BCS Mean-Field Theory and Beyond

André Becker

Faculty of Physics
TU Kaiserslautern

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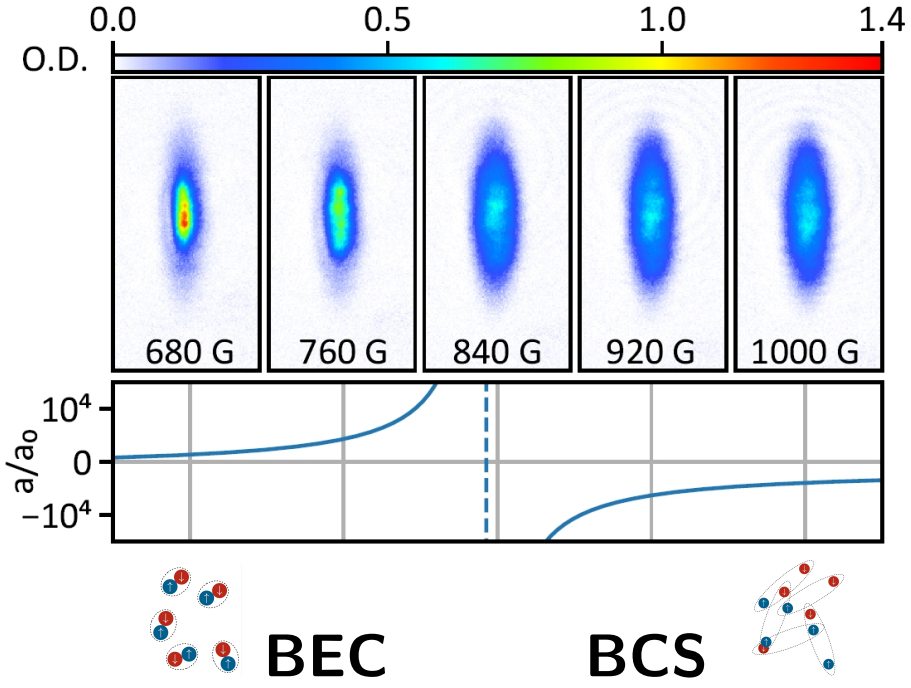
BEC-BCS Crossover

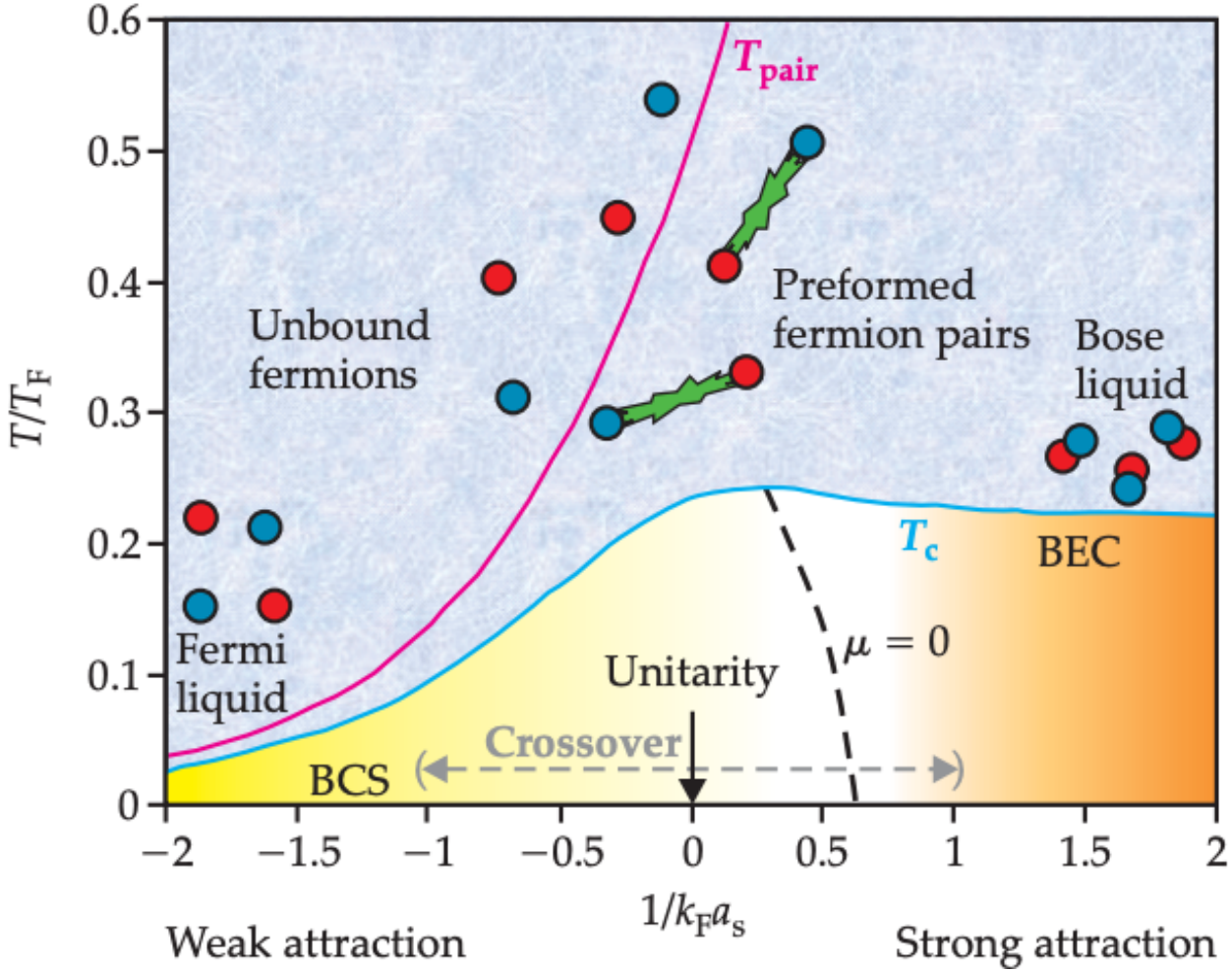
^{6}Li, $N \propto 10^5$, $T < 100$nK: confinement in combined potential of optical dipole and magnetic trap with aspect ratio $\propto 7$



