Interacting Anyons in a One-Dimensional Optical Lattice

Martin Bonkhoff

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TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

1. Identical particles
2. Anyon-Hubbard model
3. Experimental proposals
4. Gutzwiller mean-field approach
5. Mean-field decoupling

1.1 Identical Particles

- **Permutation:** $\hat{P}|\,\Psi(x_1,x_2)\,\rangle = \exp(i\theta)|\,\Psi(x_2,x_1)\,\rangle$

- **Dimensionality:**

$$D \geq 3 \quad \longrightarrow \theta = 0 \quad \text{bosons}, \quad \theta = \pi \quad \text{fermions}$$

$$D = 2 \quad \longrightarrow \text{bosons} \quad 0 \leq \theta \leq \pi \quad \text{fermions}$$

$$D = 1 \quad \longrightarrow \text{intermediate statistics ?}$$

- **Implementation:** $\left[\hat{a}_i, \hat{a}_j^\dagger\right]_{q_{i,j}^{\mp}} = \delta_{i,j}\hat{I} = \hat{a}_i\hat{a}_j^\dagger - \exp\left[\pm i\theta \mathrm{sgn}(i-j)\right]\hat{a}_j^\dagger\hat{a}_i$

1.2 Particle Exchange

- **Implementation:** $$\left[\hat{a}_i, \hat{a}_j^\dagger\right]_{q_{i,j}^{\mp}} = \delta_{i,j}\hat{I} = \hat{a}_i\hat{a}_j^\dagger - \exp\left[\pm i\theta \mathrm{sgn}(i-j)\right]\hat{a}_j^\dagger\hat{a}_i$$

- **Consequences:**

First possibility:

$$|\,\Psi\,\rangle = \hat{a}_j^\dagger\hat{a}_i^\dagger|\,0\,\rangle \xrightarrow{\text{exchange}} |\,\Psi\,\rangle = \hat{a}_k^\dagger\hat{a}_i\hat{a}_j^\dagger\hat{a}_i^\dagger|\,0\,\rangle = \exp\left(i\theta\right)\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{a}_i\hat{a}_i^\dagger|\,0\,\rangle =$$

$$\exp\left(i\theta\right)\hat{a}_k^\dagger\hat{a}_j^\dagger|\,0\,\rangle \xrightarrow{\text{translate}} \exp\left(i\theta\right)\hat{a}_j^\dagger\hat{a}_i^\dagger|\,0\,\rangle = \exp\left(i\theta\right)|\,\Psi\,\rangle$$

Second possibility:

$$|\,\Psi\,\rangle = \hat{a}_j^\dagger\hat{a}_i^\dagger|\,0\,\rangle \xrightarrow{\text{exchange}} |\,\Psi\,\rangle = \hat{a}_k^\dagger\hat{a}_j\hat{a}_j^\dagger\hat{a}_i^\dagger|\,0\,\rangle = \hat{a}_k^\dagger\hat{a}_i^\dagger\hat{a}_j\hat{a}_j^\dagger|\,0\,\rangle =$$

$$\exp\left(-i\theta\right)\hat{a}_i^\dagger\hat{a}_k^\dagger|\,0\,\rangle \xrightarrow{\text{translate}} \exp\left(-i\theta\right)\hat{a}_j^\dagger\hat{a}_i^\dagger|\,0\,\rangle = \exp\left(-i\theta\right)|\,\Psi\,\rangle$$

2. Anyon-Hubbard Model in 1D

- **Hamiltonian:**

$$\hat{H}^a = -J \sum_{j=1}^{L} (\hat{a}_j^\dagger \hat{a}_{j+1} + \text{h.c.}) + \frac{U}{2} \sum_{j=1}^{L} \hat{n}_j (\hat{n}_j - 1)$$

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right]_{q_{i,j}^{\mp}} = \delta_{i,j} \hat{I}, \quad \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right]_{q_{i,j}^{\mp}} = 0, \quad [\hat{a}_i, \hat{a}_j]_{q_{i,j}^{\pm}} = 0$$

Deformed commutators:

$$[A, B]_q = AB - qBA$$

Properties:

$$q_{i,j} = \exp\left[\pm i\theta \text{sgn}(i-j)\right]$$

$$q_{i,j} = q_{j,i}^{-1} = q_{j,i}^{*}, \quad q_{i,i} = 1 \quad \forall \quad i \in [1, L]$$

2.2 Jordan-Wigner Transformation

- **Define Jordan-Wigner anyons via:**

$$\hat{a}_{-,i} = \hat{d}_i \hat{K}_{-,i}$$

$$\hat{a}_{+,i} = \hat{d}_i \hat{K}_{+,i}$$

- **Disorder operators:**

$$\hat{K}_{-,i} = \exp\left(i\theta \sum_{k<i} \hat{n}_k \right)$$

$$\hat{K}_{+,i} = \exp\left(i\theta \sum_{l>i} \hat{n}_l \right)$$

Properties:

$$\hat{K}^{\dagger}_{j,i} = \hat{K}^{-1}_{j,i} \quad \forall \quad j = -,+ \qquad \left[\hat{K}_{j,i}, \hat{K}_{n,m}\right] = 0$$

$$\forall \quad j,n = -,+ \quad \text{and} \quad \forall \quad i,m \in [1,L]$$

2.3 Exclusion Principle

$$\left[\hat{d}_i, \hat{d}_j^\dagger\right]_k = \delta_{i,j}\hat{I}$$

$$\left[\hat{d}_i, \hat{d}_j\right]_k = 0 = \left[\hat{d}_i^\dagger, \hat{d}_j^\dagger\right]_k$$

$$\hat{n}_i = \hat{d}_i^\dagger \hat{d}_i = \hat{a}_{\pm,i}^\dagger \hat{a}_{\pm,i}$$

	Boson	Fermion
k	1	-1
Exclusion behavior	$\left(\hat{d}_i^\dagger\right)^f \neq 0, \quad f \in \mathbb{N}$	$\left(\hat{d}_i^\dagger\right)^f = 0, \quad f \geq 2$
Anyonic exclusion	$\left(\hat{a}_{\pm,i}^\dagger\right)^f \neq 0, \quad f \in \mathbb{N}$	$\left(\hat{a}_{\pm,i}^\dagger\right)^f = 0, \quad f \geq 2$

$\longrightarrow$ Exclusion principle dictated by underlying local $\mathrm{CCR}, \mathrm{CAR}$

