Higher Orders of Large-D Expansion From Variational Perturbation Theory

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1. Basic Notion of VPT

2. Anharmonic Oscillator in $D = 1$ Dimension

3. Anharmonic Oscillator in D Dimensions

1 Basic Notion of VPT

weak-coupling series: $$f_N(g) = \sum_{n=0}^{N} a_n \; g^n$$

optimization with respect to

variational parameters

strong-coupling series: $$f_M(g) = g^{\,p/q} \; \sum_{m=0}^{M} b_m \; g^{-2m/q}$$

Example:

	p	q	p/q	$2/q$
anharmonic oscillator	1	3	$1/3$	$2/3$

2.1 Perturbation Theory: Recursive Approach

Schrödinger Equation:

$$-\frac{1}{2}\,\psi''(x) + \left(\frac{1}{2}\,\omega^2 x^2 + g x^4\right)\psi(x) = E_0\,\psi(x)$$

Bender-Wu Expansion:

$$E_0 \;=\; \frac{\omega}{2} + \sum_{k=1}^{\infty} \epsilon_k\, g^k\,, \qquad \psi(x) = N \exp\left[-\frac{\omega}{2}x^2 + \phi(x)\right]$$

$$\phi(x) \;=\; \sum_{k=1}^{\infty} \phi_k(x) g^k\,, \qquad \phi_k(x) = \sum_{m=1}^{\infty} c_m^{(k)}\, x^m$$

Recursion Relations:

$$\begin{aligned}
c_m^{(k)} \;&=\; \frac{(m+1)(2m+1)}{2m\omega}\, c_{m+1}^{(k)} \\
&\quad + \sum_{l=1}^{k-1} \sum_{n=1}^{m} \frac{n(m-n+1)}{m\omega}\, c_n^{(l)} c_{m-n+1}^{(k-l)} \\
c_m^{(k)} \;&\equiv\; 0 \quad \text{for } m > k+1 \\
\epsilon_k \;&=\; -c_1^{(k)}
\end{aligned}$$

Starting Values:

$$c_1^{(1)} = -\frac{3}{4\omega^2}\,, \qquad c_2^{(1)} = -\frac{1}{4\omega}$$

2.2 Variational Resummation

Weak-Coupling Series of Ground-State Energy:

$$E_0 = \frac{\omega}{2} + g\,\frac{3}{4\omega^2} - g^2\,\frac{21}{8\omega^5} + g^3\,\frac{333}{16\omega^8} + \ldots$$

Identity:

$$V(x) = \underbrace{\frac{1}{2}\,\Omega^2 x^2}_{\text{effective harmonic oscillator}} + \underbrace{g\,x^4 + \frac{1}{2}\left(\omega^2 - \Omega^2\right)x^2}_{\text{perturbation}}$$

Ω: variational parameter

Substitution:

$$V(x) = \frac{1}{2}\left(\Omega\sqrt{1 + g\,\frac{\omega^2 - \Omega^2}{g\Omega^2}}\right)^2 x^2 + g\,x^4$$

$$\omega \to \Omega\sqrt{1 + gr}\,, \qquad r = \frac{\omega^2 - \Omega^2}{g\Omega^2}$$