Nagler et al., Rev. Sci. Instr. **89**, 093105 (2018)

BEC-BCS Crossover

Phase diagram for fermionic superfluids:



Sá de Melo, Physics Today **61**, 45 (Oct. 2008)

Table of Contents

Table of Contents

Mean-Field Theory

Mean-Field Hamiltonian

$$\hat{H} = \int \mathrm{d}^3x \sum_{\sigma=\uparrow,\downarrow} \hat{\psi}^\dagger_\sigma(\boldsymbol{x})\Big(-\frac{\hbar^2}{2M}\Delta_\mathrm{L} - \mu\Big)\hat{\psi}_\sigma(\boldsymbol{x})$$
$$+g\int \mathrm{d}^3x \hat{\psi}^\dagger_\uparrow(\boldsymbol{x})\hat{\psi}^\dagger_\downarrow(\boldsymbol{x})\hat{\psi}_\downarrow(\boldsymbol{x})\hat{\psi}_\uparrow(\boldsymbol{x})$$

anti-commutator algebra

$$\Big[\hat{\psi}_\sigma(\boldsymbol{x}), \hat{\psi}_{\sigma'}(\boldsymbol{x}')\Big]_+ = 0 = \Big[\hat{\psi}^\dagger_\sigma(\boldsymbol{x}), \hat{\psi}^\dagger_{\sigma'}(\boldsymbol{x}')\Big]_+$$
$$\Big[\hat{\psi}_\sigma(\boldsymbol{x}), \hat{\psi}^\dagger_{\sigma'}(\boldsymbol{x}')\Big]_+ = \delta_{\sigma,\sigma'}\delta(\boldsymbol{x}-\boldsymbol{x}')$$

Mean-Field Theory

Mean-Field Hamiltonian

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$$+g\int \mathrm{d}^3x \hat{\psi}^\dagger_\uparrow(\boldsymbol{x})\hat{\psi}^\dagger_\downarrow(\boldsymbol{x})\hat{\psi}_\downarrow(\boldsymbol{x})\hat{\psi}_\uparrow(\boldsymbol{x})$$

Mean-Field approach

$$\begin{aligned}\hat{\psi}^\dagger_\uparrow\hat{\psi}^\dagger_\downarrow\hat{\psi}_\downarrow\hat{\psi}_\uparrow \approx\ & \langle\hat{\psi}^\dagger_\uparrow\hat{\psi}_\uparrow\rangle\hat{\psi}^\dagger_\downarrow\hat{\psi}_\downarrow + \hat{\psi}^\dagger_\uparrow\hat{\psi}_\uparrow\langle\hat{\psi}^\dagger_\downarrow\hat{\psi}_\downarrow\rangle - \langle\hat{\psi}^\dagger_\uparrow\hat{\psi}_\uparrow\rangle\langle\hat{\psi}^\dagger_\downarrow\hat{\psi}_\downarrow\rangle \\ & -\langle\hat{\psi}^\dagger_\uparrow\hat{\psi}_\downarrow\rangle\hat{\psi}^\dagger_\downarrow\hat{\psi}_\uparrow - \hat{\psi}^\dagger_\uparrow\hat{\psi}_\downarrow\langle\hat{\psi}^\dagger_\downarrow\hat{\psi}_\uparrow\rangle + \langle\hat{\psi}^\dagger_\uparrow\hat{\psi}_\downarrow\rangle\langle\hat{\psi}^\dagger_\downarrow\hat{\psi}_\uparrow\rangle \\ & +\langle\hat{\psi}^\dagger_\uparrow\hat{\psi}^\dagger_\downarrow\rangle\hat{\psi}_\downarrow\hat{\psi}_\uparrow + \hat{\psi}^\dagger_\uparrow\hat{\psi}^\dagger_\downarrow\langle\hat{\psi}_\downarrow\hat{\psi}_\uparrow\rangle - \langle\hat{\psi}^\dagger_\uparrow\hat{\psi}^\dagger_\downarrow\rangle\langle\hat{\psi}_\downarrow\hat{\psi}_\uparrow\rangle\end{aligned}$$

Mean-Field Theory

Mean-Field Hamiltonian

$$\hat{H} = \int \mathrm{d}^3x \sum_{\sigma=\uparrow,\downarrow} \hat{\psi}^\dagger_\sigma(\boldsymbol{x})\Big(-\frac{\hbar^2}{2M}\Delta_\mathrm{L} - \mu\Big)\hat{\psi}_\sigma(\boldsymbol{x})$$
$$+g\int \mathrm{d}^3x \hat{\psi}^\dagger_\uparrow(\boldsymbol{x})\hat{\psi}^\dagger_\downarrow(\boldsymbol{x})\hat{\psi}_\downarrow(\boldsymbol{x})\hat{\psi}_\uparrow(\boldsymbol{x})$$