$\longrightarrow$ Exchange statistics is transmuted

2.4 Algebraic Relations

Phase shift property:

$$\hat{K}^{\dagger}_{j,i}\hat{d}_q\hat{K}_{j,i} = \hat{d}_q \exp(i\theta)$$

$$\forall j = -,+ \quad \forall q \in \ k,l$$

Anti-symmetry:

$$[A_i, B_j]_{q^{\pm}_{i,j}} = -q^{\pm}_{i,j}\,[B_j, A_i]_{q^{\mp}_{j,i}}$$

Algebraic structure:

$$\hat{a}_{l,i}\hat{a}_{l,j} - kq^{\pm}_{i,j}\hat{a}_{l,j}\hat{a}_{l,i} = 0,$$

$$\hat{a}^{\dagger}_{l,i}\hat{a}^{\dagger}_{l,j} - kq^{\mp}_{i,j}\hat{a}^{\dagger}_{l,j}\hat{a}^{\dagger}_{l,i} = 0,$$

$$\hat{a}_{l,i}\hat{a}^{\dagger}_{l,j} - kq^{\mp}_{j,i}\hat{a}^{\dagger}_{l,j}\hat{a}_{l,i} = \delta_{ij}\hat{I}$$

This talk:

$$k = 1 \quad \text{and} \quad \hat{K}_{-,i}$$

2.5 Bosonic Representation

Conditional hopping BH model:

$$\hat{H}_{-}^{\mathrm{OBC}} = \frac{U}{2}\sum_{i=1}^{L}\hat{n}_i\left(\hat{n}_i - 1\right) - J\sum_{i=1}^{L-1}\left[\hat{b}_i^{\dagger}\hat{b}_{i+1}\exp(i\theta\hat{n}_i) + \mathrm{h.c.}\right]$$

$$\hat{H}_{-}^{\mathrm{PBC}} = \hat{H}_{-}^{\mathrm{OBC}} - J\left[\exp(i\theta(\hat{N}-1))\hat{b}_L^{\dagger}\hat{b}_1\exp(i\theta\hat{n}_L) + \mathrm{h.c.}\right]$$

$\longrightarrow$ Density dependent Peierls phase

$\longrightarrow$ Even-odd effect as a signum of fermionic behavior $\theta = \pi$

$\longrightarrow$ Bose-Hubbard model for $\theta = 0$

2.6 Symmetries

Particle number conservation: $\left[\hat{H}, \hat{N}\right]_- = 0 \quad \longrightarrow \quad SU(L)$

Translational (PBC): $\left[\hat{H}^{\mathrm{PBC}}, \hat{U}\right]_- \neq 0 \quad \longrightarrow$ if $\theta \neq 0, \dfrac{2\pi}{N-1}, \pi$

Parity: $\left[\hat{H}, \hat{P}\right]_- \neq 0 \quad \longrightarrow$ if $\theta \neq 0, \pi$

Time inversion: $\left[\hat{H}, \hat{T}\right]_- \neq 0 \quad \longrightarrow$ if $\theta \neq 0, \pi$

Local gauge: $\left[\hat{H}, \hat{S}\right]_- \neq 0, \quad \hat{S}^\dagger \hat{b}_i \hat{S} = (-1)^i \hat{b}_i,$ if OBC or PBC L=even

$\Longrightarrow$ JW-anyons break discrete symmetries as well as translational invariance except for some parameter values

2.7 Combined Discrete Symmetries

$\longrightarrow$ If the discrete symmetries become chiral w.r.t. $\hat{H}_{\rm hop}$

i.e. $\hat{A}^\dagger \hat{H}_{\rm hop} \hat{A} = -\hat{H}_{\rm hop}$, for $\theta n_i = \dfrac{\pi}{2}$

$$\Longrightarrow \quad \left[\hat{H}, \hat{P}\hat{T}\right]_- = \left[\hat{H}, \hat{S}\hat{T}\right]_- = \left[\hat{H}, \hat{P}\hat{S}\right]_- = 0$$

$\Longrightarrow$ statistically induced phase transitions

This talk:

PBC are considered + neglection of boundary term $\mathcal{O}\left(\dfrac{1}{L}\right)$