Example: First Order

$$E_0^{(1)} = \frac{\omega}{2} + g\,\frac{3}{4\omega^2}$$

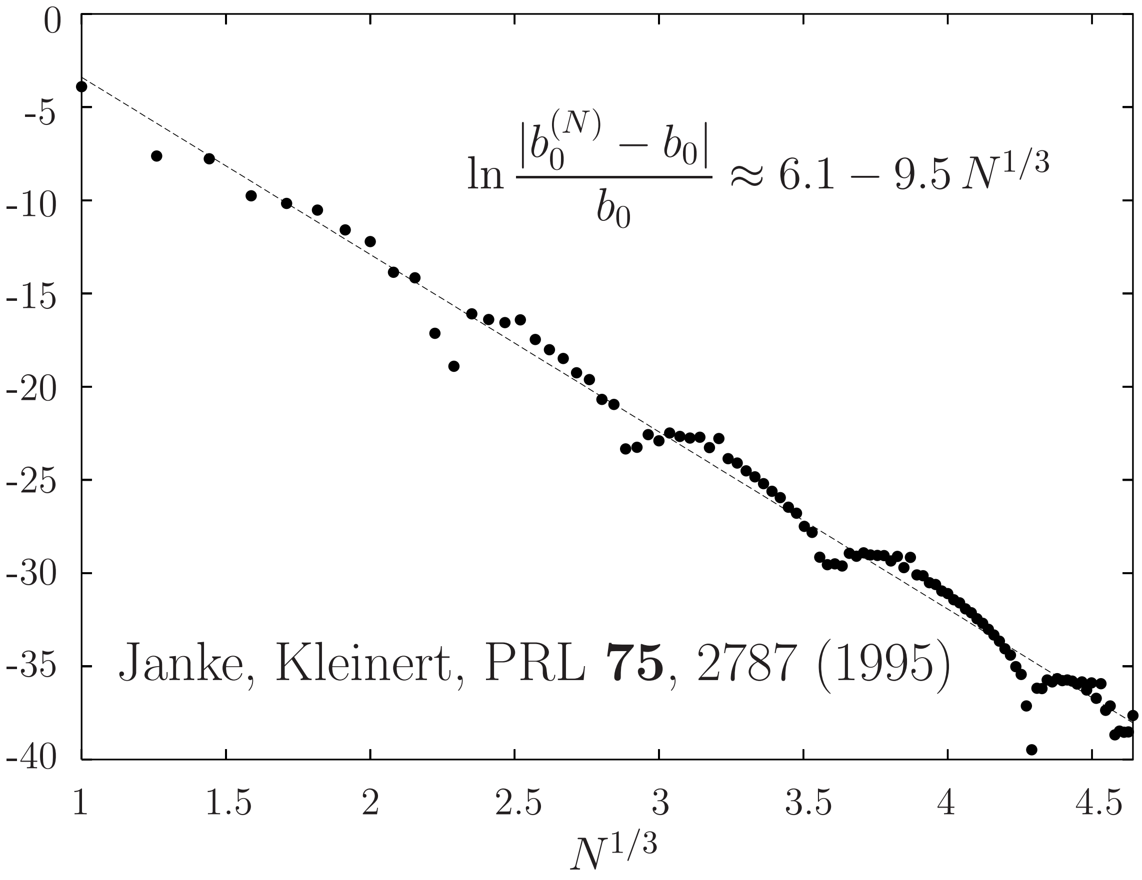
$$E_0^{(1)}(\Omega) = \frac{\Omega}{4} + \frac{\omega^2}{4\Omega} + g\,\frac{3}{4\Omega^2}$$

2.3 Strong-Coupling Results

Strong-Coupling Limit $g \to \infty$:

$$
\begin{aligned}
\frac{\partial E_0^{(1)}(\Omega)}{\partial \Omega} &= \frac{1}{4} - \frac{\omega^2}{4\Omega^2} - g\frac{3}{2\Omega^3} = 0 \\
\Rightarrow \Omega^{(1)} &= g^{1/3}\left(\Omega_0^{(1)} + \Omega_1^{(1)} g^{-2/3} + \Omega_2^{(1)} g^{-4/3} + \ldots\right) \\
\Rightarrow E_0^{(1)} &= g^{1/3}\left(b_0^{(1)} + b_1^{(1)} g^{-2/3} + b_2^{(1)} g^{-4/3} + \ldots\right)
\end{aligned}
$$

Exponential Convergence:



3.1 Perturbation Theory

$O(D)$ Symmetric Potential:

$$V(\mathbf{x}) = \frac{\omega^2}{2}\mathbf{x}^2 + g(\mathbf{x}^2)^2$$

Bender-Wu-Like Recursion Relations:

k	ϵ_k
1	$D(D+2)/4$
2	$-D(2D^2+9D+10)/8$
3	$D(8D^3+59D^2+146D+120)/16$
4	$-D(168D^4+1773D^3+7144D^2$ $+12960D+8840)/128$
5	$D(1024D^5+14325D^4+82222D^3$ $+241464D^2+360736D+216960)/256$