Mean-Field approach

$$\hat{\psi}^\dagger_\uparrow\hat{\psi}^\dagger_\downarrow\hat{\psi}_\downarrow\hat{\psi}_\uparrow \approx \overbrace{\cancel{\langle\hat{\psi}^\dagger_\uparrow\hat{\psi}_\uparrow\rangle\hat{\psi}^\dagger_\downarrow\hat{\psi}_\downarrow + \hat{\psi}^\dagger_\uparrow\hat{\psi}_\uparrow\langle\hat{\psi}^\dagger_\downarrow\hat{\psi}_\downarrow\rangle - \langle\hat{\psi}^\dagger_\uparrow\hat{\psi}_\uparrow\rangle\langle\hat{\psi}^\dagger_\downarrow\hat{\psi}_\downarrow\rangle}}^{\text{Hartree channel}}$$
$$-\overbrace{\cancel{\langle\hat{\psi}^\dagger_\uparrow\hat{\psi}_\downarrow\rangle\hat{\psi}^\dagger_\downarrow\hat{\psi}_\uparrow - \hat{\psi}^\dagger_\uparrow\hat{\psi}_\downarrow\langle\hat{\psi}^\dagger_\downarrow\hat{\psi}_\uparrow\rangle + \langle\hat{\psi}^\dagger_\uparrow\hat{\psi}_\downarrow\rangle\langle\hat{\psi}^\dagger_\downarrow\hat{\psi}_\uparrow\rangle}}^{\text{Fock channel}}$$
$$+\underbrace{\langle\hat{\psi}^\dagger_\uparrow\hat{\psi}^\dagger_\downarrow\rangle\hat{\psi}_\downarrow\hat{\psi}_\uparrow + \hat{\psi}^\dagger_\uparrow\hat{\psi}^\dagger_\downarrow\langle\hat{\psi}_\downarrow\hat{\psi}_\uparrow\rangle - \langle\hat{\psi}^\dagger_\uparrow\hat{\psi}^\dagger_\downarrow\rangle\langle\hat{\psi}_\downarrow\hat{\psi}_\uparrow\rangle}_{\text{Bogoliubov channel}}$$

Mean-Field Hamiltonian

$$\Rightarrow \hat{H} \approx \hat{H}_{\mathrm{MF}} = \int \mathrm{d}^3x \Big\{ \sum_{\sigma=\uparrow,\downarrow} \hat{\psi}^\dagger_\sigma(\boldsymbol{x}) \Big[-\frac{\hbar}{2M}\Delta_{\mathrm{L}} - \mu \Big] \hat{\psi}_\sigma(\boldsymbol{x})$$
$$+ \Big[\Delta^* \hat{\psi}_\downarrow(\boldsymbol{x}) \hat{\psi}_\uparrow(\boldsymbol{x}) + \Delta \hat{\psi}^\dagger_\downarrow(\boldsymbol{x}) \hat{\psi}^\dagger_\uparrow(\boldsymbol{x}) - \frac{\Delta^*\Delta}{g} \Big] \Big\}$$

Order parameter for pairing:

$$\Delta = g\langle \hat{\psi}_\downarrow(\boldsymbol{x}) \hat{\psi}_\uparrow(\boldsymbol{x}) \rangle, \; \Delta^* = g\langle \hat{\psi}^\dagger_\uparrow(\boldsymbol{x}) \hat{\psi}^\dagger_\downarrow(\boldsymbol{x}) \rangle$$

Mean-Field and Beyond

- **Partition function:**

$$Z = \mathrm{Tr} e^{-\beta \hat{H}} = \mathrm{Tr} e^{-\beta[\hat{H}_{\mathrm{MF}} + (\hat{H} - \hat{H}_{\mathrm{MF}})]}$$

- **Moment expansion:**

$$Z = Z_{\mathrm{MF}} \left\{ 1 - \beta \langle \hat{H} - \hat{H}_{\mathrm{MF}} \rangle_{\mathrm{MF}} + \frac{\beta^2}{2} \langle (\hat{H} - \hat{H}_{\mathrm{MF}})^2 \rangle_{\mathrm{MF}} + \ldots \right\}$$

with $\langle \cdot \rangle_{\mathrm{MF}} = \frac{\mathrm{Tr}\left\{ \cdot \; e^{-\beta \hat{H}_{\mathrm{MF}}} \right\}}{Z_{\mathrm{MF}}}$ and $Z_{\mathrm{MF}} = \mathrm{Tr} e^{-\beta \hat{H}_{\mathrm{MF}}}$

- Cumulant expansion:

$$F = -\frac{1}{\beta} \ln Z_{\mathrm{MF}}$$
$$+ \langle \hat{H} - \hat{H}_{\mathrm{MF}} \rangle_{\mathrm{MF}} - \frac{\beta}{2} \left[\langle (\hat{H} - \hat{H}_{\mathrm{MF}})^2 \rangle_{\mathrm{MF}} - (\langle \hat{H} - \hat{H}_{\mathrm{MF}} \rangle_{\mathrm{MF}})^2 \right] + \ldots$$

Order parameter as variational parameter

- **Extremalization of free energy**

$$\frac{\partial F_{\mathrm{MF}}(\Delta^*, \Delta)}{\partial \Delta^*} = 0 \Leftrightarrow \Delta = g\langle \hat{\psi}_{\downarrow}(\boldsymbol{x})\hat{\psi}_{\uparrow}(\boldsymbol{x})\rangle_{\mathrm{MF}}$$

Order parameter as variational parameter

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$$\frac{\partial F_{\mathrm{MF}}(\Delta^*, \Delta)}{\partial \Delta^*} = 0 \Leftrightarrow \Delta = g\langle \hat{\psi}_\downarrow(\boldsymbol{x})\hat{\psi}_\uparrow(\boldsymbol{x})\rangle_{\mathrm{MF}}$$

- **Selfconsistency-Equations** $\rightarrow$ Introducing dispersion $\epsilon = \frac{\hbar^2 k^2}{2M}$
  - equation for order parameter

$$\partial_{\Delta^*} F_{\mathrm{MF}}(\Delta^*, \Delta) = 0$$

$$\Rightarrow \Delta\left[\frac{1}{g} + \frac{1}{\sqrt{\pi}}\left(\frac{M}{2\pi\hbar^2}\right)^{3/2}\int_0^\infty \mathrm{d}\epsilon\sqrt{\epsilon}\frac{\tanh\left(\sqrt{(\epsilon-\mu)^2+\Delta^2}\frac{\beta}{2}\right)}{\sqrt{(\epsilon-\mu)^2+\Delta^2}}\right] = 0$$