3. Experimental Proposals

- **D = 1 : Possible realizations of anyons:**

$\longrightarrow$ Photon-assisted tunneling

T. Keilmann, S. Lanzmich, I. McCulloch, and M. Roncaglia, NC **2** 361, 2011

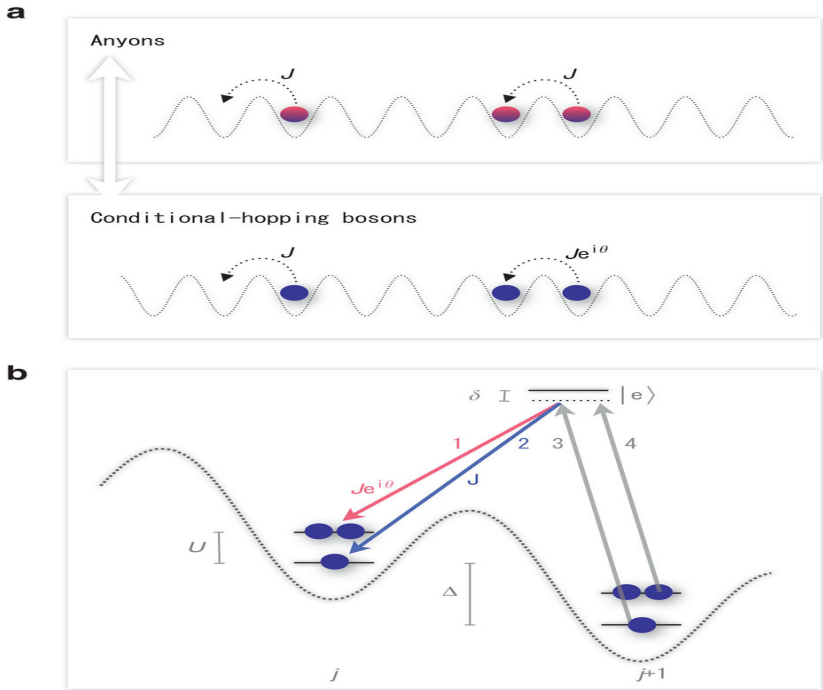
$\longrightarrow$ Raman scheme

S. Greschner and L. Santos, PRL **115** 053002, 2015

$\longrightarrow$ Lattice-shaking-induced resonant tunneling

C. Sträter, S. C. L. Srivastava, and A. Eckardt, PRL **117** ,205303, 2016

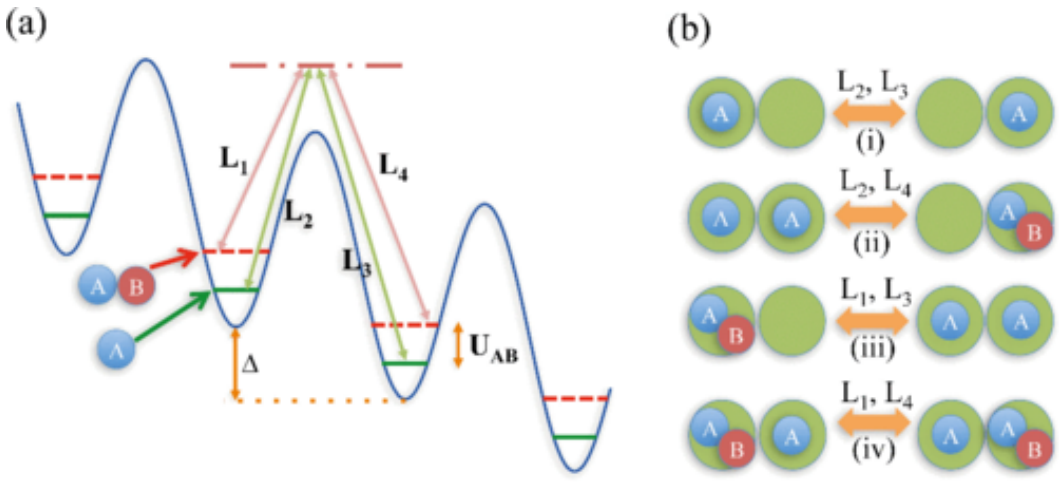
3.1 Photon-assisted tunneling



T. Keilmann, S. Lanzmich, I. McCulloch, and M. Roncaglia, NC **2** 361, 2011

- Parity violation induced by off-set Δ between adjacent sites
- Raman-coupling of excited states induces Peierls phase
- Restriction to two particles per site via Feshbach resonance

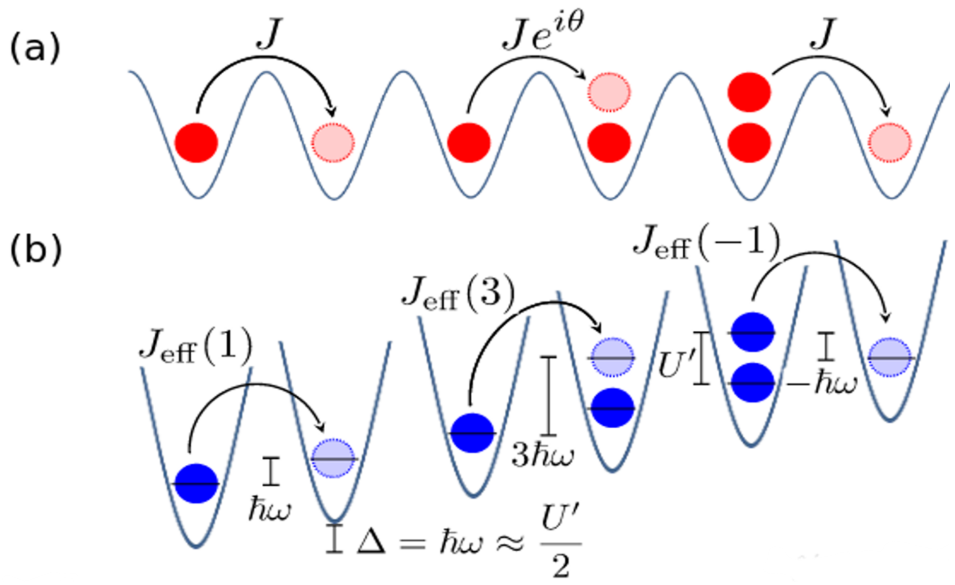
3.2 Raman scheme



S. Greschner and L. Santos, PRL **115** 053002, 2015

- Again assisted Raman-coupling to induce Peierls phase
- Effective, controllable on-site interaction via detuning
- Possible attractive interaction leads to restriction of particle number

3.3 Lattice-shaking-induced resonant tunneling



C. Sträter, S.C.L. Srivastava, and A. Eckardt, PRL **117**(2016)

- Conditional hopping via lattice tilting and periodic driving
- Effective Floquet Hamiltonian in high frequency approximation
- Adjusting statistical parameter by appropriate form of driving force

4. Gutzwiller Mean-Field Approximation

- **Product ansatz:** $$|G\rangle = \prod_j \left(\sum_{n=0}^{n_{\max}} f_n^{(j)} |n\rangle \right)$$

- **Normalization:** $$\sum_{n=0}^{n_{\max}} |f_n^{(j)}|^2 = 1$$

- **Energy:** $$E\left(\left\{ f_n^{(j)} \right\}\right) = \langle G|\hat{H}^b|G\rangle$$

- **Classical GW approach:** $$f_n^{(j)} = f_n$$

- **Modified GW approach 1:** $$f_n^{(j)} = |A_n| \exp(i\gamma_n^j)$$

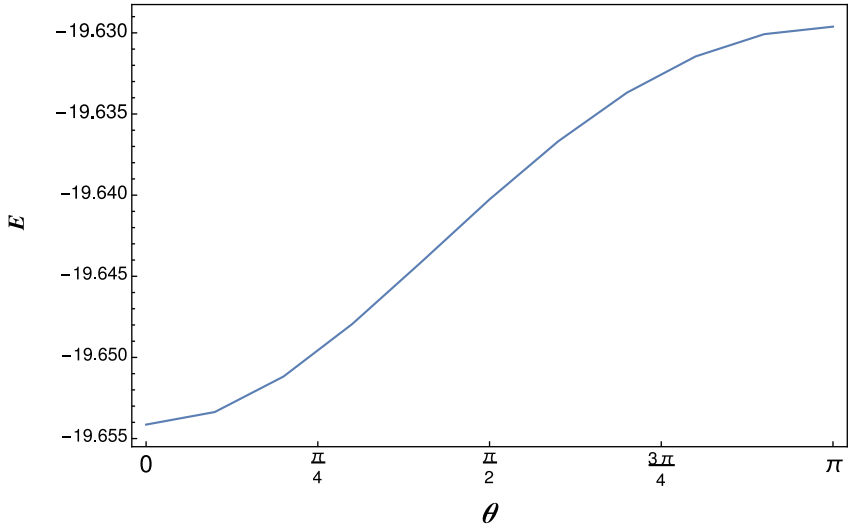
  G. Tang, S. Eggert, and A. Pelster, NJP **17**, 123016 (2015).

- **Modified GW approach 2:** $$f_n^{(j)} = |A_n| \exp(ij\beta_n + i\gamma_n)$$

4.1 Inconsistency of Modified GW Approach 1

- **Definitions:** $\beta_n^j = \gamma_n^j - \gamma_{n+1}^j, \quad \beta_n^j - \beta_n^{j+1} = \Delta\beta_n, \quad \beta_n^j + \beta_n^{j+1} = 2\beta_n$
  $\Longrightarrow$ Formally incorrect

- **Ground-state energy turns out to be independent of θ:**
  $\Longrightarrow$ Contradicts Kevin's DMRG data
  $n_0 = 0.1, \quad L = 100 \quad \frac{U}{J} = 10,$



- **Certain features of quasi-momentum distributions contradict to DMRG results:**
  $\Longrightarrow$ Monotonically decreasing peak with increasing θ
  $\Longrightarrow$ Condensation at $k \approx 0$ for small densities

