Bose-Einstein Condensate in Weak Laser Speckle Disorder

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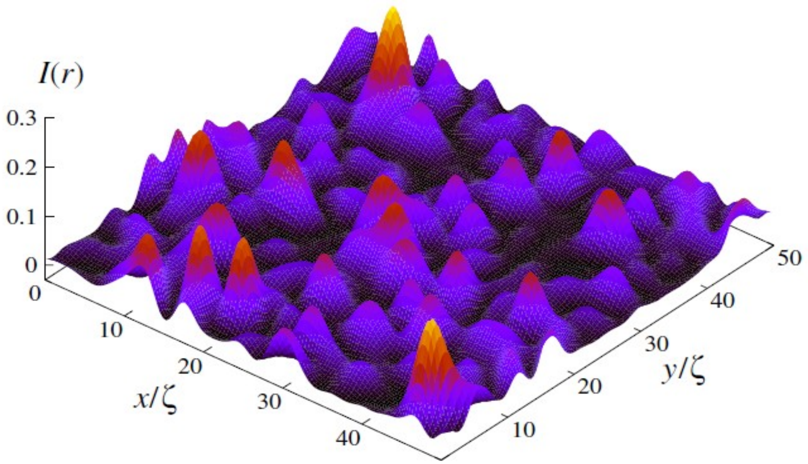

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Introduction to Physics of Laser Speckle Formation

Definition: Laser speckle – Gaussian random distribution of light electric fields A(x,y) – result of Probability Central Limit Theorem (PCLT)

$$p(A_R,A_I) = \frac{1}{2\pi\sigma^2}\exp\{-\frac{(A_R{}^2+A_I{}^2)}{2\sigma^2}\}$$

for each (x,y) and for different (x,y)- M-fold joint Gaussian probability p([A])=exp{-([z]*[z]/[CA])} with

$$[A] \equiv \{A(x_1,y_1)\ A(x_2,y_2)\ \ldots.\ A(x_M,y_M)\}$$

and

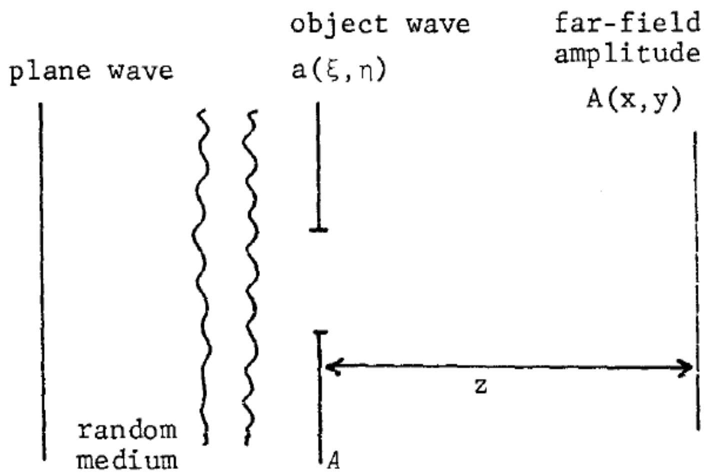
$$(C_A)_{ij} = \langle A^*(x_i,y_i)\ A(x_j,y_j)\rangle$$

PCLT Applicability Conditions:

1. Huygens-Fresnel, linear Fourier mapping in Fresnel approximation

$$z \gg (\xi^2+\eta^2)_{max}/\lambda$$

$$A(x,y) = \iint_A a(\xi,\eta)\exp\left(-\frac{2\pi i}{\lambda z}[x\xi + y\eta]\right)d\xi d\eta$$



2. Object waves are homogeneous: (i) they are completely polarized; (ii) they have uniform random distribution of phases; (iii) they have

$$C_a(\xi_1,\eta_1;\xi_2,\eta_2) \equiv \langle a^*(\xi_1,\eta_1)\ a(\xi_2,\eta_2)\rangle$$

$$= I_a\left(\frac{\xi_1+\xi_2}{2},\ \frac{\eta_1+\eta_2}{2}\right) \times C_a{}'(\xi_2-\xi_1,\eta_2-\eta_1)$$