  - particle-number equation

$$n = -\frac{1}{V}\partial_\mu F_{\mathrm{MF}}$$

Order parameter as variational parameter

Order parameter

UV-divergency $\rightarrow$ Renormalization $\frac{1}{g} = -\frac{M}{4\pi\hbar^2 a_s} + \frac{1}{\sqrt{\pi}}\left(\frac{M}{2\pi\hbar^2}\right)^{\frac{3}{2}} \int_0^\infty \mathrm{d}\epsilon \frac{1}{\sqrt{\epsilon}}$

$$\frac{M}{2\pi\hbar^2 a_S} = \frac{2}{\sqrt{\pi}}\left(\frac{M}{2\pi\hbar^2}\right)^{3/2} \int_0^\infty \mathrm{d}\epsilon\sqrt{\epsilon}\left[\frac{1}{\epsilon} - \frac{\tanh\left(\sqrt{(\epsilon-\mu)^2+\Delta^2}\frac{\beta}{2}\right)}{\sqrt{(\epsilon-\mu)^2+\Delta^2}}\right]$$

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Particle-number equation

$$n = \frac{2}{\sqrt{\pi}}\left(\frac{M}{2\pi\hbar^2}\right)^{\frac{3}{2}} \int_0^\infty \mathrm{d}\epsilon\sqrt{\epsilon}\left[1 - \frac{\epsilon\tanh\left(\sqrt{(\epsilon-\mu)^2+\Delta^2}\frac{\beta}{2}\right)}{\sqrt{(\epsilon-\mu)^2+\Delta^2}}\right]$$

Evaluation of self-consistency equation for $T = 0$

Particle Number

Approximation $\Delta = 0$

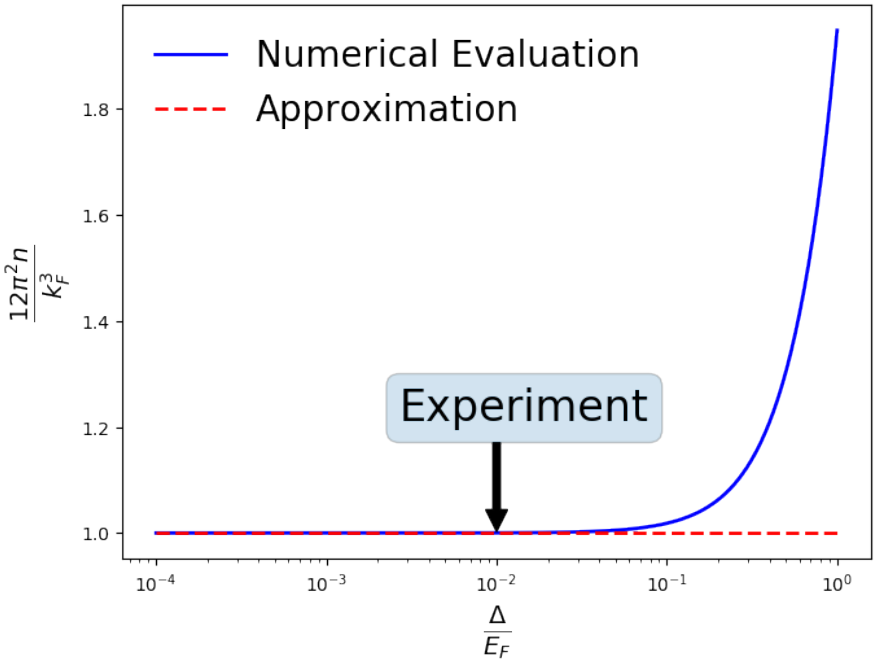
$$n = \frac{k_{\mathrm{F}}^3}{12\pi^2}, \quad E_{\mathrm{F}} = \frac{\hbar^2 k_{\mathrm{F}}^2}{2M}$$

Order Parameter

Approximation

$$\Delta = \frac{8}{e^2} E_{\mathrm{F}} e^{-\frac{\pi}{2k_F|a_s|}}$$

Orso et al., PRL **99**, 250402 (2007)



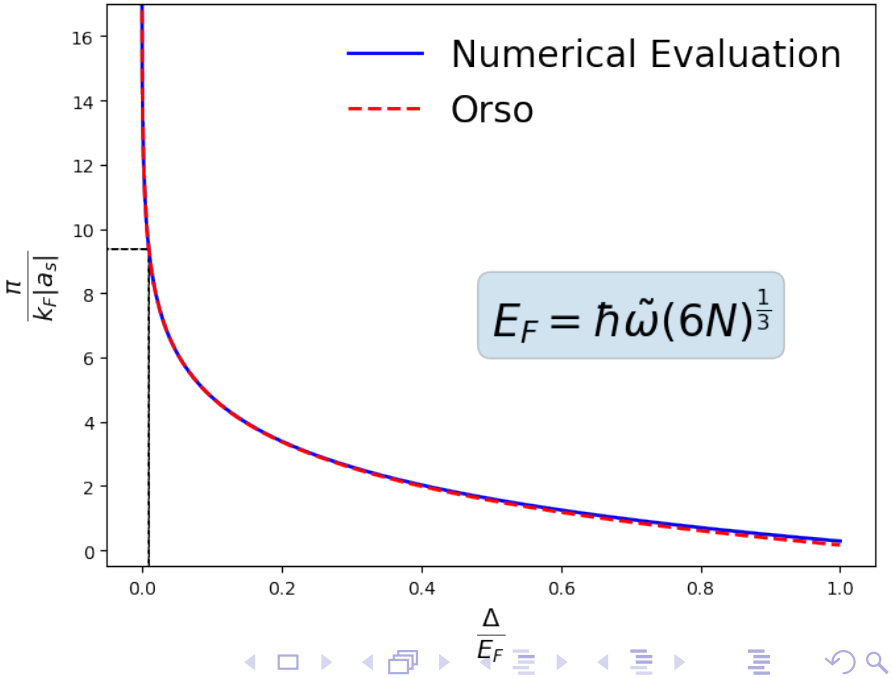


Table of Contents

Introducing harmonic trapping potential

Harmonic trapping potential

$$V(\boldsymbol{x}) = \frac{M}{2} \sum_{i=x,y,z} \omega_i^2 x_i^2$$

LDA: $\mu \to \mu(\boldsymbol{x}) = \mu - V(\boldsymbol{x})$

Introducing harmonic trapping potential

Harmonic trapping potential

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LDA: $\mu \to \mu(\boldsymbol{x}) = \mu - V(\boldsymbol{x})$