Variational Resummation:

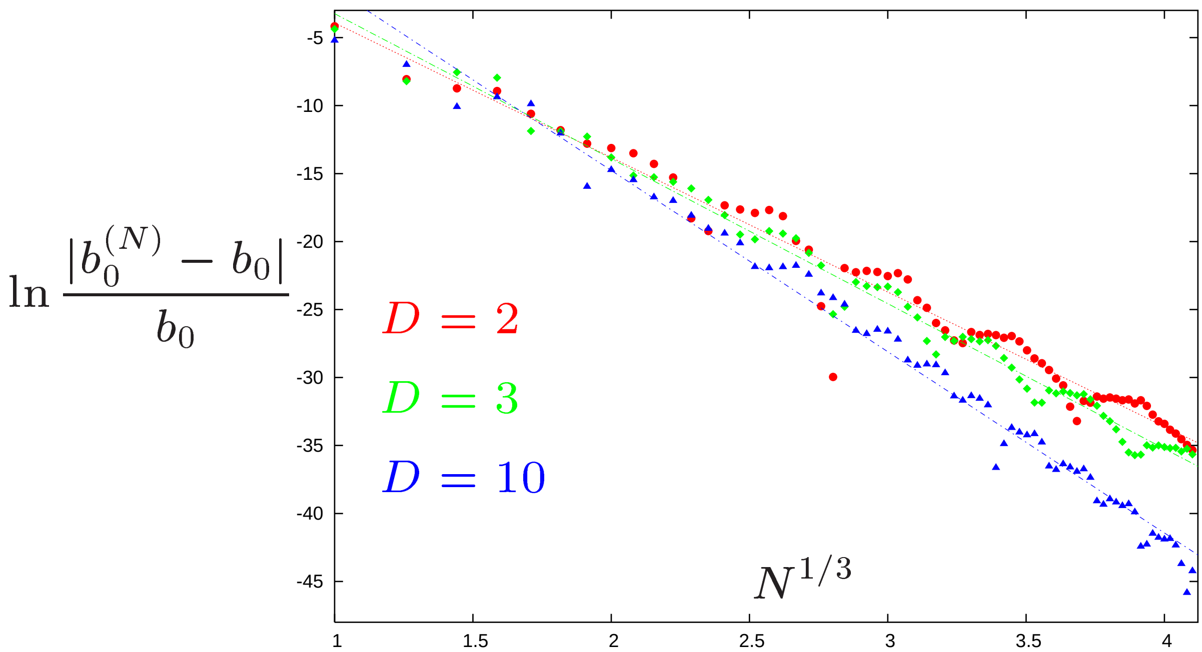
$$E^{(N)}(g,\omega) \to E^{(N)}(g,\omega,\Omega)$$

3.2 Strong-Coupling Expansion

Strong Coupling Limit $g \to \infty$:

$$
\begin{aligned}
\Omega^{(N)} &= g^{1/3}\left(\Omega_0^{(N)} + \Omega_1^{(N)} g^{-2/3} + \Omega_2^{(N)} g^{-4/3} + \dots\right) \\
E^{(N)} &= g^{1/3}\left(b_0^{(N)} + b_1^{(N)} g^{-2/3} + b_2^{(N)} g^{-4/5} + \dots\right)
\end{aligned}
$$

Exponential Convergence:



Goal:

$$
b_0 = \sum_{k=0}^{\infty} B_k \, D^{4/3-k}
$$

3.3 Hubbard-Stratonovich Transformation

Partition Function:

$$Z = \oint \mathcal{D}\mathbf{x}\, e^{-\mathcal{A}[\mathbf{x}]}$$

$$\mathcal{A}[\mathbf{x}] = \int_0^\beta d\tau \left\{ \frac{1}{2}\dot{\mathbf{x}}^2(\tau) + \frac{1}{2}\omega^2\mathbf{x}^2(\tau) + g\left[\mathbf{x}^2(\tau)\right]^2 \right\}$$