G. Tang, S. Eggert, and A. Pelster, NJP **17**, 123016 (2015)

4.2 Quasi-Momentum Distributions

- **Bosonic version:**

$$\langle \hat{n}_k^{(\mathrm{b})} \rangle = \frac{1}{L} \sum_{ij} e^{\mathrm{i}k(x_i - x_j)} \langle \hat{b}_i^\dagger \hat{b}_j \rangle \,,$$

$$\langle \hat{b}_i^\dagger \hat{b}_j \rangle = \delta_{ij} \langle \hat{n}_i \rangle + (1 - \delta_{ij}) \langle \hat{b}_i^\dagger \rangle \langle \hat{b}_j \rangle$$

- **Anyonic version:**

$$\langle \hat{n}_k^{(\mathrm{a})} \rangle = \frac{1}{L} \sum_{ij} e^{\mathrm{i}k(x_i - x_j)} \langle \hat{a}_i^\dagger \hat{a}_j \rangle \,,$$

$$\langle \hat{a}_i^\dagger \hat{a}_j \rangle \xrightarrow{i<j} \langle \hat{b}_i^\dagger e^{\mathrm{i}\theta \hat{n}_i} \rangle \Big(\prod_{i<l<j} \langle e^{\mathrm{i}\theta \hat{n}_l} \rangle \Big) \langle \hat{b}_j \rangle \,,$$

$$\langle \hat{a}_i^\dagger \hat{a}_j \rangle \xrightarrow{i=j} \langle \hat{b}_i^\dagger \hat{b}_i \rangle \,,$$

$$\langle \hat{a}_i^\dagger \hat{a}_j \rangle \xrightarrow{i>j} \langle e^{-\mathrm{i}\theta \hat{n}_j} \hat{b}_j \rangle \Big(\prod_{j<l<i} \langle e^{-\mathrm{i}\theta \hat{n}_l} \rangle \Big) \langle \hat{b}_i^\dagger \rangle$$

- **Goal:** Dependence on anyon statistical parameter θ
- **This talk:** Soft-core particles $n_{\max} = 2$

4.2 Bosonic Quasi-Momentum Distributions

- **Finite system:**

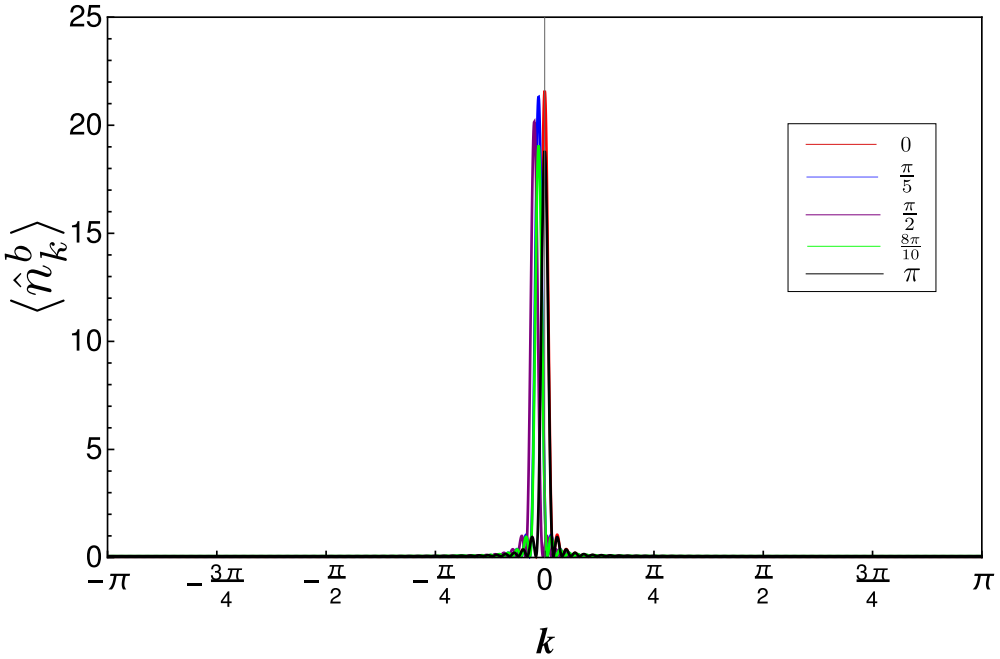
$$\langle \hat{n}_k^b \rangle = n_0 - |A_1|^2 \left[|A_0|^2 + 2\sqrt{2}|A_0||A_2|\cos\left(\frac{\theta}{2}\right) + 2|A_2|^2 \right]$$
$$+\frac{1}{L}|A_1|^2 \left[|A_0|^2 + 2\sqrt{2}|A_0||A_2|\cos\left(\frac{\theta}{2}\right) + 2|A_2|^2 \right] \frac{1-\cos\left[(k+\Delta\beta_0)L\right]}{1-\cos\left[k+\Delta\beta_0\right]}$$

- **Thermodynamic limit:**

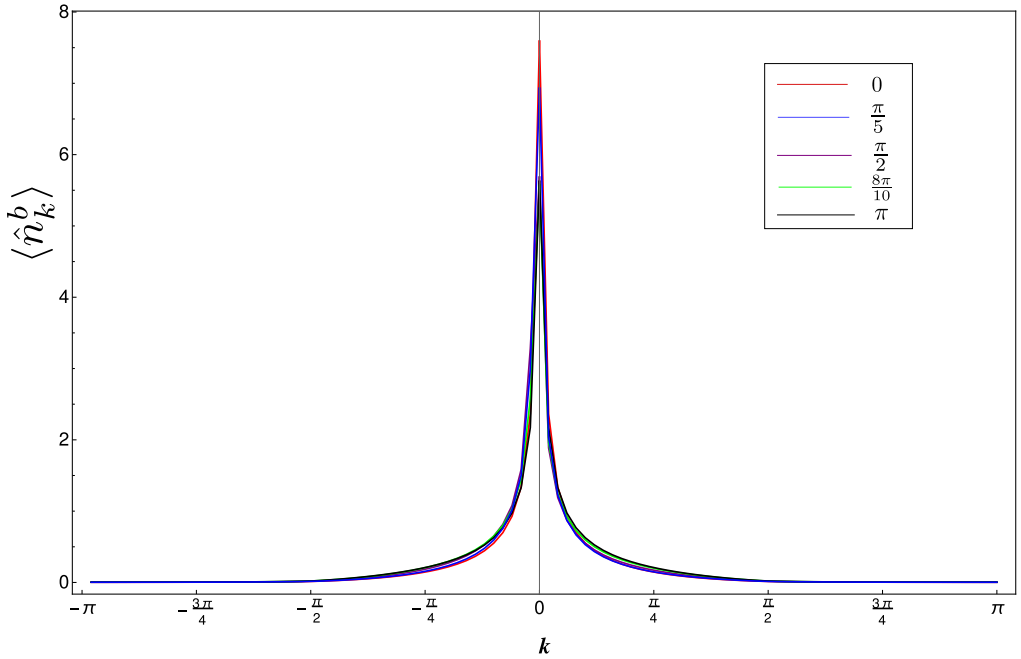
$$\langle \hat{n}_k^b \rangle_{L\to\infty} = n_0 - |A_1|^2 \left[|A_0|^2 + 2\sqrt{2}|A_0||A_2|\cos\left(\frac{\theta}{2}\right) + 2|A_2|^2 \right]$$
$$+|A_1|^2 \left[|A_0|^2 + 2\sqrt{2}|A_0||A_2|\cos\left(\frac{\theta}{2}\right) + 2|A_2|^2 \right] \delta(k+\Delta\beta_0)$$

