Retinotopic Projections: Self-Organization of Neural Maps in the Visual System

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March 4, 2005

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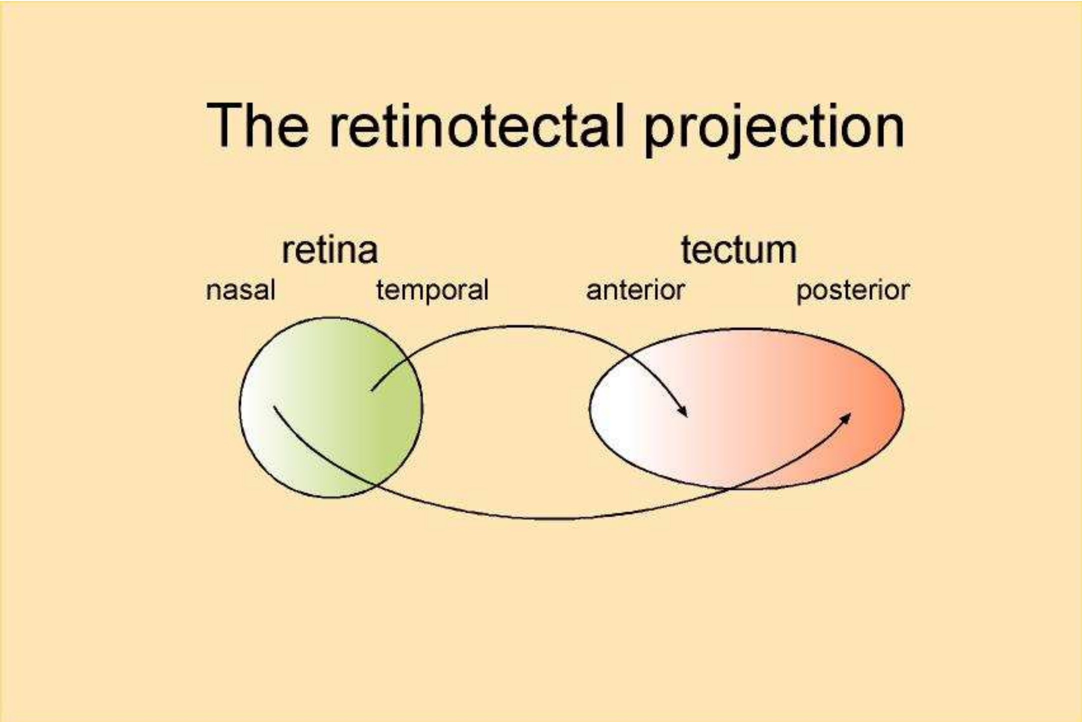
Neural Maps

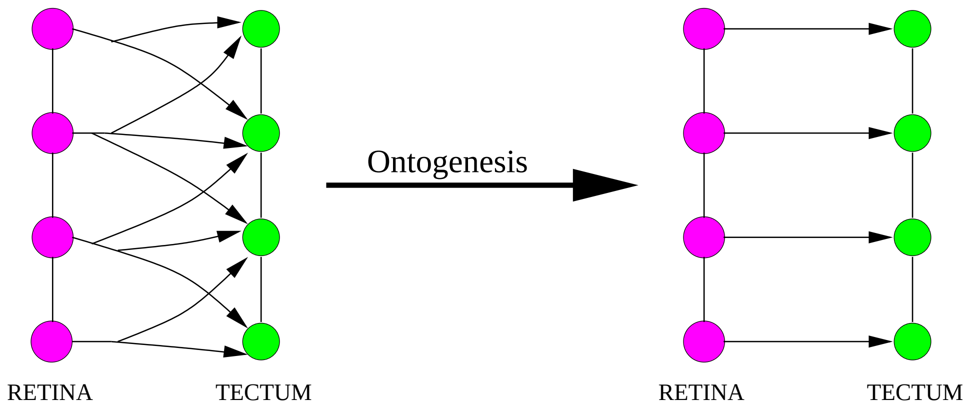
Conservation of neighbourhood relations:

Somatotopy Tonotopy Retinotopy

Visual system: retina $\longrightarrow$ tectum

retina $\longrightarrow$ thalamus $\longrightarrow$ visual cortex



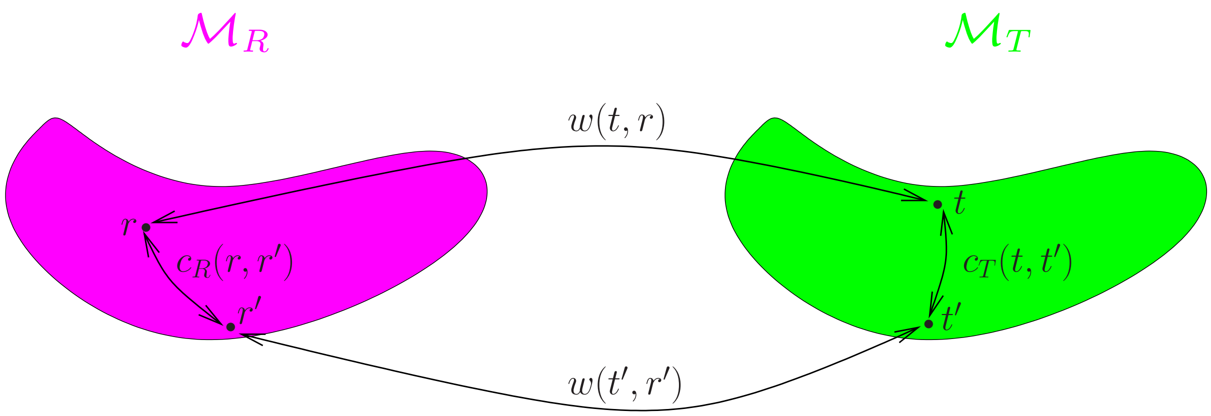


Initial distribution of synaptic contacts

Retinotopic projection in the adult animal

Model

Representation of retina and tectum by manifolds



Connection weights $w(t,r)$: $\quad \mathcal{M}_T \times \mathcal{M}_R \to \mathrm{R}^+$

Equations of Evolution: Generalization of Häussler–equations:
Häussler/v. d. Malsburg, J. Theoret. Neurobiol. **2**, 47 (1983)

$$\dot{w}(t,r) = f(t,r,w) - \frac{w(t,r)}{2M_T} \int_{t'} dt'\, f(t',r,w) - \frac{w(t,r)}{2M_R} \int_{r'} dr'\, f(t,r',w)$$

$$f(t,r,w) = \alpha + w(t,r) \int_{t'} dt' \int_{r'} dr'\, c_T(t,t') c_R(r,r') w(t',r')$$

Synergetic Analysis

$$\Delta_T \psi_{\lambda_T}^{m_T}(t) = \chi_{\lambda_T} \psi_{\lambda_T}^{m_T}(t)\,, \quad \Delta_R \psi_{\lambda_R}^{m_R}(r) = \chi_{\lambda_R} \psi_{\lambda_R}^{m_R}(r)$$

$$\Longrightarrow c_T(t,t') = \sum_{\lambda_T} \sum_{m_T} f_{\lambda_T}^{m_T} \psi_{\lambda_T}^{m_T}(t) \psi_{\lambda_T}^{m_T *}(t')$$

Stationary homogeneous solution: $w_0(t,r) = 1$

Deviation: $v(t,r) = w(t,r) - 1$

Linearization: $\hat{L} v(t,r) = \Lambda v(t,r)$

$\rightarrow v_{\lambda_T \lambda_R}^{m_T m_R}(t,r) = \psi_{\lambda_T}^{m_T}(t) \psi_{\lambda_R}^{m_R}(r)\,, \quad \Lambda_{\lambda_T \lambda_R}^{m_T m_R}(\alpha)$

Decomposition: $w(t,r) = 1 + U(t,r) + S(t,r)$

Unstable modes: $U(t,r) = U_{\lambda_T^{u'} \lambda_R^{u'}}^{m_T^{u'} m_R^{u'}} \psi_{\lambda_T^{u'}}^{m_T^{u'}}(t) \psi_{\lambda_R^{u'}}^{m_R^{u'}}(r)$

Slaving principle: $S(t,r) = h(U(t,r))$; h: center manifold

Order parameter equations:

$$\begin{aligned} \dot{U}_{\lambda_T^u \lambda_R^u}^{m_T^u m_R^u} &= \Lambda_{\lambda_T^u \lambda_R^u}^{m_T^u m_R^u} U_{\lambda_T^u \lambda_R^u}^{m_T^u m_R^u} + A_{\cdots}^{\cdots} U_{\lambda_T^{u'} \lambda_R^{u'}}^{m_T^{u'} m_R^{u'}} U_{\lambda_T^{u''} \lambda_R^{u''}}^{m_T^{u''} m_R^{u''}} \\ &\quad + B_{\cdots}^{\cdots} U_{\lambda_T^{u'} \lambda_R^{u'}}^{m_T^{u'} m_R^{u'}} U_{\lambda_T^{u''} \lambda_R^{u''}}^{m_T^{u''} m_R^{u''}} U_{\lambda_T^{u'''} \lambda_R^{u'''}}^{m_T^{u'''} m_R^{u'''}} \end{aligned}$$