Transformation of Variables:

$$\oint \mathcal{D}\sigma \exp\left\{ -\int_0^\beta d\tau \left[\frac{1}{g}\sigma^2(\tau) + 2i\mathbf{x}^2(\tau)\sigma(\tau)\right] \right\}$$

$$= \exp\left\{ -g\int_0^\beta d\tau \left[\mathbf{x}^2(\tau)\right]^2 \right\}$$

New Partition Function:

$$Z = \oint \mathcal{D}\sigma\, e^{-D\mathcal{A}[\sigma]}$$

$$\mathcal{A}[\sigma] = \frac{1}{\tilde{g}}\int_0^\beta d\tau\, \sigma^2(\tau) + \frac{1}{2}\,\mathrm{Tr}\,\ln\left[-\frac{d^2}{d\tau^2} + \omega^2 + 4i\sigma(\tau)\right]$$

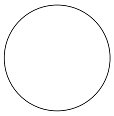
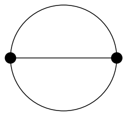
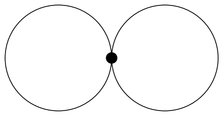
New Coupling Constant: $\tilde{g} = Dg$

Effective Potential: $V_{\text{eff}}(\sigma_0) = -\frac{1}{\beta}\ln Z$

Background Method: $1/\hbar \leftrightarrow D$

3.4 Effective Potential

Diagrammatic Result:

$V_{\text{eff}}(\sigma_0) =$ $\bullet \ +$  $+ \frac{1}{12}$  $+ \frac{1}{8}$  $+ \dots$

Analytic Result:

$$
\begin{aligned}
V_{\text{eff}}(\sigma_0) \quad &= \quad D\left\{\frac{\sigma_0^2}{\tilde{g}} + \frac{\Omega}{2}\right\} + \frac{\tilde{\Omega}}{2} - \Omega \\
&+\frac{1}{D}\left\{-\frac{5\tilde{g}^2}{4\Omega^5} + \frac{\tilde{g}^3(\tilde{\Omega}^3 + 4\tilde{\Omega}^2\Omega + 44\tilde{\Omega}\Omega^2 + 128\Omega^3)}{4\Omega^8\tilde{\Omega}(\tilde{\Omega}+2\Omega)^2}\right. \\
&-\frac{\tilde{g}^4(3\tilde{\Omega}^5 + 31\tilde{\Omega}^4\Omega + 150\tilde{\Omega}^3\Omega^2 + 392\tilde{\Omega}^2\Omega^3 + 656\tilde{\Omega}\Omega^4 + 480\Omega^5)}{\Omega^9\tilde{\Omega}^2(\tilde{\Omega}+\Omega)(\tilde{\Omega}+2\Omega)^3(\tilde{\Omega}+4\Omega)} \\
&+\frac{4\tilde{g}^5(3\tilde{\Omega}^5 + 42\tilde{\Omega}^4\Omega + 227\tilde{\Omega}^3\Omega^2 + 568\tilde{\Omega}^2\Omega^3 + 656\tilde{\Omega}\Omega^4 + 288\Omega^5)}{\Omega^{10}\tilde{\Omega}^2(\tilde{\Omega}+\Omega)^2(\tilde{\Omega}+2\Omega)^4(\tilde{\Omega}+4\Omega)} \\
&\left.-\frac{16\tilde{g}^6(\tilde{\Omega}^5 + 14\tilde{\Omega}^4\Omega + 73\tilde{\Omega}^3\Omega^2 + 160\tilde{\Omega}^2\Omega^3 + 136\tilde{\Omega}\Omega^4 + 32\Omega^5)}{\Omega^{11}\tilde{\Omega}^4(\tilde{\Omega}+\Omega)^2(\tilde{\Omega}+2\Omega)^4(\tilde{\Omega}+4\Omega)}\right\} \\
&+\mathcal{O}\left(\frac{1}{D^2}\right)
\end{aligned}
$$

Frequencies: $\Omega = \sqrt{\omega^2 + 4i\sigma_0}\,, \qquad \tilde{\Omega} = 2\sqrt{\Omega^2 + \frac{\tilde{g}}{\Omega}}$