4.2 Bosonic Quasi-Momentum Distributions

Gutzwiller MF



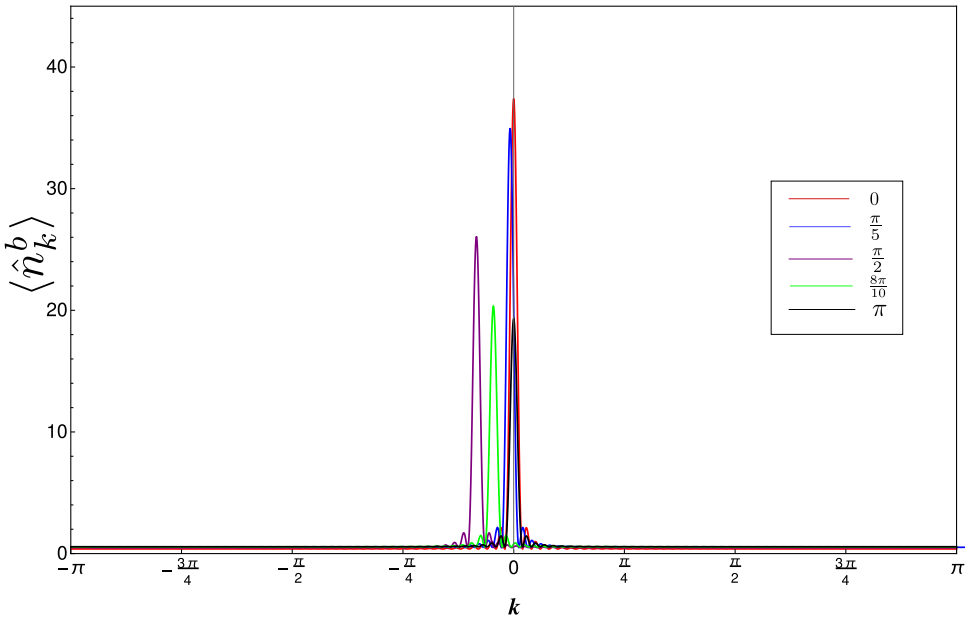
DMRG



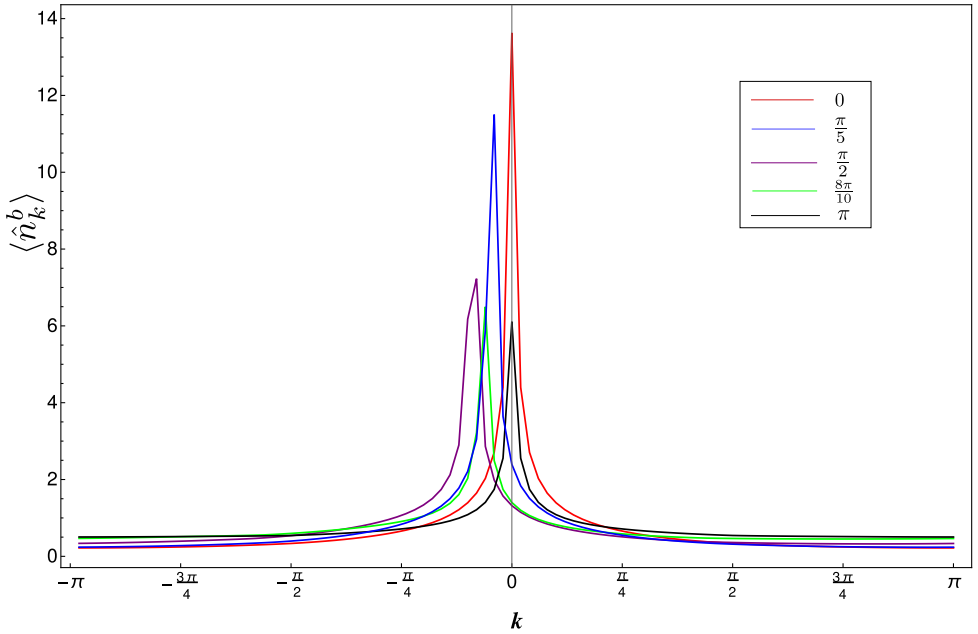
$n_0 = 0.25, \quad L = 100, \quad \frac{U}{J} = 10,$ DMRG data from Kevin