Consequences

Limitation of density cloud by Thomas-Fermi radii

How can we calculate Thomas-Fermi radii?

Fermi momentum (BCS limit)

$$k_{\mathrm{F}}(\boldsymbol{x}) = \sqrt{\frac{2M}{\hbar^2}\left[\mu - V(\boldsymbol{x})\right]}$$

vanishes at Thomas-Fermi radii $k_{\mathrm{F}}(R_{\mathrm{TF}}) = 0$

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Thomas-Fermi approximation (BEC limit)

Time-independent *Gross-Pitaevskii* equation

$$\mu\psi(\boldsymbol{x}) = \Big[\underbrace{-\frac{\hbar^2}{2(2M)}\Delta_{\mathrm{L}}}_{=0} + V(\boldsymbol{x})\Big]\psi(\boldsymbol{x}) + \underbrace{\frac{4\pi\hbar^2(0.6a_{\mathrm{s}})}{(2M)}}_{\equiv\tilde{g}}|\psi(\boldsymbol{x})|^2\psi(\boldsymbol{x})$$

Calculation of Thomas-Fermi radii

BCS-limit: $k_{\mathrm{F}}(R_{\mathrm{TF}}) = 0 \Rightarrow E_{\mathrm{F}} = \frac{M}{2}\omega_i^2 R_{\mathrm{TFi}}^2$

$$\Rightarrow R_{\mathrm{TF}i} = \sqrt{\frac{2E_{\mathrm{F}}}{M\omega_i^2}}$$

Calculation of Thomas-Fermi radii

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BEC-limit: $|\psi_{\mathrm{TF}}(\boldsymbol{x})|^2 \equiv n_{\mathrm{TF}}(\boldsymbol{x}) = \frac{\mu - V(\boldsymbol{x})}{\tilde{g}} \Rightarrow n_{\mathrm{TF}}(\boldsymbol{x}) = \frac{\mu - V(\boldsymbol{x})}{\tilde{g}} \geq 0$

$$\Rightarrow R_{\mathrm{TF}i} = 15^{\frac{1}{5}} \left(\frac{0.6 a_{\mathrm{s}} \frac{N}{2}}{L} \right)^{\frac{1}{5}} L$$

with oscillator length $L \equiv \sqrt{\frac{\hbar}{(2M)\tilde{\omega}}}$

Which particle cloud is the smaller one?

Which particle cloud is the smaller one?

Fermi pressure vs. Boson-Boson interaction

Comparison of Thomas-Fermi radii

- *particle number:* $N = 10^6$
- *frequencies:* $\omega_x = 2\pi \times 195$ Hz, $\omega_y = 2\pi \times 22.6$ Hz and $\omega_z = 2\pi \times 129$ Hz
- *scattering length:* $a_S = 4510 \cdot a_0$ (fermionic), $0.6 \cdot a_S = 2706 \cdot a_0$ (bosonic)

BCS

- $R_{\mathrm{TF},x} = 37$ µm
- $R_{\mathrm{TF},y} = 315$ µm
- $R_{\mathrm{TF},z} = 55$ µm

BEC

- $R_{\mathrm{TF},x} = 29$ µm
- $R_{\mathrm{TF},y} = 68$ µm
- $R_{\mathrm{TF},z} = 34$ µm

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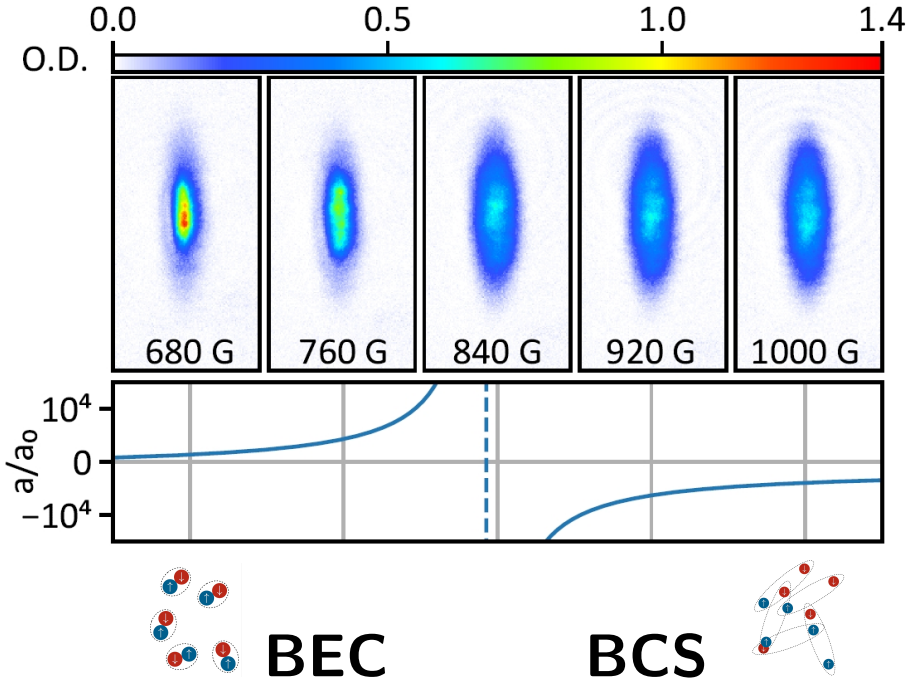
Result:

BCS-limit: particle cloud $\Rightarrow$ **larger volume** compared to *BEC-limit*

BEC-BCS Crossover

^{6}Li, $N \propto 10^5$, $T < 100$nK: confinement in combined potential of optical dipole and magnetic trap with aspect ratio $\propto 7$



Widera et al., Rev. Sci. Instr. **89**, 093105 (2018)

Table of Contents

Temperature dependence of particle-number equation

Starting from **Fermi-Dirac statistics**:

$$
\begin{aligned}
n(T) &= \frac{4}{\sqrt{\pi}}\left(\frac{M}{2\pi\hbar^2}\right)^{\frac{3}{2}}\int_0^\infty \mathrm{d}\epsilon\sqrt{\epsilon}\frac{1}{e^{\beta[\epsilon-\mu(T)]}+1} \\
&= \left(\frac{Mk_\mathrm{B}T}{2\pi\hbar^2}\right)^{\frac{3}{2}}\xi^+_{\frac{3}{2}}\left(e^{\beta\mu(T)}\right)
\end{aligned}
$$

Temperature dependence of particle-number equation

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$$= \left(\frac{Mk_\mathrm{B}T}{2\pi\hbar^2}\right)^{\frac{3}{2}}\xi^+_{\frac{3}{2}}\left(e^{\beta\mu(T)}\right)$$

LDA:

$$n(\boldsymbol{x},T) = \left(\frac{Mk_\mathrm{B}T}{2\pi\hbar^2}\right)^{\frac{3}{2}}\xi^+_{\frac{3}{2}}\left(e^{\beta[\mu(T)-V(\boldsymbol{x})]}\right)$$

Approximation of temperature dependence for fermions

Sommerfeld expansion

$$\xi_\nu^+(e^{\beta\mu}) = \frac{1}{\Gamma(\nu)}\left\{\frac{(\beta\mu)^\nu}{\nu} + 2\sum_{k=1}^{\infty}(\nu-1)(\nu-2)\cdot\ldots \right.$$
$$\cdot(\nu-2k+1)(\beta\mu)^{\nu-2k}\zeta(2k)\left(1-\frac{1}{2^{2k-1}}\right)$$
$$\left. +(-1)^{\nu-1}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-\beta\mu k}}{k^\nu}\right\}$$

Considering $T \to 0$:

$$\xi_\nu^+(e^{\beta\mu}) = \frac{1}{\Gamma(\nu+1)}\left(\frac{\mu}{k_\mathrm{B}T}\right)^\nu\left\{1+\nu(\nu-1)\zeta(2)\left(\frac{k_\mathrm{B}T}{\mu}\right)^2\right.$$
$$\left. +\frac{7}{4}\nu(\nu-1)(\nu-2)(\nu-3)\zeta(4)\left(\frac{k_\mathrm{B}T}{\mu}\right)^4+\ldots\right\}$$

Sommerfeld expansion for particle-number density

2nd order Sommerfeld expansion

Assumption: $\frac{T}{T_{\mathrm{TF}}} \ll 1$

$$\xi^{+}_{\frac{3}{2}}\left(e^{\beta[\mu(T)-V(\boldsymbol{x})]}\right) = \frac{1}{\Gamma(\frac{5}{2})}\left(\frac{\mu(T)-V(\boldsymbol{x})}{k_{\mathrm{B}}T}\right)^{\frac{3}{2}} \cdot \left[1+\frac{3}{2}\cdot\frac{1}{2}\underbrace{\zeta(2)}_{\frac{\pi^2}{6}}\left(\frac{k_{\mathrm{B}}T}{\mu(T)-V(\boldsymbol{x})}\right)^{2}+\ldots\right]$$

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Temperature dependent particle-number density

$$n(\boldsymbol{x},T) = \frac{4}{3\sqrt{\pi}}\left(\frac{M}{2\pi\hbar^2}\right)^{\frac{3}{2}}\left\{\left[\mu(T)-V(\boldsymbol{x})\right]^{\frac{3}{2}}+\frac{\pi^2}{8}\frac{(k_{\mathrm{B}}T)^2}{\sqrt{\mu(T)-V(\boldsymbol{x})}}\right\}$$

Temperature dependency of...

- **Chemical potential:**

$$\mu(T) = E_F \left[1 - \frac{\pi^2}{3} \left(\frac{T}{T_\mathrm{F}} \right)^2 \right]$$

- **Thomas-Fermi radii:**

$$R_{\mathrm{TF},i}(T) = \sqrt{\frac{2E_\mathrm{F}}{M\omega_i^2}} \left[1 - \frac{\pi^2}{6} \left(\frac{T}{T_\mathrm{F}} \right)^2 \right]$$

$\Rightarrow$ Shrinking of Thomas-Fermi radii for increasing temperature

Temperature dependent density cloud in x-y direction

Performing integration within Thomas-Fermi radius in z-direction:

$$n(x, y, T) = \int_{-\sqrt{\frac{2\chi(x,y,T)}{M\omega_z^2}}}^{\sqrt{\frac{2\chi(x,y,T)}{M\omega_z^2}}} \mathrm{d}z\, n(x, y, z, T)$$

with $\chi(x, y, T) \equiv \mu(T) - \frac{M}{2}(\omega_x^2 x^2 + \omega_y^2 y^2)$