H. Haken: Synergetics, 3rd ed., Springer 2004

1d-Manifolds: String

Eigenfunctions:

$$\psi_{\lambda_T}(t) = \frac{1}{\sqrt{L_T}} \exp\left(i\frac{2\pi}{L_T}\lambda_T\, t\right)$$

$$\psi_{\lambda_R}(r) = \frac{1}{\sqrt{L_R}} \exp\left(i\frac{2\pi}{L_R}\lambda_R\, r\right)$$

$$\lambda_T, \lambda_R = \ldots, -1, 0, +1, \ldots$$

$c_T(t-t')$, $c_R(r-r')$ monotonically decreasing:

$$\Lambda_{\max} = -\alpha + f_1^T f_1^R \;\rightarrow\; (\lambda_T^u, \lambda_R^u) = (\pm 1, \pm 1)$$

Order parameter equations:

$$\begin{aligned}\dot{U}_{\lambda_T^u \lambda_R^u} &= \Lambda_{\lambda_T^u \lambda_R^u} U_{\lambda_T^u \lambda_R^u} + A (U_{\lambda_T^u \lambda_R^u})^2 U_{-\lambda_T^u - \lambda_R^u} \\ &\quad + B U_{\lambda_T^u \lambda_R^u} U_{-\lambda_T^u \lambda_R^u} U_{\lambda_T^u - \lambda_R^u}\end{aligned}$$

Transformation to real variables:

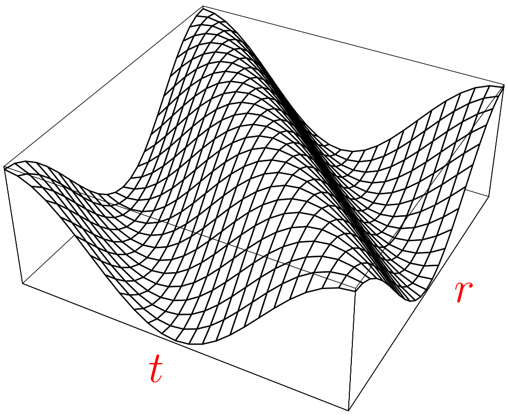
$$\begin{aligned} u_1 &= U_{11} + U_{-1-1}\,, & u_2 &= i(U_{11} - U_{-1-1}) \\ u_3 &= U_{1-1} + U_{-11}\,, & u_4 &= i(U_{1-1} - U_{-11}) \end{aligned}$$

Constant phase shift angles:

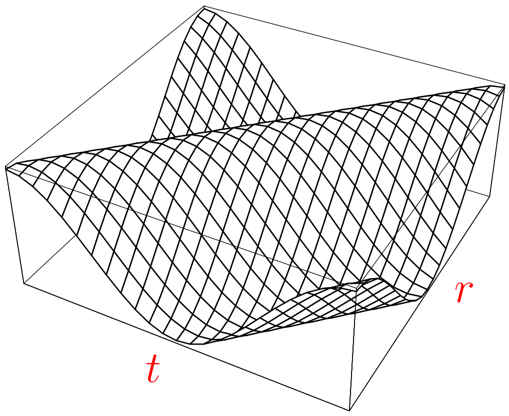
$$\frac{u_1}{u_2} = \text{const}\,, \quad \frac{u_3}{u_4} = \text{const}$$

$$U(t,r) = \xi \cos\left(\frac{2\pi}{L_T}t + \frac{2\pi}{L_R}r - \psi\right)$$
$$+\eta \cos\left(\frac{2\pi}{L_T}t - \frac{2\pi}{L_R}r - \varphi\right)$$

Retinotopic diagonal modes:

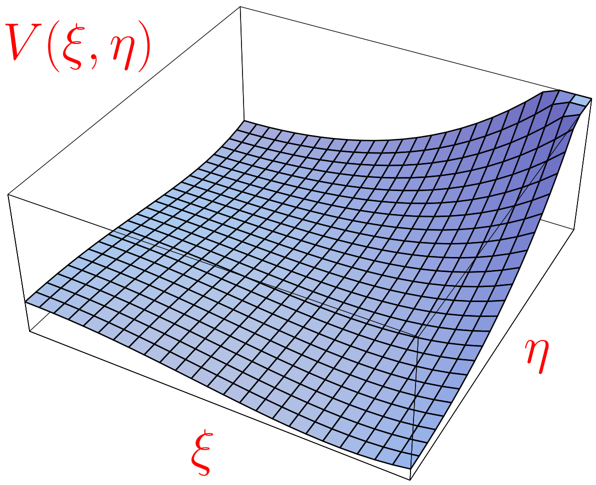


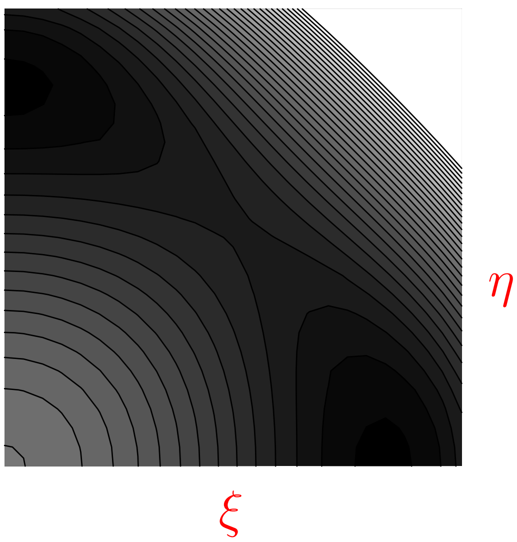
$(\lambda_T^u, \lambda_R^u) = (1, 1)$



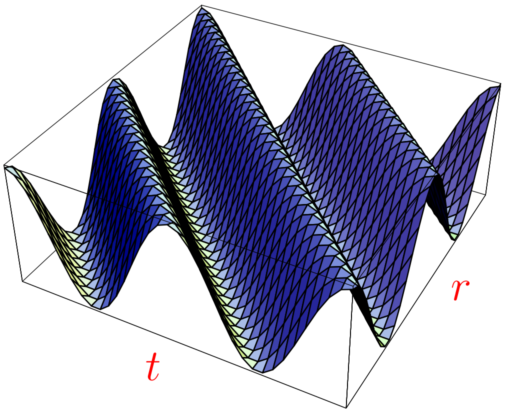
$(\lambda_T^u, \lambda_R^u) = (1, -1)$

Potential dynamics: $\dot{\xi} = -\dfrac{\partial V(\xi,\eta)}{\partial \xi}\,, \quad \dot{\eta} = -\dfrac{\partial V(\xi,\eta)}{\partial \eta}$

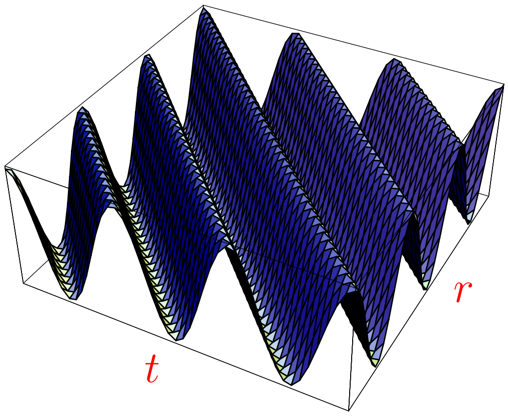




Sharpening of the projection by higher modes:



$(\lambda_T, \lambda_R) = (2, 2)$



$(\lambda_T, \lambda_R) = (3, 3)$

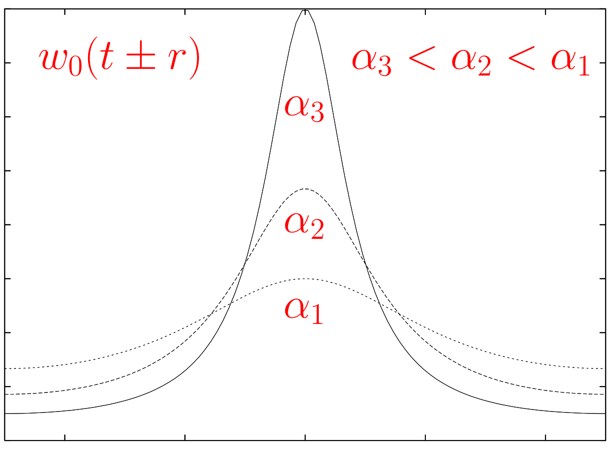
Stationary solutions:

$$w_0(t \pm r) = \sum_\lambda \varepsilon^{|\lambda|} \exp\left[i2\pi\lambda\left(\frac{t}{L_T} \pm \frac{r}{L_R}\right)\right]$$

$$\varepsilon = \frac{\gamma - \alpha}{\gamma}, \quad \gamma = f_1^T f_1^R$$

$$\alpha = \gamma \quad \rightarrow \quad w_0(t, r) = 1$$

$$\alpha = 0 \quad \rightarrow \quad w_0(t, r) = \delta\left(\frac{t}{L_T} \pm \frac{r}{L_R}\right)$$

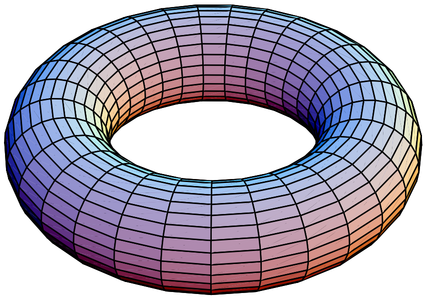


2-d Manifolds: Plane

Plane

$+$

periodic boundary conditions $\Longrightarrow$



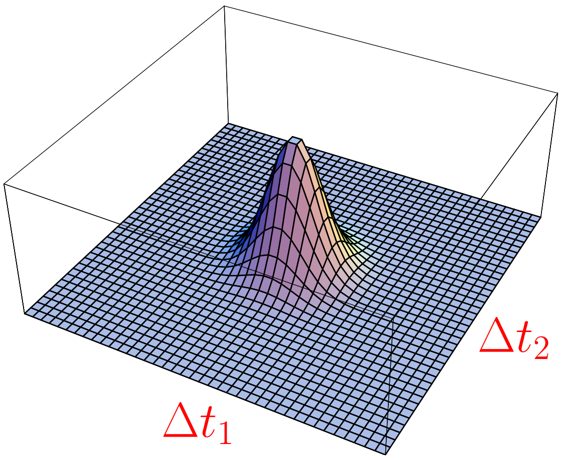
Eigenfunctions:

$$\psi_{\lambda_T}(t) = \frac{1}{\sqrt{L_{T1}L_{T2}}}\exp\left(i\frac{2\pi}{L_{T1}}\lambda_{T1}t_1 + i\frac{2\pi}{L_{T2}}\lambda_{T2}t_2\right)$$

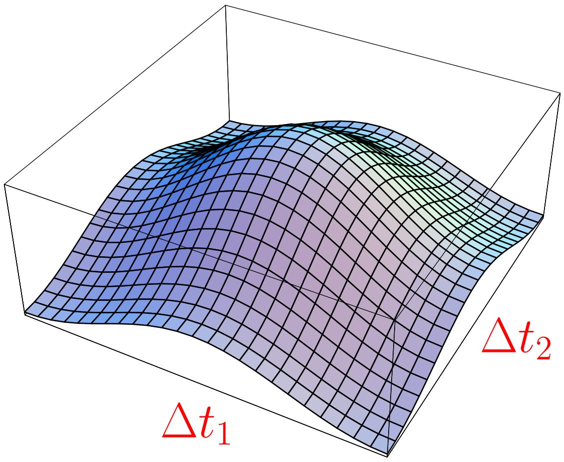
$$\psi_{\lambda_R}(r) = \frac{1}{\sqrt{L_{R1}L_{R2}}}\exp\left(i\frac{2\pi}{L_{R1}}\lambda_{R1}r_1 + i\frac{2\pi}{L_{R2}}\lambda_{R2}r_2\right)$$

$$\lambda_{T1}, \lambda_{T2}, \lambda_{R1}, \lambda_{R2} = \ldots, -1, 0, 1, \ldots$$

