Scattering theory and equation of state of a spherical 2D Bose gas

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Workshop on Prospects of
Quantum Bubbles Physics

07/04/2022

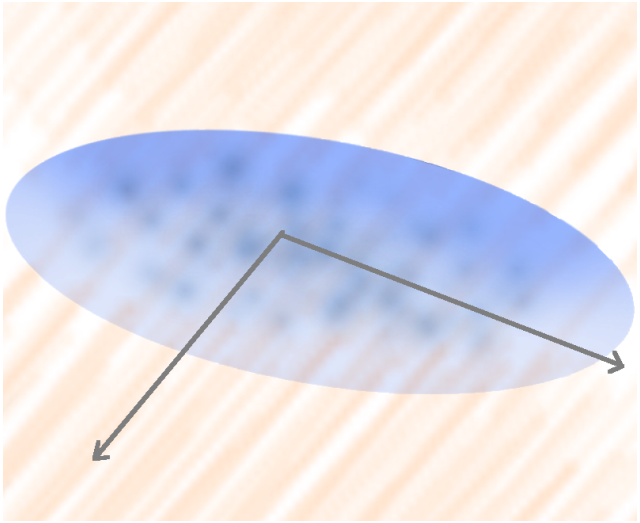
based on
[A. Tononi, PRA **105**, 023324 (2022)]

Outline

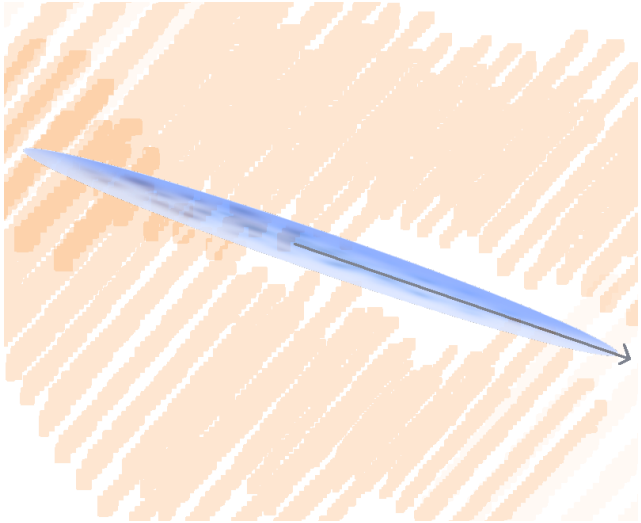
- Introduction and motivation
- Equation of state of a 2D spherical Bose gas
- Derivation of the equation through scattering theory
- Application: hydrodynamic modes
- Conclusions

Low-dimensional quantum gases

(2D)



(1D)



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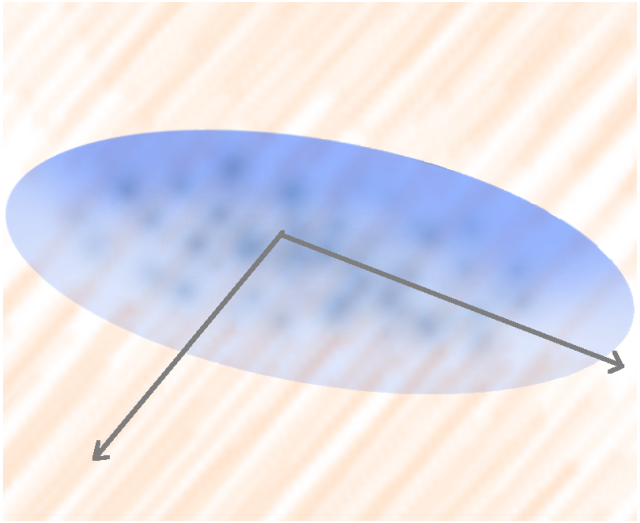