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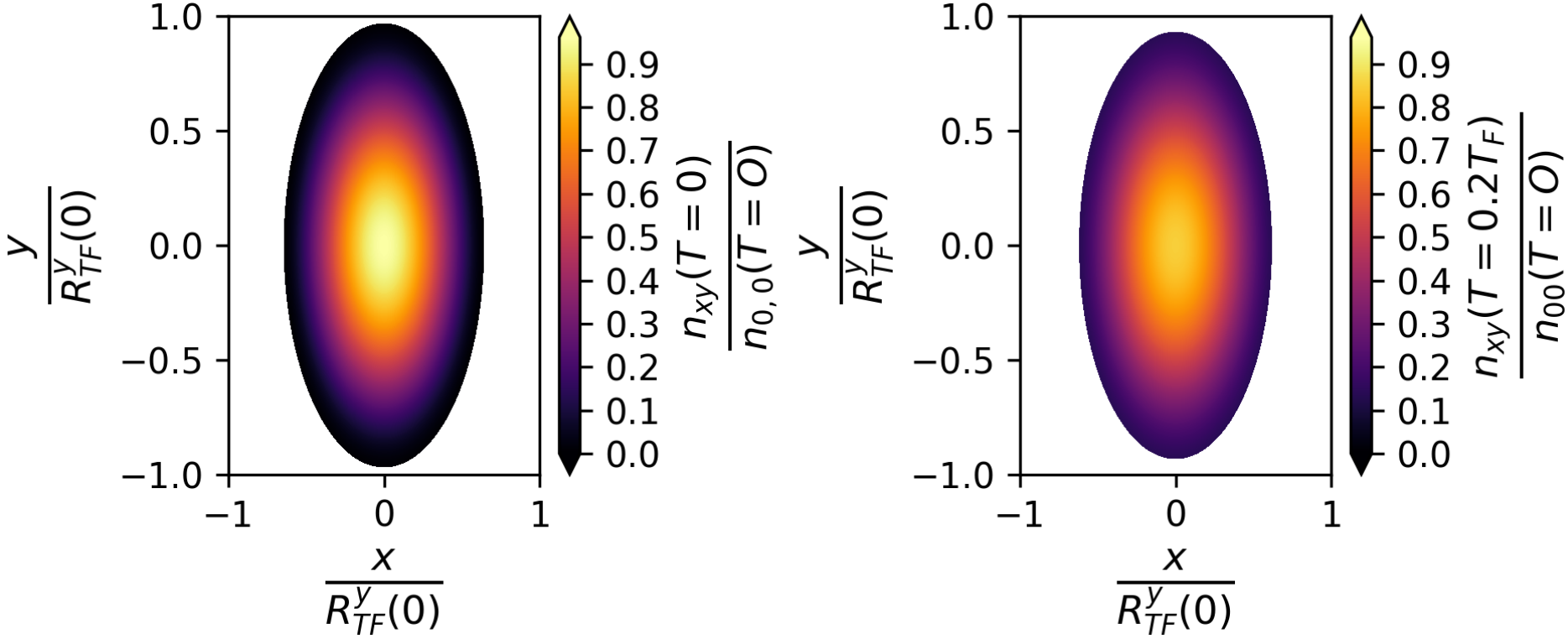
with $\chi(x,y,T) \equiv \mu(T) - \frac{M}{2}(\omega_x^2 x^2 + \omega_y^2 y^2)$

2nd order expression:

$$n(x,y,T) = \frac{M}{12\pi\hbar^3\omega_z}\left\{3\left[\mu(T) - \frac{M}{2}(\omega_x^2 x^2 + \omega_y^2 y^2)\right]^2 + \pi^2(k_\mathrm{B}T)^2\right\}$$



$T = 0$ vs. $T = 0.2\,T_{\mathrm{F}}$

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Temperature dependent density cloud in x direction

Performing integration within Thomas-Fermi radius in y-direction:

$$n(x,T) = \int_{-\sqrt{\frac{2\xi(x,T)}{M\omega_y^2}}}^{\sqrt{\frac{2\xi(x,T)}{M\omega_y^2}}} \mathrm{d}y\, n(x,y,T)$$