$$c_T(t-t') = c_{T1}(t_1 - t_1')c_{T2}(t_2 - t_2')$$



isotropic



anisotropic

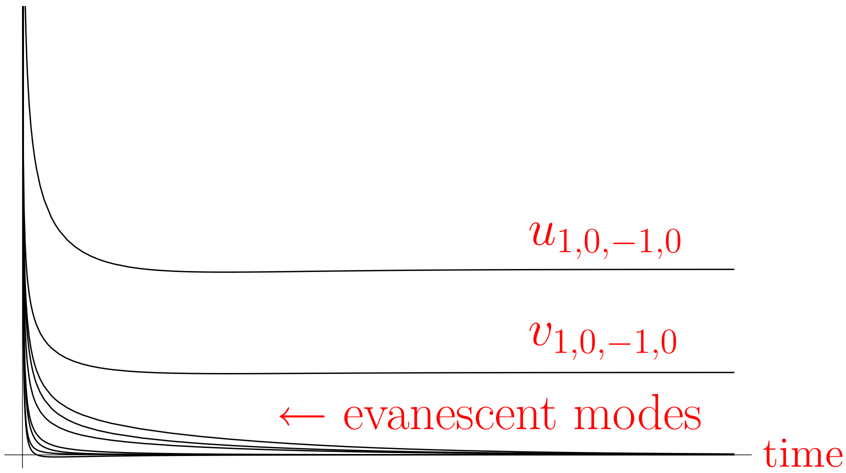
$\longrightarrow$ Unstable modes: $\lambda_T^u, \lambda_R^u = (\pm 1, 0), (0, \pm 1)$

Transformation to real variables:

$$\begin{aligned} u_{\lambda_T^u \lambda_R^u} &= U_{\lambda_T^u \lambda_R^u} + U_{-\lambda_T^u - \lambda_R^u} \\ v_{\lambda_T^u \lambda_R^u} &= i(U_{\lambda_T^u \lambda_R^u} - U_{-\lambda_T^u - \lambda_R^u}) \end{aligned}$$

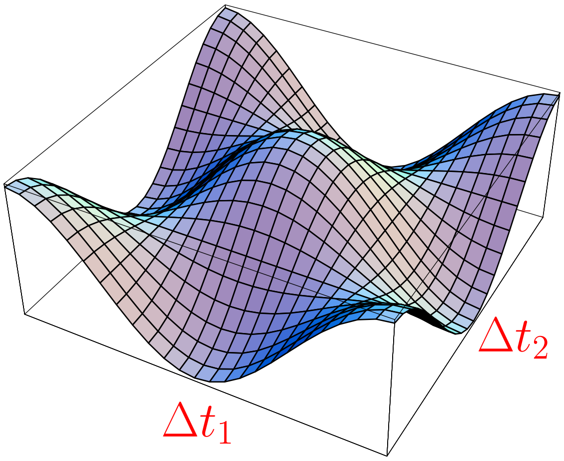
$$\begin{aligned} \Longrightarrow U(t, r) &= u_{1,0,-1,0} \cos\left[\frac{2\pi}{L}(t_1 - r_1)\right] \\ &+ v_{1,0,-1,0} \sin\left[\frac{2\pi}{L}(t_1 - r_1)\right] \\ &+ \ldots \end{aligned}$$

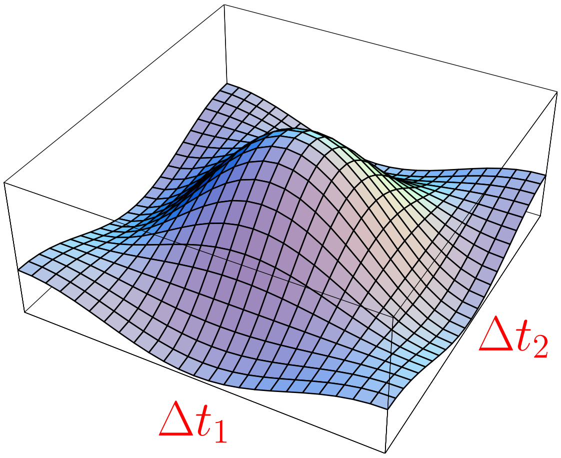
$$u_{1,0,-1,0}(0) > u_{\lambda_T^u \lambda_R^u}(0)\,, \quad (\lambda_T^u, \lambda_R^u) \neq (1, 0, -1, 0)$$



Retinotopic States

$\longrightarrow$ Unstable modes: $(\lambda_T^u, \lambda_R^u) = (\pm 1, \pm 1, \pm 1, \pm 1)$





$$\Longrightarrow U(t,r) = u_1 \cos\left[\frac{2\pi}{L}(t_1 - r_1)\right] \cos\left[\frac{2\pi}{L}(t_2 - r_2)\right] + u_2 \sin\left[\frac{2\pi}{L}(t_1 - r_1)\right] \sin\left[\frac{2\pi}{L}(t_2 - r_2)\right] + \ldots$$

Sharpening: