole mode frequency shift

$$\omega_Q(\varepsilon) = \omega_Q + \Delta\omega_Q = \omega_Q - \frac{\varepsilon^2}{2\omega_Q} \frac{f_{Q,3}(\omega_Q, \omega_B, u_{\rho 0}, u_{z0}, p, k, \lambda)}{(\omega_B - 2\omega_Q)(\omega_B + 2\omega_Q)}$$





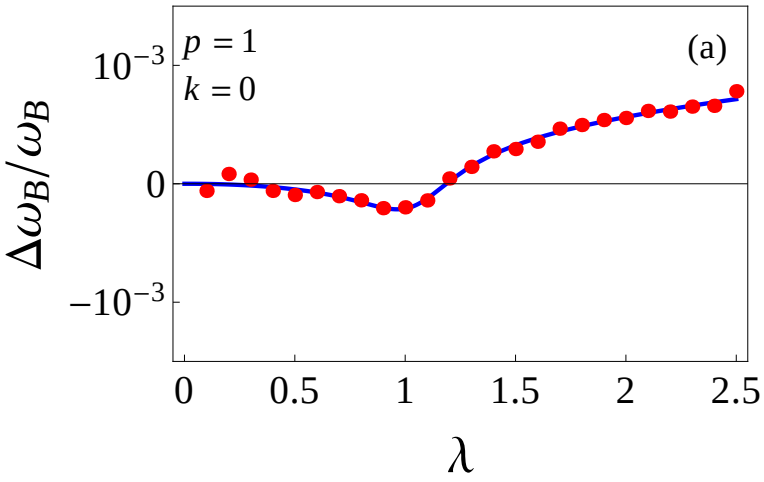
- ⋆ In a similar way, we study dynamics of a cylindrically-symmetric BEC system when initially only the breathing mode is excited

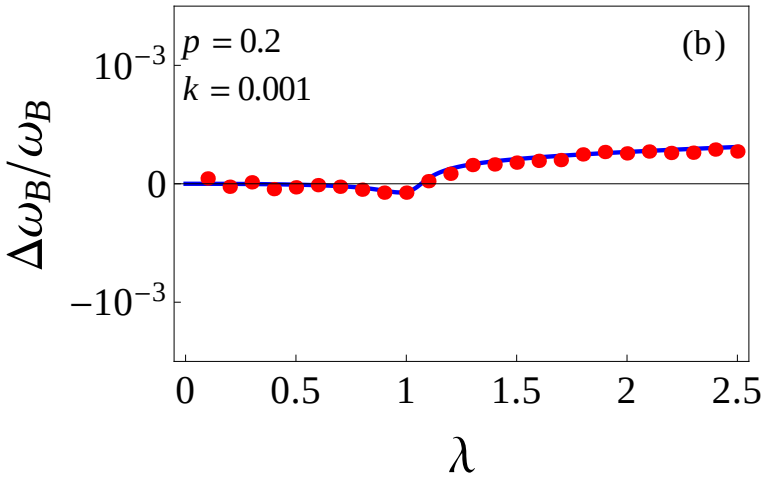
$$\mathbf{u}(0) = \mathbf{u}_0 + \varepsilon \mathbf{u}_B \,, \quad \dot{\mathbf{u}}(0) = \mathbf{0} \,.$$

- ⋆ Applying again the Poincaré-Lindstedt perturbation theory

$$\omega_B(\varepsilon) = \omega_B + \Delta\omega_B = \omega_B - \varepsilon^2 \frac{(\mathbf{u}_B^L)^T \mathbf{I}_{B,3}}{2\omega_B}$$

$$\Delta\omega_B = -\varepsilon^2 \frac{f_{B,3}(\omega_Q, \omega_B, u_{\rho 0}, u_{z0}, p, k, \lambda)}{2\omega_B(2\omega_B - \omega_Q)(2\omega_B + \omega_Q)}$$