Autocorrelation Function of Laser Speckle

Using Fresnel formula one obtains autocorrelation function:

$$C_A(x_1,y_1;x_2,y_2) = \iint_{-\infty}^{\infty} I_a(\xi',\eta')\exp\{-\frac{2\pi i}{\lambda z}(\xi'(x_2-x_1)+\eta'(y_2-y_1))\}d\xi'd\eta' \times \iint_{-\infty}^{\infty} C_a'(\xi'',\eta'')\exp\{-\frac{2\pi i}{\lambda z}(\xi''(\frac{x_1+x_2}{2})+\eta''(\frac{y_1+y_2}{2}))\}d\xi''d\eta''.$$

and

$$C_A(x_1,y_1;x_2,y_2) \simeq I_A(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2})\, C_A'(x_2-x_1,\, y_2-y_1).$$

For delta correlated $C_a'(\xi_2-\xi_1,\eta_2-\eta_1)$ a function $I_a(\frac{\xi_1+\xi_2}{2},\frac{\eta_1+\eta_2}{2})$ is constant and thus

$$R_I(x_1,y_1;x_2,y_2)=\langle I(x_1,y_1)I(x_2,y_2)\rangle=\langle I\rangle^2\left[1+|C_A'(x_2-x_1,y_2-y_1)|^2\right]$$

for intensity $I(x,y)=|A(x,y)|^2$ has

$$|C_A'(x_2-x_1,y_2-y_1)|^2=\mathrm{sinc}^2(L_x\Delta/\lambda z)\,\mathrm{sinc}^2(L_y\Delta/\lambda z)\;;\;\mathrm{sinc}(x)=\sin(\pi x)/(\pi x)$$

- for 2D rectangle (Goodman);

$$|C_A'(r)|^2=|2J_1(\pi Dr/\lambda z)/(\pi Dr/\lambda z)|^2$$

- for 2D volume circle (Goodman);

$$|C_A'(r)|^2=|3[\sin(\pi Dr/\lambda z)-(\pi Dr/\lambda z)\cos(\pi Dr/\lambda z)]/(\pi Dr/\lambda z)^3|^2$$

- for 3D volume sphere (Abdullaev, Pelster).

Experimental Realization of Laser Speckle

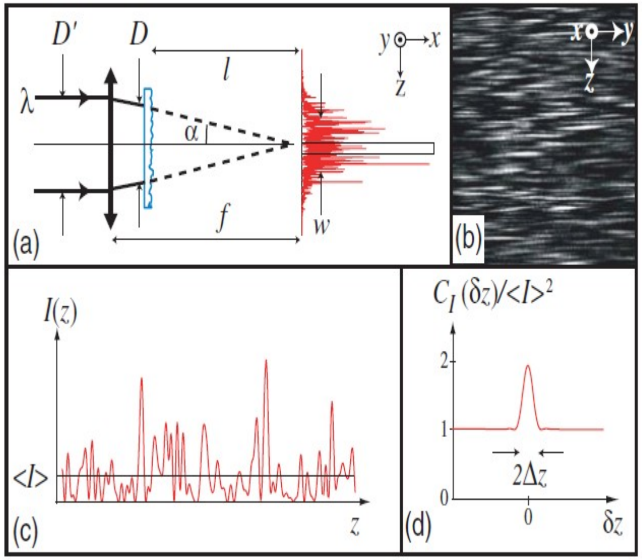


Figure 1. (a) Experimental realization of the speckle pattern.

from D. Clement et al., New J. Phys. 8,165 (2006).