4.2 Bosonic Quasi-Momentum Distributions

Gutzwiller MF



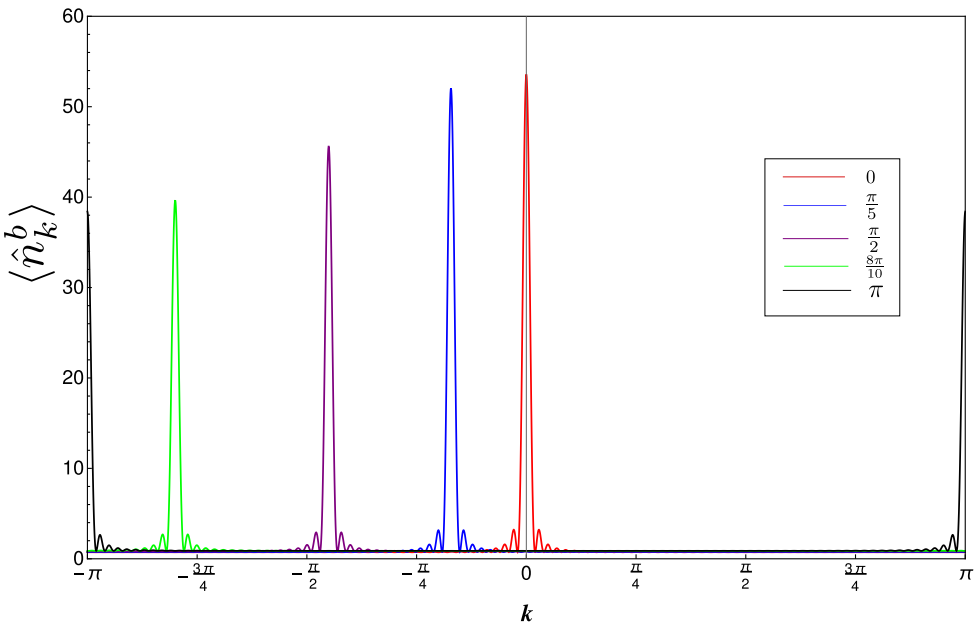
DMRG



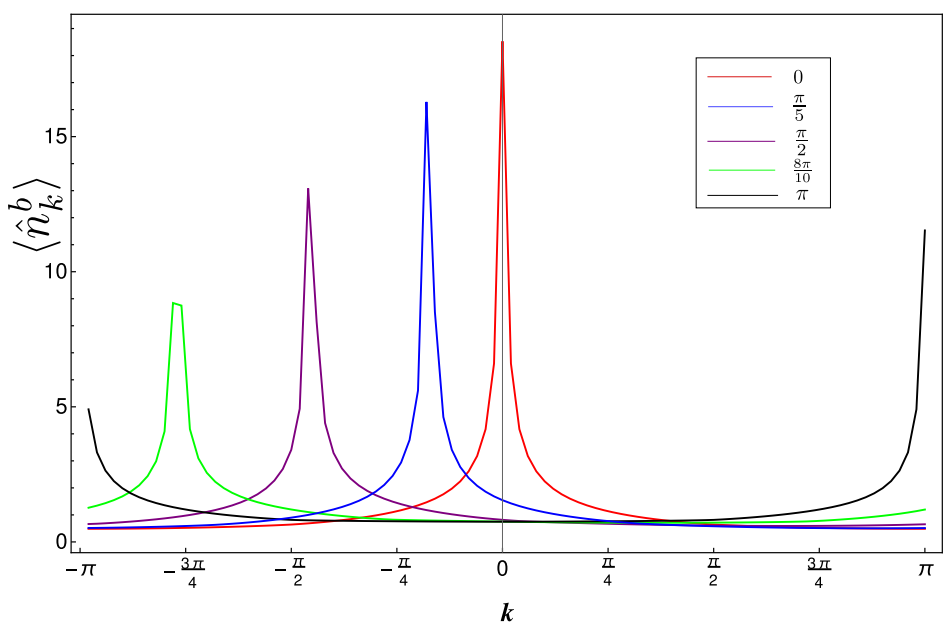
$n_0 = 0.75, \quad L = 100, \quad \frac{U}{J} = 10,$ DMRG data from Kevin

4.2 Bosonic Quasi-Momentum Distributions

Gutzwiller MF



DMRG



$n_0 = 1.25, \quad L = 100, \quad \frac{U}{J} = 10,$ DMRG data from Kevin

4.3 Anyonic Quasi-Momentum Distributions

- **Thermodynamic limit:**

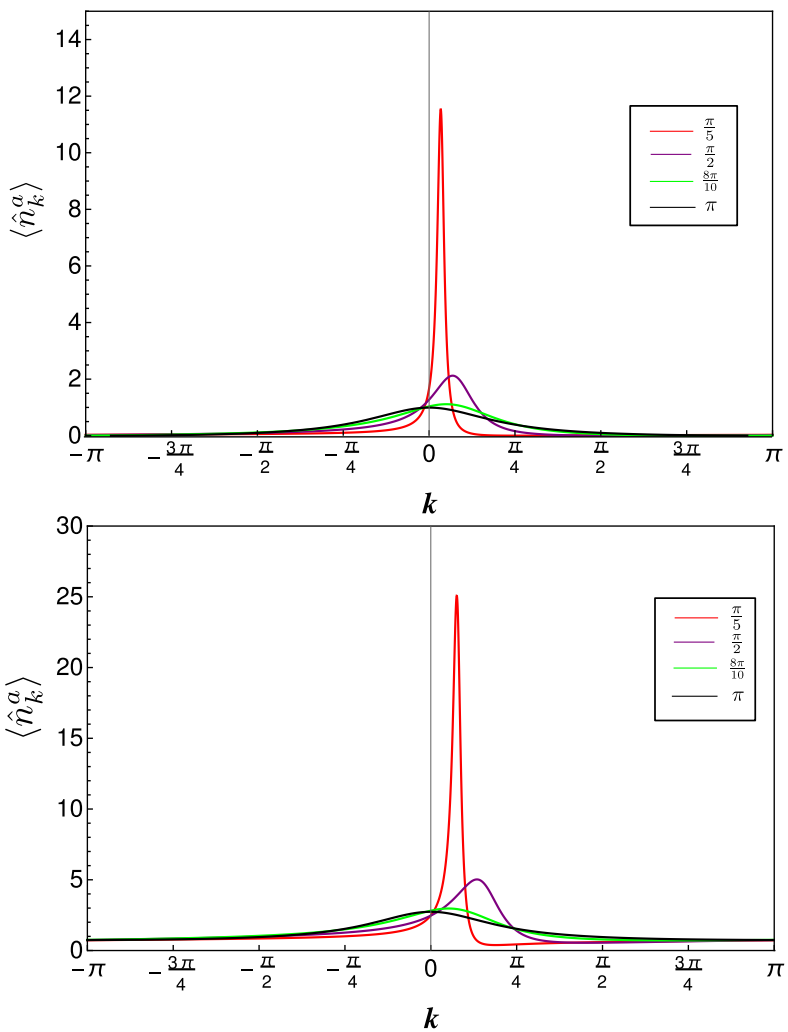
$$\langle \hat{n}_k^a \rangle_{L\to\infty} = n_0 - 2|C| \left\{ \frac{|z|\cos\left[\mathrm{Arg(C)} - \mathrm{Arg(z)}\right] - \cos\left[k + \Delta\beta_0 - \mathrm{Arg(C)}\right]}{|z|^2 - 2|z|\cos\left[k + \Delta\beta_0 - \mathrm{Arg(C)}\right] + 1} \right\}$$

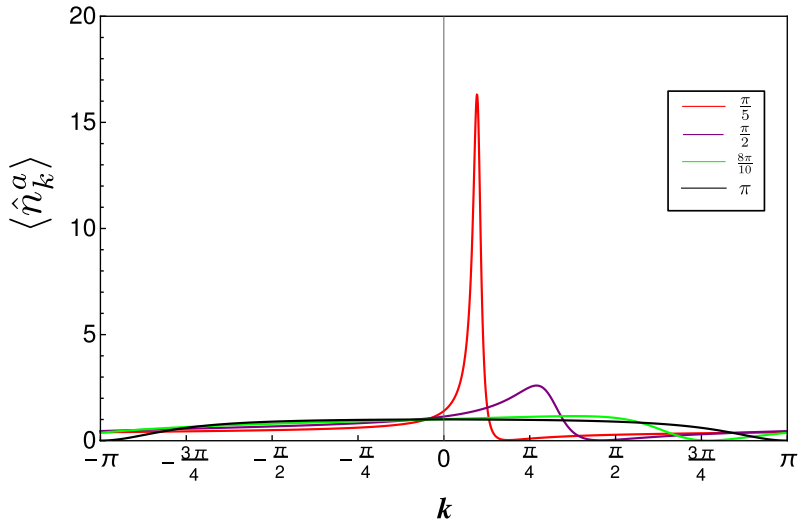
$$C = |A_1|^2 \left(|A_0|^2 + \sqrt{2}|A_0||A_2| \left\{ \exp\left[i\Delta\gamma\right] + \exp\left[-i(\Delta\gamma - \theta)\right] \right\} + 2|A_2|^2 \exp\left[i\theta\right] \right)$$

$$z = |A_0|^2 + |A_1|^2 \exp\left[i\theta\right] + |A_2|^2 \exp\left[2i\theta\right]$$