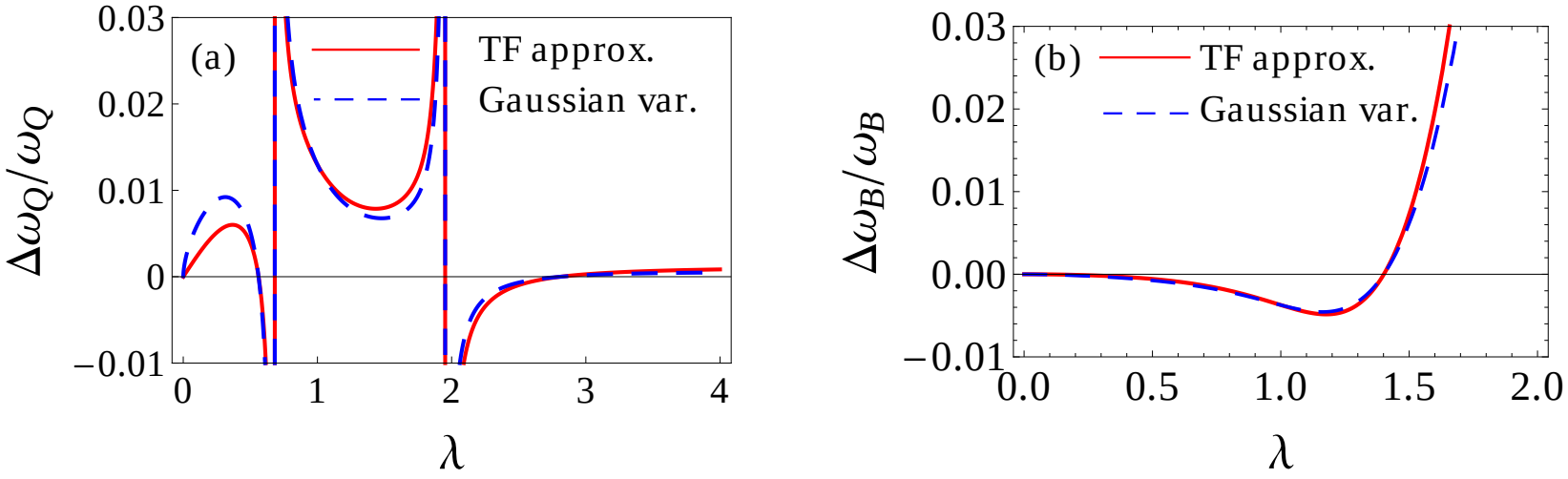




Limit $p \to \infty$ and $k = 0$

$$\omega_{B,Q}^2 = 2 + \frac{3}{2}\lambda^2 \pm \frac{1}{2}\sqrt{16 - 16\lambda^2 + 9\lambda^4}$$

- ⋆ The condition for a geometric resonance $\omega_B = 2\omega_Q$ yields trap aspect ratios $\lambda_{1,2} = (\sqrt{125} \pm \sqrt{29})/\sqrt{72}$, or $\lambda_1 \approx 0.683$ and $\lambda_2 \approx 1.952$.
  F. Dalfovo, C. Minniti, and L. Pitaevskii, Phys. Rev. A. **56**, 4855 (1997).



- ⋆ $k = 0$

- The second-order perturbative solution $\mathbf{u}_0 + \varepsilon \mathbf{u}_1(t) + \varepsilon^2 \mathbf{u}_2(t)$ can be written in the form

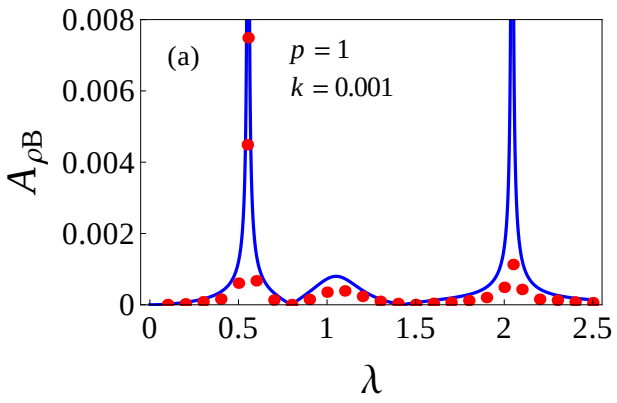
$$\mathbf{u}_0 + \begin{pmatrix} A_{\rho Q} \\ A_{zQ} \end{pmatrix} \cos \omega_Q t + \begin{pmatrix} A_{\rho B} \\ A_{zB} \end{pmatrix} \cos \omega_B t + \ldots$$