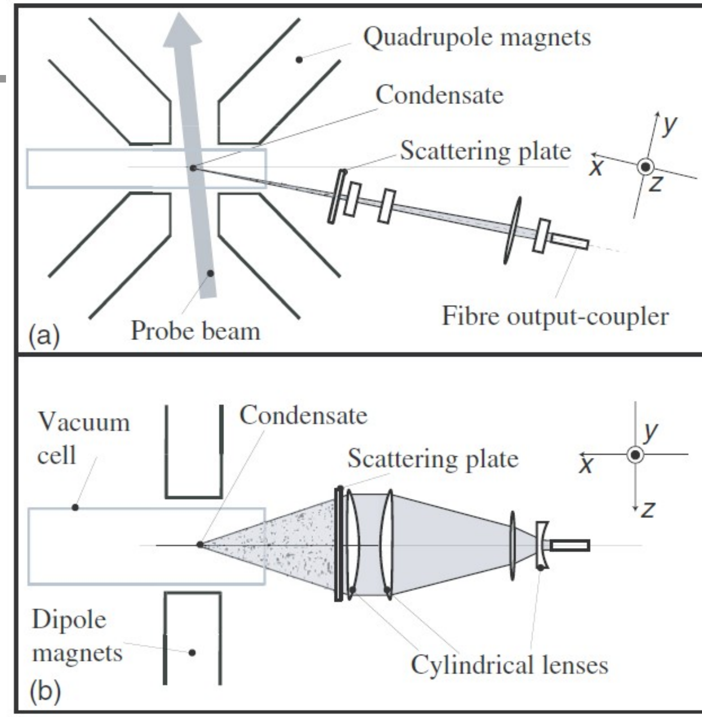
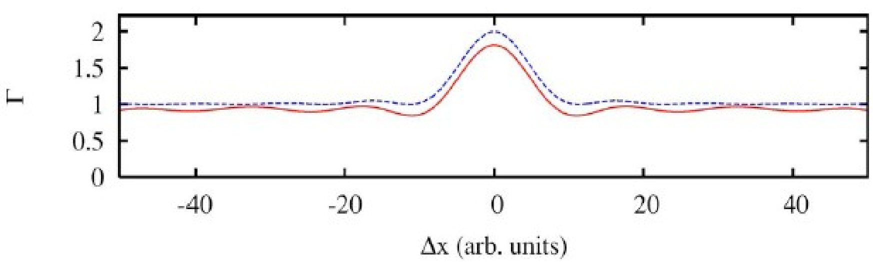


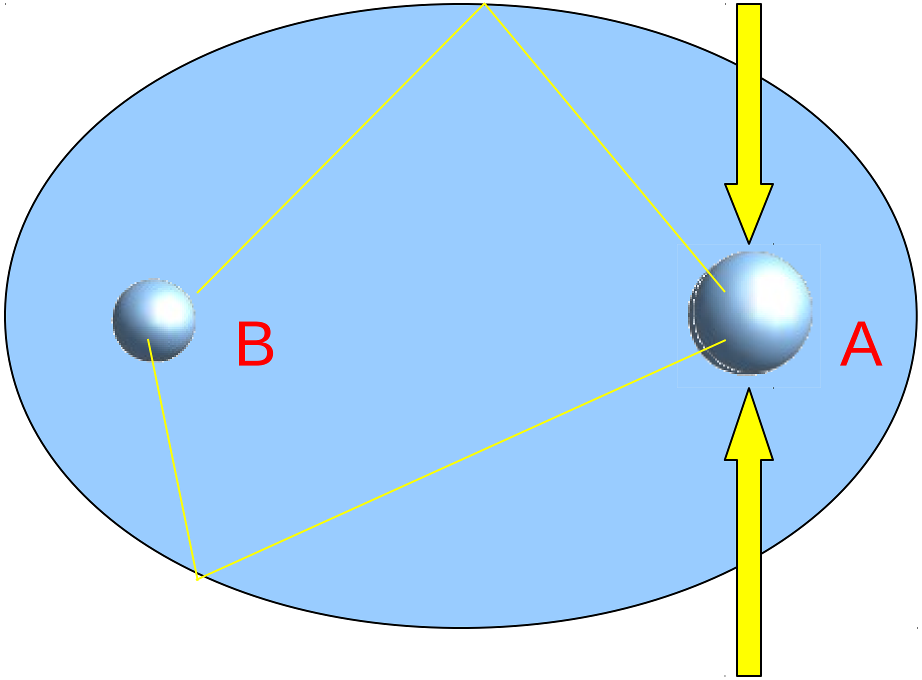
Figure 3. Optical setup used to create the speckle potential.



$$\Gamma(\Delta x) \equiv \langle I(x) I(x+\Delta x)\rangle = 1 + \mathrm{sinc}(D\Delta x)^2$$

Correlation function is taken from M. Modugno, Phys. Rev. A 73 013606 (2006).