$\Longrightarrow$ Deformation to smoothened Fermi edge is observable for $\lim\limits_{\theta\to\pi} \langle \hat{n}_k^a \rangle$

4.3 Anyonic Quasi-Momentum Distributions





$$n_0 = 0.25, 0.75, 1.25; \quad L = 100, \quad \frac{U}{J} = 10$$

5. Mean-Field Decoupling

- **Decomposition of ladder operators:** $\hat{a}_i \longrightarrow \langle \hat{a}_j \rangle + \delta \hat{a}_j$

- **Neglection of fluctuations:** $\delta \hat{a}_j^2 \approx 0$

- **Locality:** $\hat{H}_{\mathrm{MF}} = \sum_{j=1}^{L} \hat{H}_j$

- **Self-consistency relations:** $\langle \hat{a}_j \rangle = \alpha_j \longrightarrow \min_{\alpha_j} \left[E_{\mathrm{MF}}(\alpha_j) \right]$

- **Classical Decoupling:** $\alpha_j \longrightarrow \alpha$ i.e. homogeneity is assumed

- **Modified Decoupling:** $\alpha_j \longrightarrow \mathrm{Abs}(\alpha) \exp[ij \mathrm{Arg}(\alpha)]$

5. Mean-Field Decoupling

$\longrightarrow$ Use bi-partite structure of underlying lattice

$$\hat{H} = \frac{U}{2}\sum_{j=1}^{L}\hat{n}_j\left(\hat{n}_j - 1\right) - J\sum_{j=1}^{\frac{L}{2}}\left[\hat{c}^\dagger_{2j-1}\hat{b}_{2j} + \hat{c}^\dagger_{2j}\hat{b}_{2j+1} + \mathrm{h.c.}\right] - \mu\sum_{j=1}^{L}\hat{n}_j$$

$$\hat{c}^\dagger_j = \hat{b}^\dagger_j \exp(i\theta\hat{n}_j)$$

$\longrightarrow$ Decompose ladder operators $\langle\hat{b}_j\rangle + \delta\hat{b}_j, \quad \langle\hat{c}_j\rangle + \delta\hat{c}_j$

$\longrightarrow$ Consistency w.r.t. product state $\langle\hat{c}^\dagger_j\rangle \neq \langle\hat{b}^\dagger_j\rangle\langle\exp(i\theta\hat{n}_j)\rangle$

5. Mean-Field Decoupling

$$\longrightarrow \hat{H}_{\mathrm{MF}} = \hat{H}_0 - J\sum_{j=1}^{\frac{L}{2}} \left[B_o^* \left(\eta_o, \nu_e, \hat{n}_{2j}\right) \hat{b}_{2j} + A_e^* \left(\eta_e, \nu_o, \hat{n}_{2j+1}\right) \hat{b}_{2j+1} + \mathrm{h.c.} \right]$$

$\longrightarrow$ Self-consistency equations, b_l, c_l $\sim$ variational parameters

$$\begin{aligned}
\langle \hat{b}_{2j} \rangle &= b_{2j} = \mathrm{Abs}(b)\exp(i2j\nu_e), \\
\langle \hat{c}_{2j} \rangle &= c_{2j} = \mathrm{Abs}(c)\exp(i2j\eta_e), \\
\langle \hat{b}_{2j+1} \rangle &= b_{2j+1} = \mathrm{Abs}(b)\exp\left[i(2j+1)\nu_o\right], \\
\langle \hat{c}_{2j+1} \rangle &= c_{2j+1} = \mathrm{Abs}(c)\exp\left[i(2j+1)\eta_o\right],
\end{aligned}$$

$\Longrightarrow$ Linearize self-consistency map around trivial solution $b_l, c_l \approx 0$

5. Mean-Field Decoupling

$\longrightarrow$ Perturbation theory $\quad |\,\Psi\,\rangle \approx \prod_{i=1}^{L} |\,n\,\rangle_i + |\,\Psi_1\,\rangle$

$\longrightarrow$ Assume equality of variational parameter amplitudes on both partitions

$\longrightarrow$ Consistency relations for variational parameter phases

$\longrightarrow$ Two types of superfluid depending on $\quad n_1, n_2 \in \mathbb{Z}$

W. Zhang, E. Fan, T. C. Scott, and Y. Zhang, `arXiv:1511.01712`

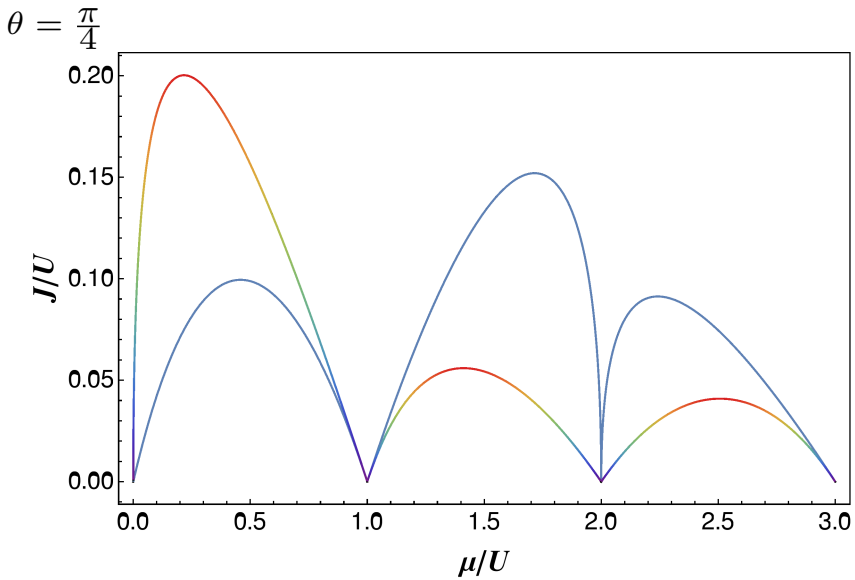
$$\langle \hat{b}_{2j} \rangle = \mathrm{Abs}(b), \qquad \langle \hat{c}_{2j} \rangle = \mathrm{Abs}(c)$$