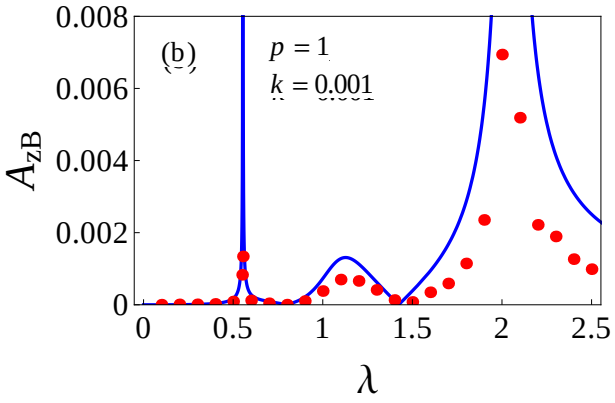
- Quadrupole mode amplitude

$$A_{\rho Q} = \varepsilon u_{\rho Q} + \varepsilon^2 \mathcal{A}_{\rho Q 2} \frac{u_{\rho Q}^2}{\omega_Q^2}, \qquad A_{zQ} = c_1 A_{\rho Q}$$

- Breathing mode amplitude

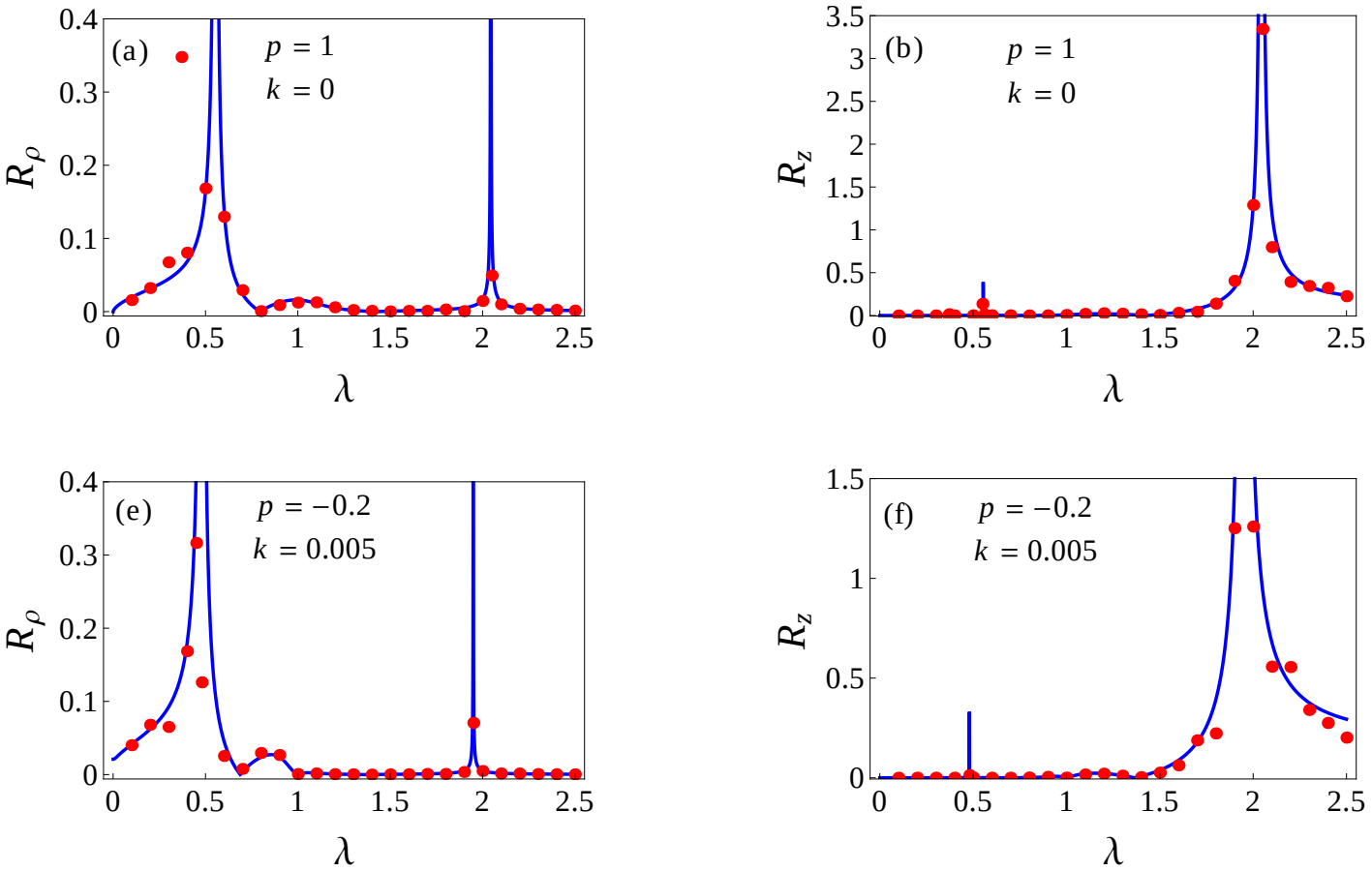
$$A_{\rho B} = \varepsilon^2 \mathcal{A}_{\rho B 2} \frac{u_{\rho Q}^2 (\omega_B^2 - 2\omega_Q^2)}{\omega_B^2 (\omega_B^2 - 4\omega_Q^2)}, \qquad A_{zB} = c_2 A_{\rho B}$$