Proposition for Experimental Realization of 3D Speckle



Cross section of ellipsoidal reflective cavity with spheres A and B in its focuses. A contains small size absolute spheric light reflector in the center and volume optic inhomogeneities. Incident laser beams (thick yellow arrows), after reflection from reflector, scatter additionally from inhomogeneities producing individual wavelets (thin yellow lines), which are collected in B (in point, where BEC exists).

Gross - Pitaevskii Perturbation Theory for BEC Depletion

1. Gross - Pitaevskii equation

$$\left[-\frac{\hbar^2}{2m}\Delta + U(\mathbf{x}) + \int d^3x' |\Psi(\mathbf{x}')|^2 V_{\rm int}(\mathbf{x}-\mathbf{x}')\right]\Psi(\mathbf{x}) = \mu\Psi(\mathbf{x}),$$

2. Perturbative solution

$$\Psi(\mathbf{x}) = \psi_0(\mathbf{x}) + \psi_1(\mathbf{x}) + \psi_2(\mathbf{x}) + \cdots ,$$

3. Statistical properties of disorder

$$\langle U(\mathbf{x})\rangle = 0,$$
$$\langle U(\mathbf{x})U(\mathbf{x}')\rangle = R(\mathbf{x}-\mathbf{x}').$$

$$\langle U(\mathbf{k})U(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')R(\mathbf{k}).$$

4. Disorder ensemble averages

$$n_0 = \Psi_0^2 + 2\Psi_0\langle\Psi_2(\mathbf{x})\rangle + \cdots,$$
$$n = \Psi_0^2 + 2\Psi_0\langle\Psi_2(\mathbf{x})\rangle + \langle\Psi_1(\mathbf{x})^2\rangle + \cdots .$$

5. Condensate depletion

$$n - n_0 = n\int \frac{d^3k}{(2\pi)^3}\frac{R(\mathbf{k})}{[\hbar^2\mathbf{k}^2/2m + 2nV_{\rm int}(\mathbf{k})]^2} + \cdots .$$

$$\text{For } V_{inter}(\vec{k}) = g \text{, where } g = \frac{4\pi\hbar^2 a}{m},$$

$$R(\vec{k}) = \frac{R}{16}\left[(2k\sigma)^3 - 12(2k\sigma) + 16\right] \text{ for 3D isotropic speckle}$$

with σ as correlation length of disorder.

Equation for Sound Velocity

- **Two-fluid model:** $n = n_S + n_N\,, \qquad \mathbf{j} = n_S \mathbf{v}_S + n_N \underbrace{\mathbf{v}_N}_{=\mathbf{0}}$
- **Continuity equation:** $\frac{\partial}{\partial t} n + \nabla \mathbf{j} = 0$
- **Euler equation:** $M \frac{\partial}{\partial t} \mathbf{v}_S + \nabla \left(\mu + \frac{1}{2} M \mathbf{v}_S^2 \right) = 0$
- **Linearization around equilibrium:**

$$n_S = n_{S,\mathrm{eq}} + \delta n_S(\mathbf{x},t)\;, \qquad \mathbf{v}_S = \delta \mathbf{v}_S(\mathbf{x},t)$$

$$n = n_{\mathrm{eq}} + \delta n(\mathbf{x},t)\;, \qquad \mu = \mu\left(n_{\mathrm{eq}}\right) + \left.\frac{\partial \mu}{\partial n}\right|_{\mathrm{eq}} \delta n(\mathbf{x},t)$$