3.5 Weak-Coupling Regime

Extrimization with Respect to σ_0:

$$\sigma_0 = -i\left(s_1\tilde{g} + s_2\tilde{g}^2 + s_3\tilde{g}^3 + \ldots\right)$$

Results for Coefficients:

$$\begin{aligned} s_1 &= \frac{1}{2\omega} \\ s_2 &= -\frac{1}{2\omega^4} - \frac{1}{D\omega^4} \\ s_3 &= \frac{5}{4\omega^7} + \frac{45}{8D\omega^7} + \frac{25}{4D^2\omega^7} \end{aligned}$$

Weak-Coupling Series Reproduced:

k	ϵ_k
1	$D(D+2)/4$
2	$-D(2D^2+9D+10)/8$
3	$D(8D^3+59D^2+146D+120)/16$
4	$-D(168D^4+1773D^3+7144D^2$ $+12960D+8840)/128$
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3.6 Strong-Coupling Regime

Extremization with Respect to σ_0:

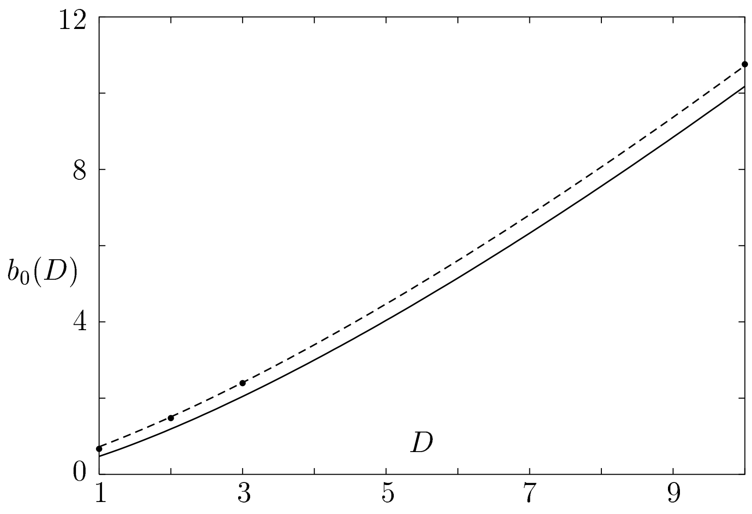
$$\sigma_0 = -i\tilde{g}^{2/3}\left(S_1 + S_2\tilde{g}^{-2/3} + \ldots\right)$$

Result for Leading Coefficient:

$$S_1 = \frac{1}{2^{4/3}} + \left(3^{1/2}2^{1/6} - 2^{5/3}\right)\frac{1}{6D} + \left(27\sqrt{6} - \frac{73}{2}\right)^{1/3}\frac{1}{18D^2} + \mathcal{O}\left(\frac{1}{D^3}\right)$$

Strong-Coupling Coefficient: $\quad b_0(D) = \sum_{k=0}^{\infty} B_k D^{4/3-k}$

$$\begin{aligned} B_0 &= \frac{3\cdot 2^{1/3}}{8} \approx 0.4724703937105774 \\ B_1 &= \frac{3^{1/2} - 2^{1/2}}{2^{1/6}} \approx 0.2831607943221791 \\ B_2 &= -\frac{239}{18\,2^{2/3}\left(25 + 12\sqrt{6}\right)} \approx -0.1537760559399285 \end{aligned}$$



4.7 Large-D Expansion from VPT

Optimization with Respect to Variational Parameter:

$$\frac{\partial E^{(N)}(\Omega_0)}{\partial \Omega_0} = 0$$

Ansatz: $\Omega_0^{(N)} = (Dg)^{1/3} \left(C_0^{(N)} + \frac{C_1^{(N)}}{D} + \frac{C_2^{(N)}}{D^2} + \ldots \right) + \ldots$

Result: $E^{(N)}(\Omega_0^{(N)}) = D^{4/3} g^{1/3} \left(B_0^{(N)} + \frac{B_1^{(N)}}{D} + \ldots \right) + \ldots$

$\Rightarrow B_0$ **Exact, Exponential Convergence for** $B_k^{(N)}$ **(**$k \geq 1$**):**