- Radial and axial ratio

$$R_\rho = \frac{A_{\rho B}}{A_{\rho Q}} \propto \frac{\omega_B^2 - 2\omega_Q^2}{\omega_B^2 - 4\omega_Q^2}, \qquad R_z = \frac{A_{zB}}{A_{zQ}} \propto \frac{\omega_B^2 - 2\omega_Q^2}{\omega_B^2 - 4\omega_Q^2}$$



- If geometry of the trap is tuned so that $\omega_B = \omega_Q\sqrt{2}$, then $A_{\rho B} = A_{zB} = 0$.

- In a similar way, we can initially excite only the breathing mode

$$\mathbf{u}_0 + \begin{pmatrix} A_{\rho B} \\ A_{zB} \end{pmatrix} \cos\omega_B t + \begin{pmatrix} A_{\rho Q} \\ A_{zQ} \end{pmatrix} \cos\omega_Q t + \dots$$

- breathing mode amplitude

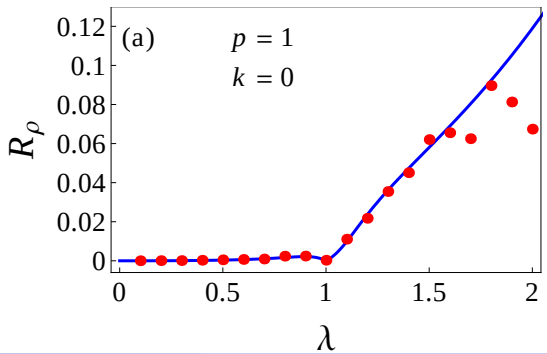
$$A_{\rho B} = \varepsilon u_{\rho B} + \varepsilon^2 \mathcal{A}_{\rho B 2} \frac{u_{\rho B}^2}{\omega_B^2}, \qquad A_{zB} = c_2 A_{\rho B}$$

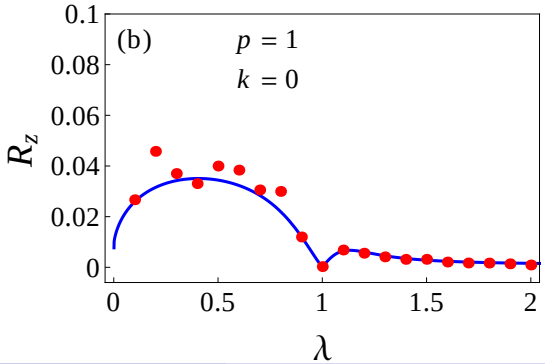
- Quadrupole mode amplitude

$$A_{\rho Q} = \varepsilon^2 \mathcal{A}_{\rho Q 2} \frac{u_{\rho B}^2 (2\omega_B^2 - \omega_Q^2)}{\omega_Q^2 (4\omega_B^2 - \omega_Q^2)} \qquad A_{zQ} = c_1 A_{\rho Q}$$