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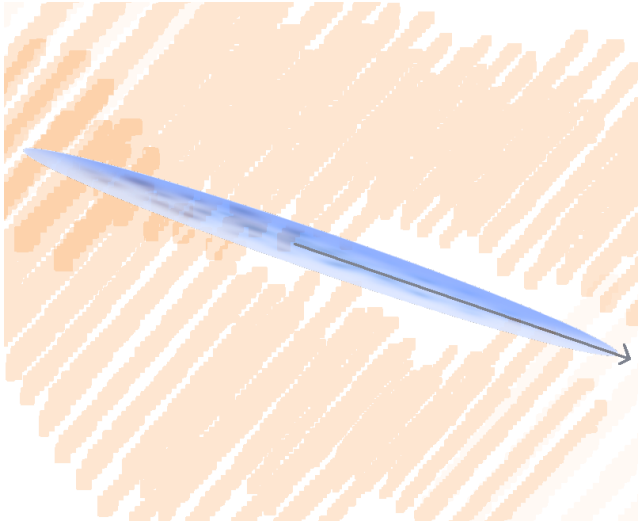

Quantum gases have been studied **consistently** only in *"flat"* low-dimensional configurations

What about *curved* geometries?

Shell-shaped quantum gases (rf-induced adiabatic potentials)

Theoretical proposal of [Zobay, Garraway, PRL **86**, 1195 (2001)]**:**

confine the atoms with $B_0(\vec{r})$, and $B_{rf}(\vec{r}, t)$, yielding

$$U(\vec{r}) = M_F \sqrt{\left[\sum_i \frac{m}{2} \omega_i{}^2 x_i^2 - \hbar\Delta\right]^2 + (\hbar\Omega)^2}$$

ω_i: frequencies of the bare harmonic trap

Δ: detuning from the resonant frequency

Ω: Rabi frequency between coupled levels

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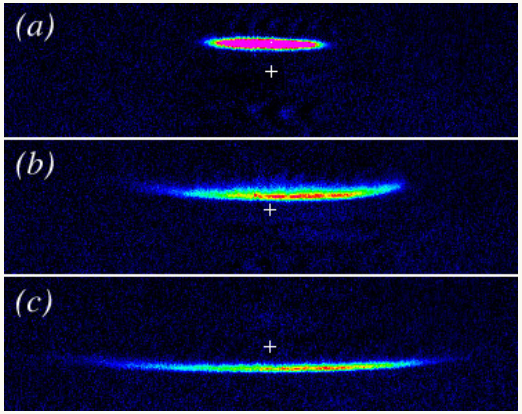
Minimum of $U(\vec{r})$ for

$$\omega_x{}^2 x^2 + \omega_y{}^2 y^2 + \omega_z{}^2 z^2 = \frac{2\hbar\Delta}{m}.$$

Shell-shaped quantum gases

On Earth...



[Colombe *et al.*, EPL **67**, 593 (2004)]

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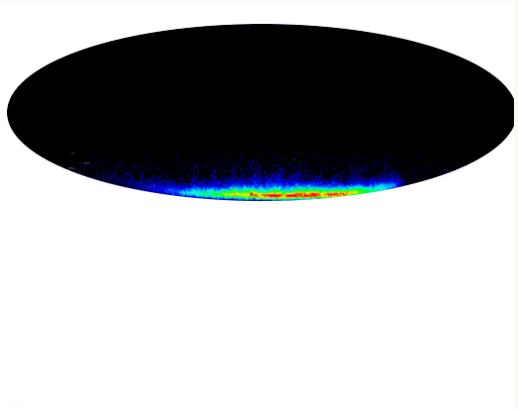

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Shell-shaped quantum gases, in microgravity

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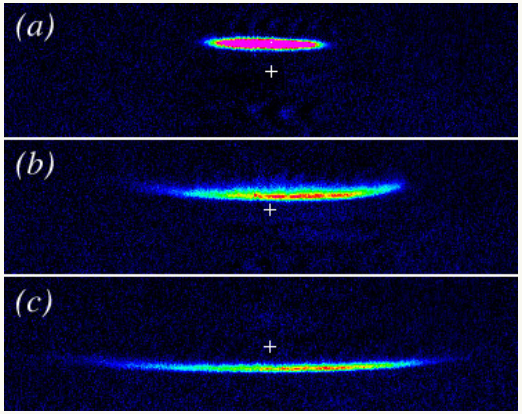