with $\xi(x,T) = \mu(T) - \frac{M}{2}\omega_x^2 x^2$

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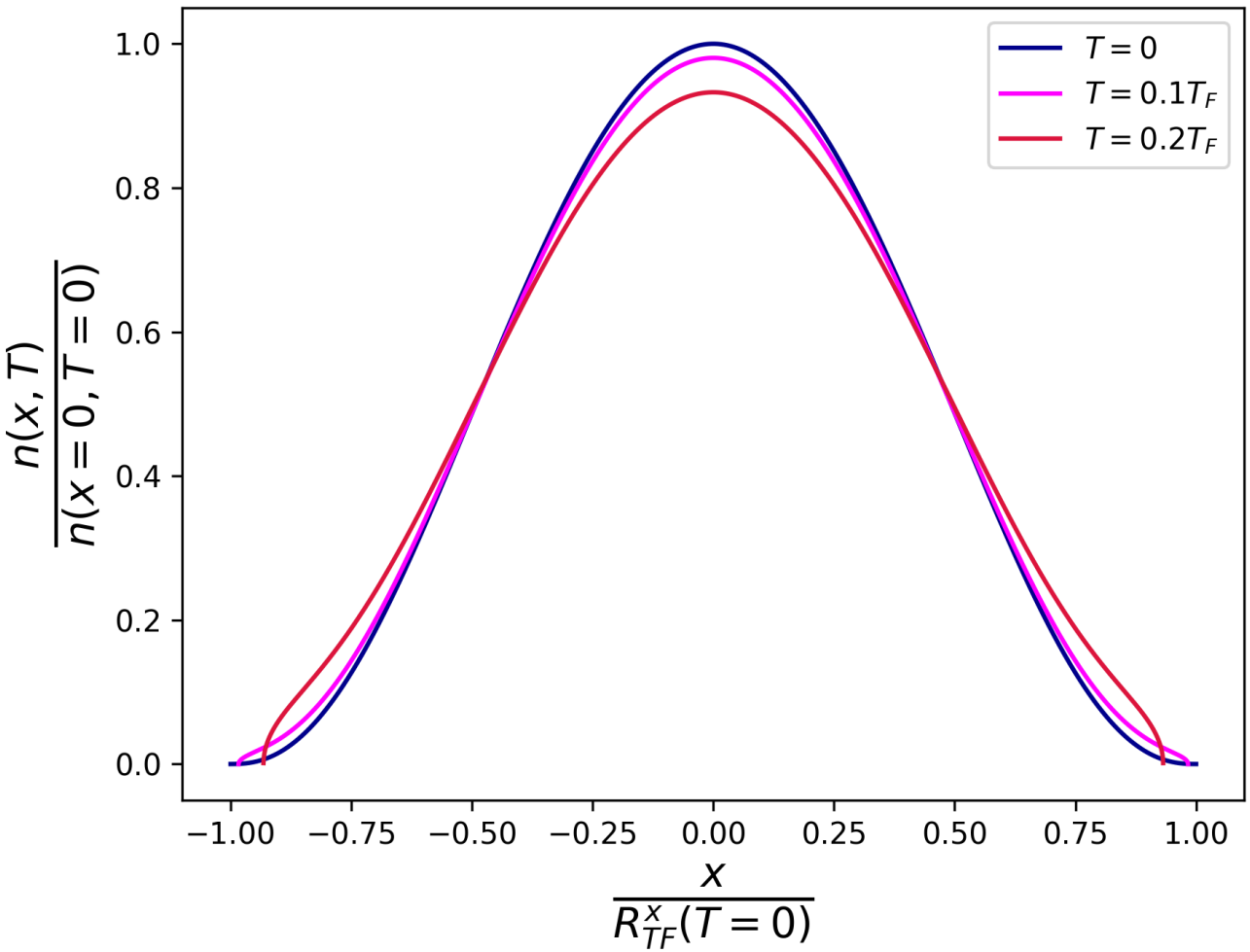
2nd order expression:

$$n(x,T) = \sqrt{\frac{M}{2}}\frac{\sqrt{\mu(T) - \frac{M}{2}\omega_x^2 x^2}}{\pi\hbar^3\omega_y\omega_z}\left\{\frac{8}{15}\left[\mu(T) - \frac{M}{2}\omega_x^2 x^2\right]^2 + \frac{\pi^2}{3}(k_\mathrm{B}T)^2\right\}$$



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Calculation of temperature and particle number

- *Thomas-Fermi radius:*
$$R_{\mathrm{TF},i}(T) \to \mu(T) = \frac{M}{2}\omega_i^2 R_{\mathrm{TF},i}(T)^2$$
- *Particle-number density* in x direction:
$$n(x,T) = \sqrt{\frac{M}{2}}\frac{\sqrt{\mu(T)-\frac{M}{2}\omega_x^2 x^2}}{\pi\hbar^3\omega_y\omega_z}\left\{\frac{8}{15}\left[\mu(T)-\frac{M}{2}\omega_x^2 x^2\right]^2 + \frac{\pi^2}{3}(k_{\mathrm{B}}T)^2\right\}$$

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$$n(x,T) = \sqrt{\frac{M}{2}}\frac{\sqrt{\mu(T)-\frac{M}{2}\omega_x^2 x^2}}{\pi\hbar^3\omega_y\omega_z}\left\{\frac{8}{15}\left[\mu(T)-\frac{M}{2}\omega_x^2 x^2\right]^2 + \frac{\pi^2}{3}(k_{\mathrm{B}}T)^2\right\}$$

Temperature:

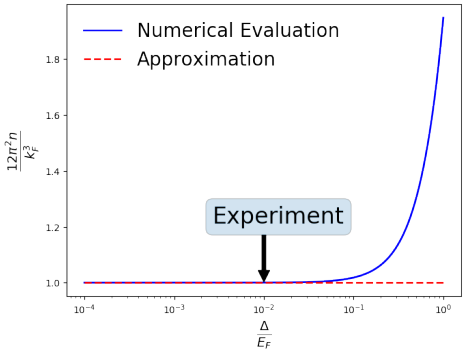
$$\Rightarrow n(0,T) = \frac{M}{2}\frac{\omega_x R_{\mathrm{TF},x}(T)}{\pi\hbar^3\omega_y\omega_z}\left\{\frac{8}{15}\left[\frac{M}{2}\omega_x^2 R_{\mathrm{TF},x}(T)^2\right]^2 + \frac{\pi^2}{3}(k_{\mathrm{B}}T)^2\right\}$$

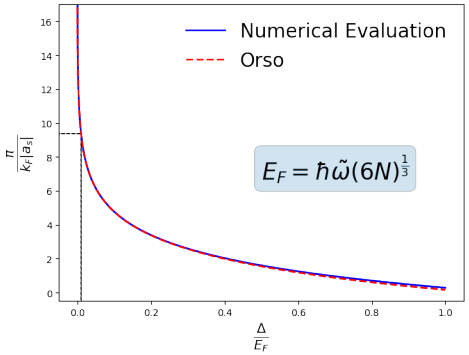
Particle-number:

$$\frac{M}{2}\omega_i^2 R_{\mathrm{TF},x}(T)^2 = \mu(T) = E_F\left\{1-\frac{\pi^2}{3}\left(\frac{T}{T_{\mathrm{F}}}\right)^2\right\} \Rightarrow E_{\mathrm{F}} = \hbar\tilde{\omega}(6N)^{\frac{1}{3}}$$

Summary

1. **Mean-Field Theory**





2. **Local Density Approximation**

   Thomas-Fermi radii comparison

   BCS-limit: particle cloud $\Rightarrow$ **larger volume** compared to *BEC-limit*

3. **Temperature Dependence**

Temperature and particle number:

$$n(0, T), R_{\mathrm{TF},x}(T) \Leftrightarrow N, T$$