- Radial and axial ratio

$$R_\rho = \frac{A_{\rho Q}}{A_{\rho B}} \propto \frac{2\omega_B^2 - \omega_Q^2}{4\omega_B^2 - \omega_Q^2}, \qquad R_z = \frac{A_{zQ}}{A_{zB}} \propto \frac{2\omega_B^2 - \omega_Q^2}{4\omega_B^2 - \omega_Q^2}$$





Conclusions

- ⋆ We have studied the dynamics as well as collective excitations of a BEc by changing the trap anisotropy.
- ⋆ We have discussed in detail the stability of a Bose-Einstein condensate in an axially-symmetric trap.
- ⋆ We have used a perturbative expansion and a Poincaré-Lindstedt analysis.
- ⋆ We numerically observe and analytically describe strong nonlinear effects.
- ⋆ We have compared our analytical results of the frequency shift for $p \to \infty$ and $k = 0$ with
  F. Dalfovo, C. Minniti, and L. Pitaevskii, Phys. Rev. A. **56**, 4855 (1997).

Vortex BEC

- ⋆ Hamid Al-Jibbouri and Axel Pelster, *Collective Excitations of a BEC with a Single Vortex*, In preparation

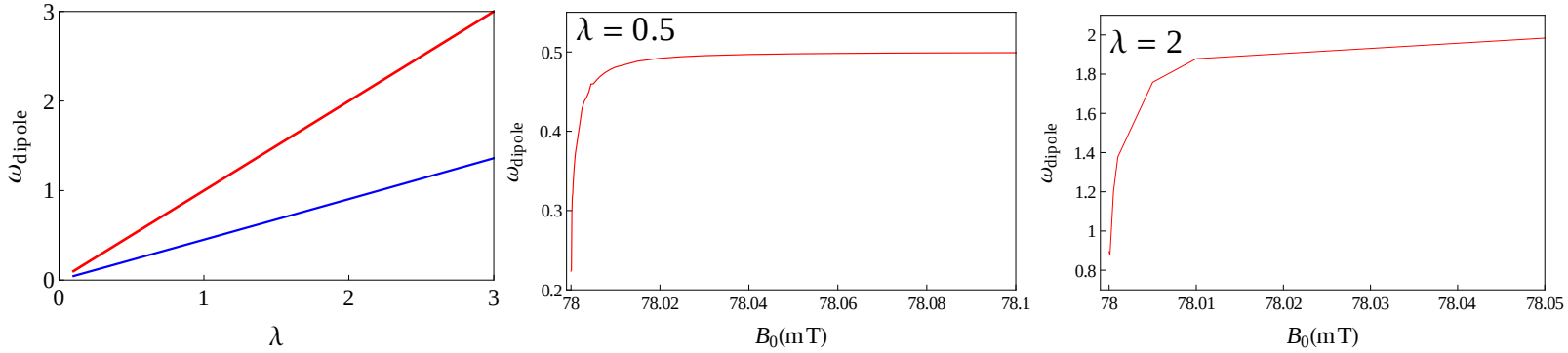
- ⋆ Density profiles of an expanding condensate with a central vortex after turning off the trapping potential for $P = 1000$ and $\lambda = 5$.

Dipole Mode

- We study the changes of dipole mode frequency of a ^{7}Li BEC due to the Feshbach resonance
  E. R. F. Ramos, F. E. A. dos Santos, M. A. Caracanhas, and V. S. Bagnato, Phys. Rev. A **85**, 033608 (2012)

$$V_{\text{ext.}}(\mathbf{r}) = V_0 + \frac{M\omega_r^2}{2}\left(r^2 + \lambda^2 z^2\right), \qquad V_0 = B_0 \mu_{\text{B}}$$

$$a_{\text{s}} = a_{\text{nr}}\left(1 + \frac{\Delta}{B_0 - B_{\text{res}} + \frac{M\omega_r^2}{2\mu_{\text{B}}}\left(r^2 + \lambda^2 z^2\right)}\right)$$



⋆ What about the left-side of Feshbach resonance?

Thank you for your attention