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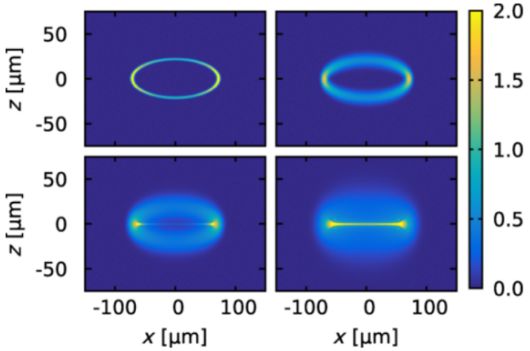
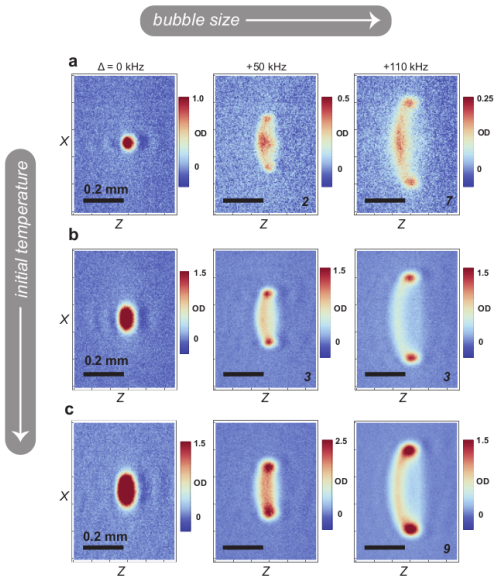
Quantum Bubbles in Microgravity

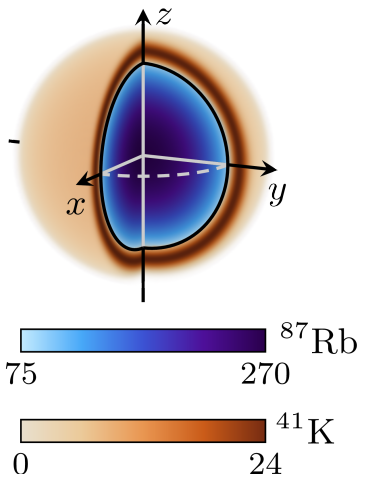
A. Tononi, F. Cinti, and L. Salasnich
Phys. Rev. Lett. **125**, 010402 – Published 29 June 2020



...in microgravity:



[Carollo *et al.*, arXiv:2108.05880]



[Wolf *et al.*, arXiv:2110.15247]

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Equation of state of a spherical Bose gas

Implementing the Bogoliubov theory, we calculated T_{BEC}, n_0/n, Ω.

[AT, Salasnich, PRL **123**, 160403 (2019)]

*: [AT, PRA **105**, 023324 (2022)],

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Recently, through the analysis of scattering theory*...

equation of state:

$$n = \frac{m\mu}{4\pi\hbar^2} \ln\left\{\frac{4\hbar^2[1-\alpha(\mu)]}{m\mu\, a_s^2\, e^{2\gamma+1+\alpha(\mu)}}\right\},$$

with $\alpha(\mu) = 1 - \frac{\mu}{\mu + E_1^B + \epsilon_1}$,

$E_l^{\mathrm{B}} = \sqrt{\epsilon_l(\epsilon_l + 2\mu)}$,

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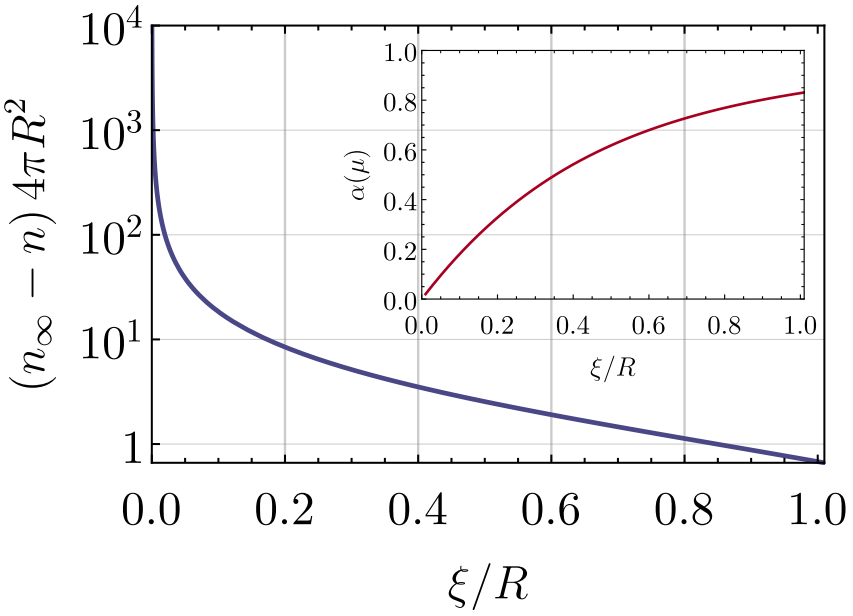

(n_∞ means $R = \infty \leftrightarrow \alpha = 0$)

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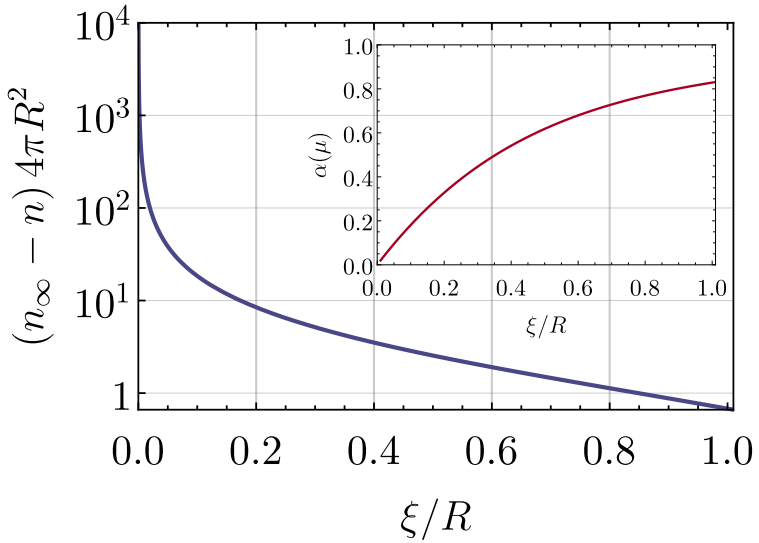

Comments:

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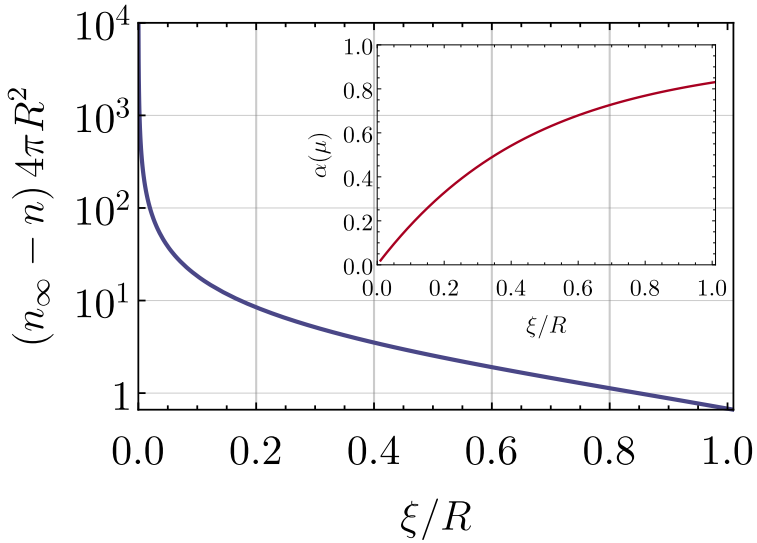