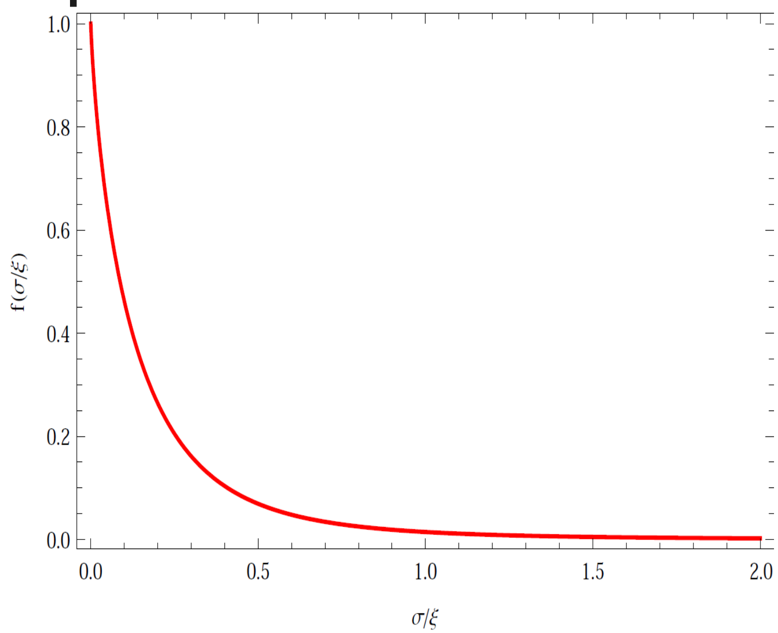
- **Wave equation:**

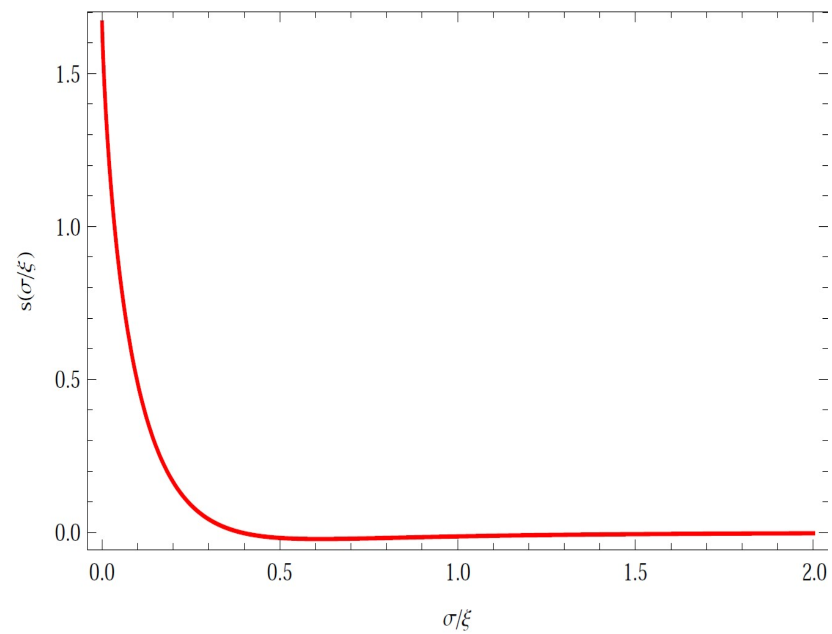
$$\frac{\partial^2}{\partial t^2} \delta n(\mathbf{x},t) - \frac{1}{M} \left.\frac{\partial \mu}{\partial n}\right|_{\mathrm{eq}} \nabla \left[n_{S,\mathrm{eq}}(\mathbf{x}) \nabla \delta n(\mathbf{x},t) \right] = 0$$

$$\frac{c}{\sqrt{n V_{\mathrm{int}}(\mathbf{0})/m}} = 1 + 2 \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})}{[\hbar^2 \mathbf{k}^2/2m + 2n V_{\mathrm{int}}(\mathbf{k})]^2} \times \left\{ \frac{\hbar^2 \mathbf{k}^2 V_{\mathrm{int}}(\mathbf{k})/2m V_{\mathrm{int}}(\mathbf{0})}{\hbar^2 \mathbf{k}^2/2m + 2n V_{\mathrm{int}}(\mathbf{k})} - (\hat{\mathbf{q}}\,\hat{\mathbf{k}})^2 \right\} + \cdots ,$$

For 3D isotropic speckle $V_{inter}(\vec{k}) = V_{inter}(0) = g.$

BEC Depletion and Sound Velocity in 3D Isotropic Speckle





$$n - n_0 = n_{HM}\, f\left(\frac{\sigma}{\xi}\right), \text{ where } n_{HM} = \left[\frac{m^2 R}{8\pi^{3/2}\hbar^4}\right]\sqrt{\frac{n}{a}},$$

$$f\left(\frac{\sigma}{\xi}\right) = \frac{\sigma}{\sqrt{2\pi}\,\xi}\left[4 - \left(\frac{8\sigma^2}{\xi^2} + 6\right)\ln\left(1 + \frac{\xi^2}{2\sigma^2}\right) + \frac{4\xi}{\sqrt{2}\sigma}\arctan\left(\frac{\xi}{\sqrt{2}\sigma}\right)\right]$$