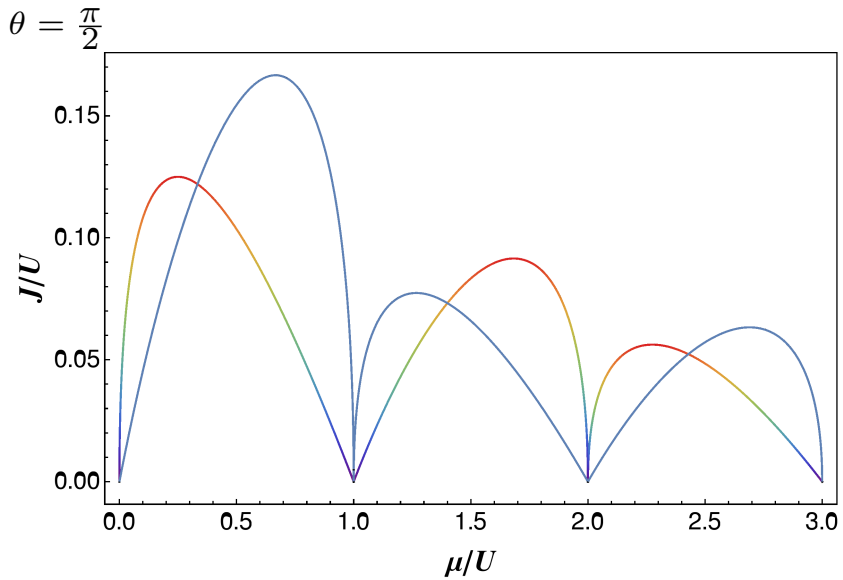
$$\langle \hat{b}_{2j+1} \rangle = \mathrm{Abs}(b)(-1)^{n_2-n_1}, \quad \langle \hat{c}_{2j+1} \rangle = \mathrm{Abs}(c)(-1)^{n_2+n_1}$$

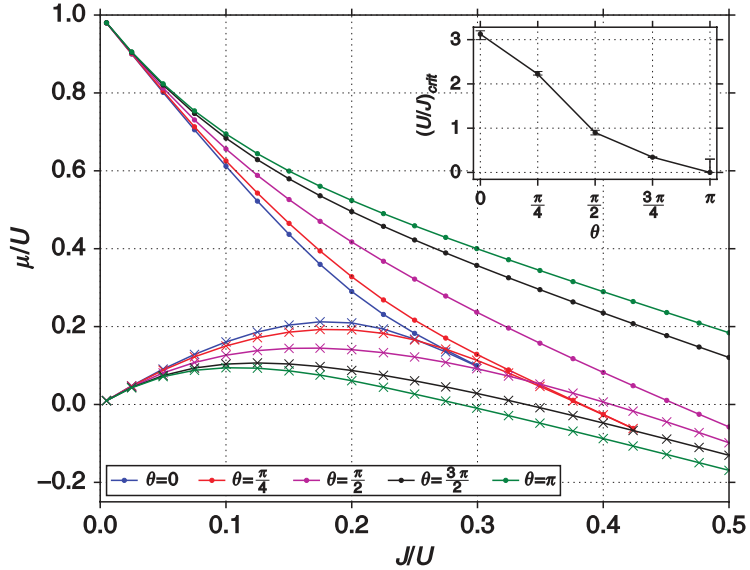
$\longrightarrow \mathrm{J}_{\mathrm{crit}}$ from instabillity of self-consistency equations $\quad \alpha = A\alpha$

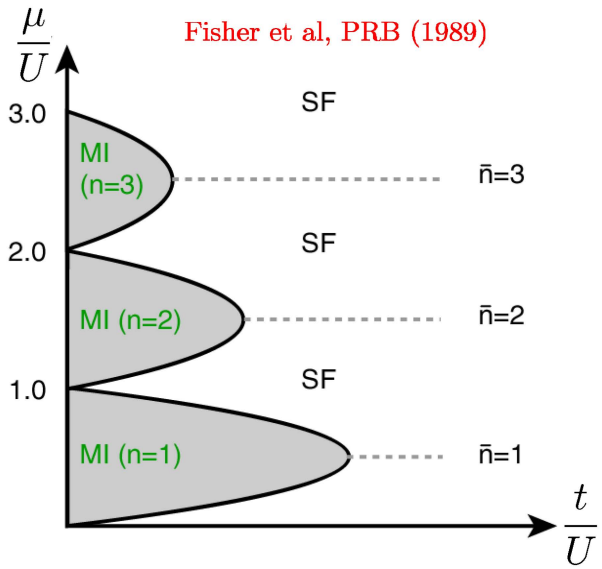
with $\quad \alpha = (\mathrm{Abs}(b), \mathrm{Abs}(c))^T, \qquad$ i.e. $\quad \max\left[\mathrm{Re}\left(\lambda_\pm\right)\right] = 1$

5.1 Phase Diagramm









S. Lanzmich, et al., NC **2**, 2011

Conclusion

$\longrightarrow$ Independence of exclusion principle and exchange statistics

$\longrightarrow$ Smooth transmutation between bosonic and fermionic exchange statistics

$\longrightarrow$ Consistent modified Gutzwiller approach

$\longrightarrow$ Superfluid condensation at finite momentum

$\longrightarrow$ Non-trivial deformation of Mott lobes

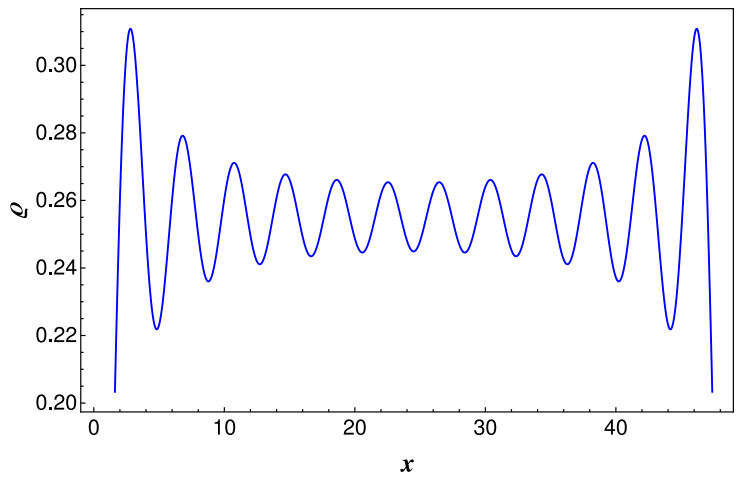
$\longrightarrow$ Emergence of new superfluid phases

Outlook

$\longrightarrow$ Strong coupling expansion to check phase boundary

J.K. Freericks and H. Monien, PRB **53**, 2691 (1996)

$\longrightarrow$ Friedel oscillation (for open boundary conditions)



C. Sträter, S.C.L. Srivastava, and A. Eckardt, PRL **117**, 205303 (2016)

$\longrightarrow$ Nearest neighbour interactions inducing topological phases (Shijie, Kevin)

F. Lange, S. Ejima, and H. Fehske, `arXiv:1612.00605`

$\longrightarrow$ Investigation of fermionic anyons

$\longrightarrow$ Finite temperature + dynamics beyond hard-core limit