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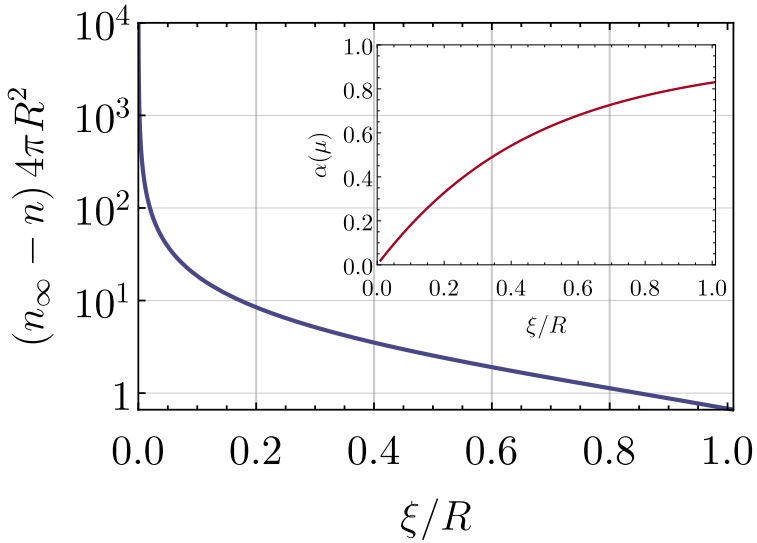

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- extandable (in principle) to other geometries

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Let us see how the equation of state is derived

Bogoliubov theory of a spherical gas

Uniform bosons on the surface of the sphere

$$\mathcal{Z} = \int \mathcal{D}[\bar{\psi}, \psi]\, e^{-\frac{S[\bar{\psi},\psi]}{\hbar}}, \qquad \Omega = -\frac{1}{\beta} \ln(\mathcal{Z})$$

where

$$S[\bar{\psi}, \psi] = \int_0^{\beta\hbar} d\tau \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, R^2 \sin\theta\, \mathcal{L}(\bar{\psi}, \psi)$$

is the Euclidean action, and

$$\mathcal{L} = \bar{\psi}(\theta, \varphi, \tau)\left(\hbar\partial_\tau + \frac{\hat{L}^2}{2mR^2} - \mu\right)\psi(\theta, \varphi, \tau) + \frac{g_0}{2}|\psi(\theta, \varphi, \tau)|^4$$

is the Euclidean Lagrangian.

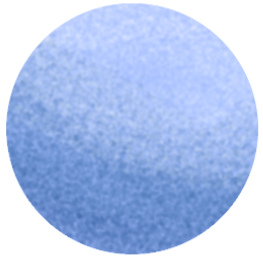
Bogoliubov theory of a spherical gas

ψ_0

Bogoliubov theory:

$$\psi(\theta,\varphi,\tau) = \psi_0 + \eta(\theta,\varphi,\tau)$$

$\eta(\theta,\varphi,\tau)$



Performing the Gaussian integral on $\sim \eta^2$ terms, we get

$$\Omega = -\left(4\pi R^2\right)\frac{\mu^2}{2g_0} + \frac{1}{2}\sum_{l=1}^{\infty}\sum_{m_l=-l}^{l}\left(E_l^{\mathrm{B}} - \epsilon_l - \mu\right),$$

with $E_l^{\mathrm{B}} = \sqrt{\epsilon_l(\epsilon_l + 2\mu)}$, and $\epsilon_l = \hbar^2 l(l+1)/(2mR^2)$.

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Problem: the zero-point energy diverges logarithmically at large l:

$$\frac{1}{2}\int_1^{l_c} \mathrm{d}l\,(2l+1)\left(E_l^{\mathrm{B}} - \epsilon_l - \mu\right) \sim \ln(l_c)$$

Solution: g_0 scales with l_c!

To see this, we need to discuss scattering theory

Scattering theory on the sphere

For a particle with reduced mass on the sphere, the interacting scattering problem reads [Zhang, Ho, J. Phys. B **51**, 115301 (2018)]

$$\hat{H}_0 \Psi^{\mu}_{\nu}(\theta,\varphi) = \mathcal{E}_\nu \Psi^{\mu}_{\nu}(\theta,\varphi), \qquad \textbf{when} \quad \theta > r_0/R$$

with $\hat{H}_0 = \frac{\hat{L}^2}{mR^2}$. For s-wave scattering, we can write

$$\Psi^0_\nu(\theta,\varphi) \propto P^0_\nu(\cos\theta) + \frac{f_0(\mathcal{E}_\nu)}{4i}\left[P^0_\nu(\cos\theta) + \frac{2i}{\pi} Q^0_\nu(\cos\theta)\right],$$

and imposing $\Psi^0_\nu(a_s/R,\varphi) = 0$:

$$f_0(\mathcal{E}_\nu) = -\frac{4}{\cot\delta_0(\mathcal{E}_\nu) - i}, \qquad \cot\delta_0(\mathcal{E}_\nu) = \frac{2}{\pi}\ln\left(\frac{\nu\, a_s\, e^{\gamma_E}}{2R}\right)$$

We identify (**it is a shortcut**, see [AT, PRA **105**, 023324 (2022)] for all steps)

$$g_0 \approx f_0(\mathcal{E}_{l_c}) \approx -\frac{2\pi\hbar^2}{m}\frac{1}{\ln\left[l_c\, a_s e^{\gamma_E}/(2R)\right]}$$

Regularized equation of state

Putting $g_0 = -\frac{2\pi\hbar^2}{m}\frac{1}{\ln\left[l_c\, a_s e^{\gamma_E}/(2R)\right]}$ into

$$\Omega = -\left(4\pi R^2\right)\frac{\mu^2}{2g_0} + \frac{1}{2}\int_1^{l_c} \mathrm{d}l\,(2l+1)\,(E_l^{\mathrm{B}} - \epsilon_l - \mu),$$

the $\ln(l_c)$ divergence disappears, and we obtain the equation of state:

$$n = -\frac{1}{4\pi R^2}\frac{\partial\Omega}{\partial\mu} = \frac{m\mu}{4\pi\hbar^2}\,\ln\left\{\frac{4\hbar^2[1-\alpha(\mu)]}{m\mu\, a_s^2\, e^{2\gamma+1+\alpha(\mu)}}\right\}$$

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Knowing the equation of state **and** the superfluid density, we extend the Landau two-fluid model to the spherical case.

[AT, Pelster, Salasnich, PRR **4**, 013122 (2022)]

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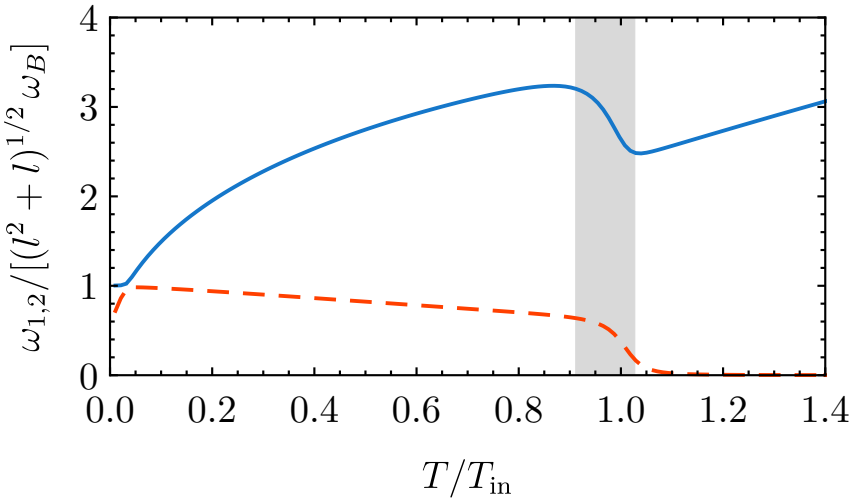
Knowing the equation of state **and** the superfluid density, we extend the Landau two-fluid model to the spherical case.

Frequencies of the hydrodynamic modes:

$$\omega_{1,2}^2 = \left[\frac{l(l+1)}{R^2}\right]\left[\frac{v_A^2+v_L^2}{2} \pm \sqrt{\left(\frac{v_A^2+v_L^2}{2}\right)^2 - v_L^2 v_T^2}\right]$$

ω_1, ω_2 are the main quantitative probe of superfluid BKT transition



$$v_{\{A,T\}} = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_{\{\tilde{s},T\}}}, \quad v_L = \sqrt{\frac{\rho_s T \tilde{s}^2}{\rho_n \tilde{c}_V}}$$

[AT, Pelster, Salasnich, PRR **4**, 013122 (2022)]

Superfluid BKT transition in a spherical superfluid

$$\frac{dK^{-1}(\theta)}{d\ell(\theta)} = -4\pi^3 y^2(\theta)$$

$$\frac{dy(\theta)}{d\ell(\theta)} = [2 - \pi K(\theta)]\, y(\theta)$$

RG scale: $\ell(\theta) = \ln[2R\sin(\theta/2)/\xi]$

Renormalization group equations describing how the superfluid density ($\propto K$) is renormalized by the thermally excited vortices with chemical potential $\sim -\ln(y)$