$$\frac{c}{c_0} = 1 + \frac{n_{HM}}{2n}\, s\left(\frac{\sigma}{\xi}\right), \text{ where } c_0 = \sqrt{\frac{ng}{m}},\ \xi = \sqrt{\frac{\hbar^2}{2mng}},$$

$$s\left(\frac{\sigma}{\xi}\right) = \frac{\sqrt{2}\sigma}{\pi\xi}\left[\frac{14}{3} - \left(\frac{28\sigma^2}{3\xi^2} + 4\right)\ln\left(1 + \frac{\xi^2}{2\sigma^2}\right) + \frac{5\xi}{3\sqrt{2}\sigma}\arctan\left(\frac{\xi}{\sqrt{2}\sigma}\right)\right]$$

Khalatnikov – Landau Derivation of Normal Fluid Density

Two fluid model: origin of superfluid density – condensate, origin of normal fluid density – gas of quasiparticles

1. Boost for one quasiparticle:

Reference of frame K:
liquid has zero velocity as also one quasiparticle with energy $\varepsilon(\vec{p})$.
Reference of frame K_0
with centre of coordinate with quasiparticle does boost with velocity $-\vec{v}$.
Energies relative two frames: $E = \varepsilon(\vec{p}) - \vec{p}\,\vec{v} + \frac{mv^2}{2}$.

2. Mean momentum of quasiparticle after boost:

$$\vec{P} = \int \vec{p}\; n(\varepsilon(\vec{p}) - \vec{p}\,\vec{v}) \frac{Vd^3p}{(2\pi\hbar)^3} \simeq -\int \vec{p}\,(\vec{p}\,\vec{v}) \frac{dn(\varepsilon)}{d\epsilon} \frac{Vd^3p}{(2\pi\hbar)^3} = \vec{v} \int p_z^2 \left(\frac{-dn(\varepsilon)}{d\epsilon} \right) \frac{Vd^3p}{(2\pi\hbar)^3} \;\; for \;\; \vec{v} \to 0.$$

$$Hence,\ normal\ fluid\ density\ \rho_n = \int p_z^2 \left(\frac{-dn(\varepsilon)}{d\epsilon} \right) \frac{d^3p}{(2\pi\hbar)^3}$$

3. Assumption: particles of condensate after scattering with disorder are quasiparticles. Then averaged over the ensemble of disorder

$$\langle \rho_n \rangle = \int p_z^2 \left(\frac{-d\langle n(\varepsilon)\rangle}{d\epsilon} \right) \frac{d^3p}{(2\pi\hbar)^3}.$$

$$For\ m\langle n(\varepsilon)\rangle = mn_0 (2\pi\hbar)^3 \delta(\vec{p}) + \frac{mn_0 R(\vec{k})}{\left(\frac{\hbar^2 k^2}{2m} + 2gn_0 \right)}\ one\ obtains$$

$$\langle \rho_n \rangle = mn_0 \int k_z^2 \frac{R(\vec{k})}{k^2 \left(\frac{\hbar^2 k^2}{2m} + 2gn_0 \right)} \frac{d^3k}{(2\pi)^3}.\ For\ isotropic\ case\ k^2 = k_x^2 + k_y^2 + k_z^2 = 3k_z^2,$$

$$thus\ the\ condensate\ depletion\ \langle n \rangle - n_0 = 3\langle \rho_n \rangle.\ One\ finds\ \langle \rho_n \rangle = \frac{\langle \rho_n \rangle_{LR}}{4},\ where\ \langle \rho_n \rangle_{LR}$$

obtained within linear response theory.

Summary and Conclusion

1. The 'Gaussian' speckle is result of Huygens-Fresnel, Fourier mapping of homogeneous object ways.
2. Spatially varying part of autocorrelation function of speckle can not be approximated by Gaussian, since it contains zeros.
3. Ellipsoidal optical cavity is proposed for experimental realization of 3D speckle.
4. Density depletion and sound velocity is found for BEC in weak 3D isotropic speckle.
5. Khalatnikov-Landau derivation of normal fluid density: particles of condensate scattered by disorder consists gas of normal fluid quasiparticles – alternative physical insight to problem.

Thanks for attention.