# 1/N-Expansion for Interacting Bose Gas

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# Model

Action in imaginary time

$$S=\int_0^\beta d\tau\int d^dr\left[\sum_{\alpha=1}^{N}b_\alpha^*\left(\partial_\tau-\mu-\frac{\nabla^2}{2m}\right)b_\alpha+\frac{g}{2}\left(\sum_{\alpha=1}^{N}|b_\alpha|^2\right)^2\right]$$

Partition function:

$$Z=\int\left[\prod_\alpha\mathcal{D}b_\alpha^*\mathcal{D}b_\alpha\right]e^{-S}$$

Large $N$ limit is taken by letting $N\to\infty$ with $Ng$ fixed

# Hubbard-Stratonovich transformation

$$S' = \int_0^\beta d\tau \int d^d r \left[ \sum_{\alpha=1}^{N} b_\alpha^* \left( \partial_\tau - \mu - \frac{\nabla^2}{2m} + i\lambda \right) b_\alpha + \frac{1}{2g}\lambda^2 \right]$$

Integrating out $N-1$ fields:

$$S_{\rm eff} = (N-1)\mathrm{Tr}\ln\left( \partial_\tau - \mu - \frac{\nabla^2}{2m} + i\lambda \right)$$
$$+ \int_0^\beta d\tau \int d^d r \left[ b^* \left( \partial_\tau - \mu - \frac{\nabla^2}{2m} + i\lambda \right) b + \frac{1}{2g}\lambda^2 \right]$$

$\Longrightarrow$ The leading order corresponds to the Hartree approximation (saddle-point level)

# Gaussian fluctuations around the saddle-point approximation: Bogoliubov theory and beyond

$$b = b_0 + \tilde{b}, \qquad i\lambda = \lambda_0 + i\tilde{\lambda}$$

$$\begin{aligned} S_{\rm eff} &= S_{\rm eff}^{\rm SPA} + \frac{1}{2}{\rm Tr}\ln\left[\delta(\tau-\tau')\delta^d(\mathbf{r}-\mathbf{r}') - Ng\; G_0(\tau-\tau',\mathbf{r}-\mathbf{r}')G_0(\tau'-\tau,\mathbf{r}'-\mathbf{r})\right] \\ &+ \frac{1}{2}\int_0^\beta d\tau\int d^d r\int_0^\beta d\tau'\int d^d r'\;\Psi^\dagger(\tau,\mathbf{r})\mathbf{M}(\tau-\tau',\mathbf{r}-\mathbf{r}')\Psi(\tau,\mathbf{r}') \end{aligned}$$

$$\hat{G}_0(i\omega_n,\mathbf{p}) = \frac{1}{i\omega_n + \mu - \lambda_0 - \frac{\mathbf{p}^2}{2m}}$$

$$\Psi^\dagger(\tau,\mathbf{r}) = \begin{bmatrix} \tilde{b}^*(\tau,\mathbf{r}) & \tilde{b}(\tau,\mathbf{r}) \end{bmatrix}, \qquad \Psi(\tau,\mathbf{r}) = \begin{bmatrix} \tilde{b}(\tau,\mathbf{r}) \\ \tilde{b}^*(\tau,\mathbf{r}) \end{bmatrix},$$

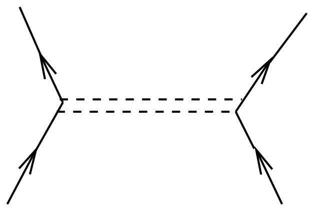
$$\hat{\mathbf{M}}(i\omega_n,\mathbf{p}) = \begin{bmatrix} -i\omega_n - \mu + \lambda_0 + \frac{\mathbf{p}^2}{2m} + |b_0|^2\hat{\Gamma}(i\omega_n,\mathbf{p}) & b_0^2\,\hat{\Gamma}(i\omega_n,\mathbf{p}) \\ (b_0^*)^2\hat{\Gamma}(i\omega_n,\mathbf{p}) & i\omega_n - \mu + \lambda_0 + \frac{\mathbf{p}^2}{2m} + |b_0|^2\hat{\Gamma}(i\omega_n,\mathbf{p}) \end{bmatrix},$$

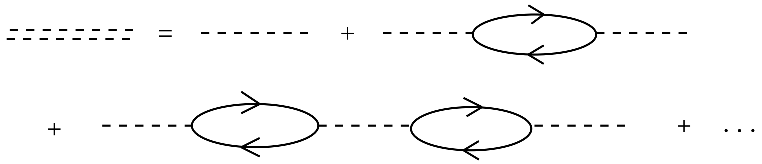
$$\hat{\Gamma}(i\omega_n,\mathbf{p}) = \frac{g}{1 - Ng\hat{\Pi}(i\omega_n,\mathbf{p})}$$

Polarization Bubble:

$$\hat{\Pi}(i\omega_n, \mathbf{p}) = \frac{1}{\beta} \sum_{m=-\infty}^{\infty} \int \frac{d^d q}{(2\pi)^d} \hat{G}_0(i\omega_n + i\omega_m, \mathbf{p} + \mathbf{q}) \hat{G}_0(i\omega_m, \mathbf{q})$$

Graphical representation of the vertex function:





# Propagators

$$\hat{\mathbf{G}}(i\omega_n, \mathbf{p}) = \begin{bmatrix} \hat{\mathbf{G}}_{\tilde{b}*\tilde{b}}(i\omega_n, \mathbf{p}) & \hat{\mathbf{F}}_{\tilde{b}*\tilde{b}*}(i\omega_n, \mathbf{p}) \\ \hat{\mathbf{F}}_{\tilde{b}\tilde{b}}(i\omega_n, \mathbf{p}) & \hat{\mathbf{G}}_{\tilde{b}*\tilde{b}}(-i\omega_n, \mathbf{p}) \end{bmatrix}$$

$$\hat{\mathbf{G}}_{\tilde{b}*\tilde{b}}(i\omega_n, \mathbf{p}) = \frac{i\omega_n + \lambda_0 - \mu + \frac{\mathbf{p}^2}{2m} + |b_0|^2\hat{\Gamma}(i\omega_n, \mathbf{p})}{\omega_n^2 + \left[\frac{\mathbf{p}^2}{2m} + \lambda_0 - \mu + |b_0|^2\hat{\Gamma}(i\omega_n, \mathbf{p})\right]^2 - |b_0|^4\hat{\Gamma}^2(i\omega_n, \mathbf{p})}$$

$$\hat{\mathbf{F}}_{\tilde{b}*\tilde{b}*}(i\omega_n, \mathbf{p}) = -\frac{(b_0^*)^2\hat{\Gamma}(i\omega_n, \mathbf{p})}{\omega_n^2 + \left[\frac{\mathbf{p}^2}{2m} + \lambda_0 - \mu + |b_0|^2\hat{\Gamma}(i\omega_n, \mathbf{p})\right]^2 - |b_0|^4\hat{\Gamma}^2(i\omega_n, \mathbf{p})}$$

Generalized Bogoliubov spectrum:

$$E(\mathbf{p}) = \sqrt{\frac{\mathbf{p}^4}{4m^2} + \frac{nN}{m}\mathbf{p}^2\, \hat{\Gamma}[E(\mathbf{p}), \mathbf{p}]}$$

Polarization bubble in the high-temperature approximation:

$$\hat{\Pi}(i\omega_n, \mathbf{p}) = -A_d m^2 T a^{d-2}[(pa)^{2-d} + d - 3] \left[\left(p^2 - 2mi\omega_n\right)^{d-3} + e^{-i\pi(d-2)}\left(p^2 + 2mi\omega_n\right)^{d-3}\right]$$

$$A_d = 2^{2-d}\pi^{-d/2}e^{i\pi(d-2)/2}\Gamma(d/2-1)\Gamma(3-d)$$

$$d = 3 \Longrightarrow \hat{\Pi}(i\omega_n, \mathbf{p}) = -i\frac{Tm^2}{2\pi p}\ln\left(\frac{2mi\omega_n + p^2}{2mi\omega_n - p^2}\right)$$

$$d = 2 \Longrightarrow \hat{\Pi}(i\omega_n, \mathbf{p}) = -\frac{T}{\pi}\frac{p^2\ln(pa)}{(i\omega_n)^2 - \frac{p^4}{4m^2}}$$

*Some care is needed at* $\mathbf{d} = 2$*, since the condensate is zero for any finite temperature*

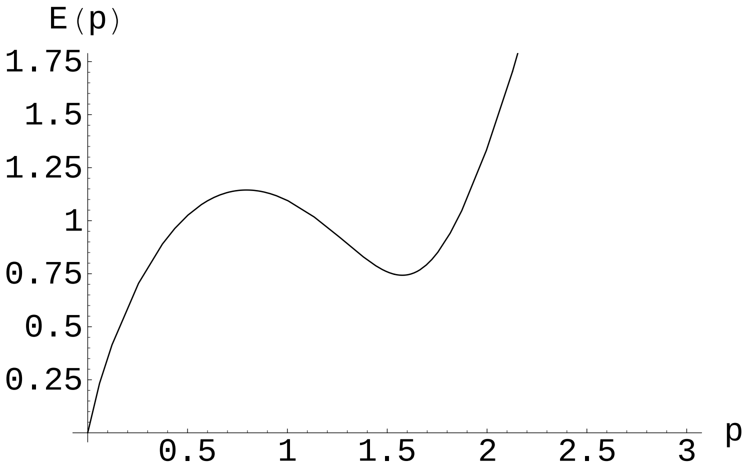
Bare coupling constant:

$$g = \frac{2\pi/m}{\ln\left(\frac{2e^{-\gamma}}{Nna^2}\right)} \quad (d = 2), \qquad g = \frac{4\pi^{d/2}a^{d-2}}{\Gamma(d/2 - 1)m} \quad (d > 2)$$

# Roton-like behavior at $d = 2$

Excitation spectrum at $d = 2$:

$$E(p) = \sqrt{\frac{p^4}{4m^2} + \frac{Ngnp^2}{m}\left[1 - \frac{Tm}{\pi n}\ln(pa)\right]}$$




The coefficient of the $p^2$-term changes sign at

$$p_0 = a^{-1}e^{\pi n/Tm}$$

# Low momentum behavior at finite temperature

$\underline{2 < d < 4:}$

$$E(p) \approx \sqrt{\frac{n}{\alpha_d m^3 T}} p^{(6-d)/2} \qquad \text{where} \qquad \alpha_d = \frac{\Gamma(2-d/2)\Gamma^2(d/2-1)}{2^{d-2}\pi^{d/2}}$$

$\underline{d = 2:}$

$$E(p) \approx \sqrt{\frac{Ngn}{m}} a^{-Tm/2\pi n} p^{1-Tm/2\pi n}$$

The continuously varying exponent in the $d = 2$ case is reminiscent of the Kosterlitz-Thouless (KT) physics. However, true KT behavior would emerge only for $N = 1$.

# Conclusions

- The $1/N$-expansion through functional integrals provides a simple, physically appealing, and controlled way for studying non-relativistic Bose systems.
- The analysis was made for any dimensions $2 \leq d < 4$.
- While the $d = 3$ result is known, being essentially equivalent to RPA, the $d = 2$ results are new.
- At $d = 2$ rotons are thermally induced.