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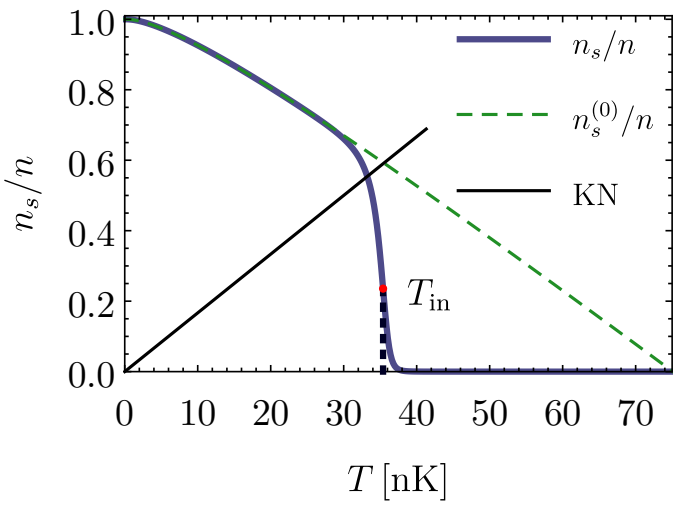
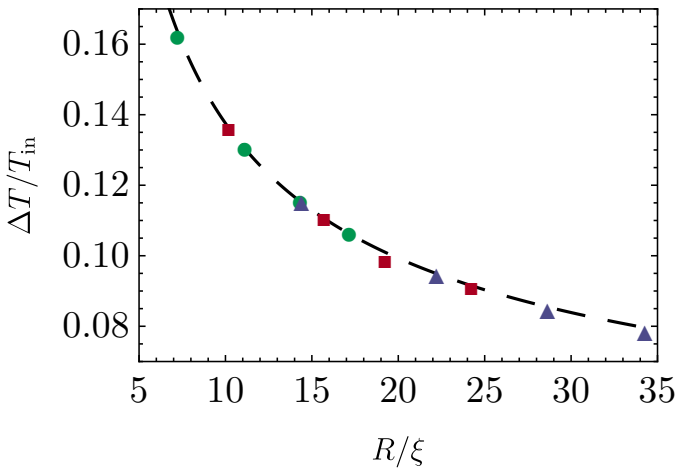
Finite system size $\Rightarrow$ **smooth** vanishing of n_s



$\Delta T/T_{\rm in} \propto \ln^{-2}(R/\xi)$



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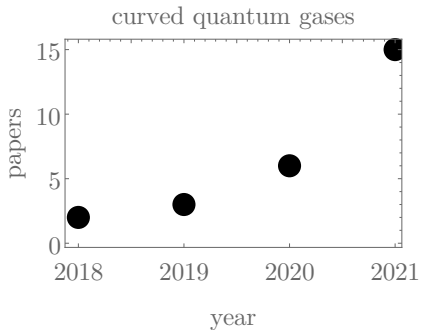
The scientific community has just started exploring shell-shaped BECs



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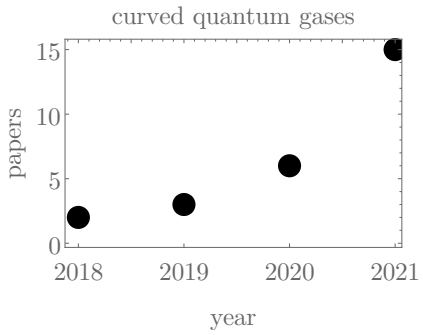

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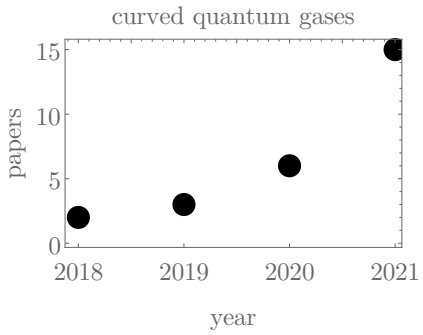

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Thank you for your attention!

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