Noise-Dependent Stability of the Synchronized State in a Coupled System of Active Rotators

Sebastian F. Brandt[1], Axel Pelster[2], and Ralf Wessel[1]

[1]Department of Physics, Washington University in St. Louis
[2]Fachbereich Physik, Universität Duisburg-Essen, Campus Duisburg

arXiv: 0802.1105

Phase Dynamics of Excitable Systems

- Kuramoto Model for active rotators

$$\dot{\phi}(t) = \omega - a \sin \phi(t)$$

- Langevin Equation

$$\dot{\phi}_i(t) = \omega - a \sin \phi_i(t) - \sum_{j=1}^{n} W_{ij}(\phi_j - \phi_i) + \eta_i(t)$$

$$W_{ij}(\phi) = w_{ij} \sin \phi$$

$$\langle \eta_i(t) \rangle = 0, \qquad \langle \eta_i(t_1) \eta_j(t_2) \rangle = 2\sigma \delta(t_1 - t_2) \delta_{ij}$$

- Fokker-Planck Equation

$$\frac{\partial}{\partial t} P(\boldsymbol{\phi}, t) = -\sum_{i=1}^{n} \frac{\partial}{\partial \phi_i} \left[D_i(\boldsymbol{\phi}) P(\boldsymbol{\phi}, t) \right] + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial \phi_i \partial \phi_j} \left[D_{ij}(\boldsymbol{\phi}) P(\boldsymbol{\phi}, t) \right]$$

$$D_i(\boldsymbol{\phi}) = \omega - a \sin \phi_i - \sum_{j=1}^{n} w_{ij} \sin(\phi_j - \phi_i) \,, \qquad D_{ij}(\boldsymbol{\phi}) = \delta_{ij} \sigma$$

Single Rotator

- Fokker-Planck Equation

$$\frac{\partial}{\partial t}P(\phi,t) = -\frac{\partial}{\partial \phi}\left[D(\phi)P(\phi,t)\right] + \sigma\frac{\partial^2}{\partial \phi^2}P(\phi,t)$$

- Potential

$$D(\phi) = \omega - a\sin\phi\,, \quad V(\phi) = -\omega\phi - a\cos\phi + c$$

- Probability Current

$$S(\phi,t) = D(\phi)P(\phi,t) - \sigma\frac{\partial}{\partial \phi}P(\phi,t)$$

- Continuity Equation

$$\frac{\partial}{\partial t}P(\phi,t) + \frac{\partial}{\partial \phi}S(\phi,t) = 0$$

General Stationary Solution

- Constant Probability Current → Solution

$$P(\phi) = Ce^{-\frac{V(\phi)}{\sigma}} - \frac{S}{\sigma}\int_0^{\phi} d\phi' e^{\frac{V(\phi')-V(\phi)}{\sigma}}$$

- Determine Constants through Normalization and Boundary Conditions

$$C = \frac{e^{-\frac{V(2\pi)}{\sigma}}}{\det}\int_0^{2\pi} d\phi\, e^{\frac{V(\phi)}{\sigma}}\,, \quad S = \frac{\sigma}{\det}\left[e^{-\frac{V(2\pi)}{\sigma}} - e^{-\frac{V(0)}{\sigma}}\right]$$

$$\det = \begin{vmatrix} \int_0^{2\pi} d\phi\, e^{-\frac{V(\phi)}{\sigma}} & \int_0^{2\pi} d\phi \int_0^{\phi} d\phi' e^{\frac{V(\phi')-V(\phi)}{\sigma}} \\ e^{-\frac{V(2\pi)}{\sigma}} - e^{-\frac{V(0)}{\sigma}} & \int_0^{2\pi} d\phi\, e^{\frac{V(\phi)-V(2\pi)}{\sigma}} \end{vmatrix}$$

Mean Frequency of Active Rotator

- Integral Representation

$$\bar{\omega} = 2\pi S = \frac{2\pi\sigma \left(1 - e^{-\frac{2\pi\omega}{\sigma}}\right)}{\int_0^{2\pi} d\phi' \, e^{-\frac{\omega}{\sigma}\phi'} \int_0^{2\pi} d\phi \, e^{\frac{a}{\sigma}[\cos(\phi+\phi')-\cos\phi]}}$$

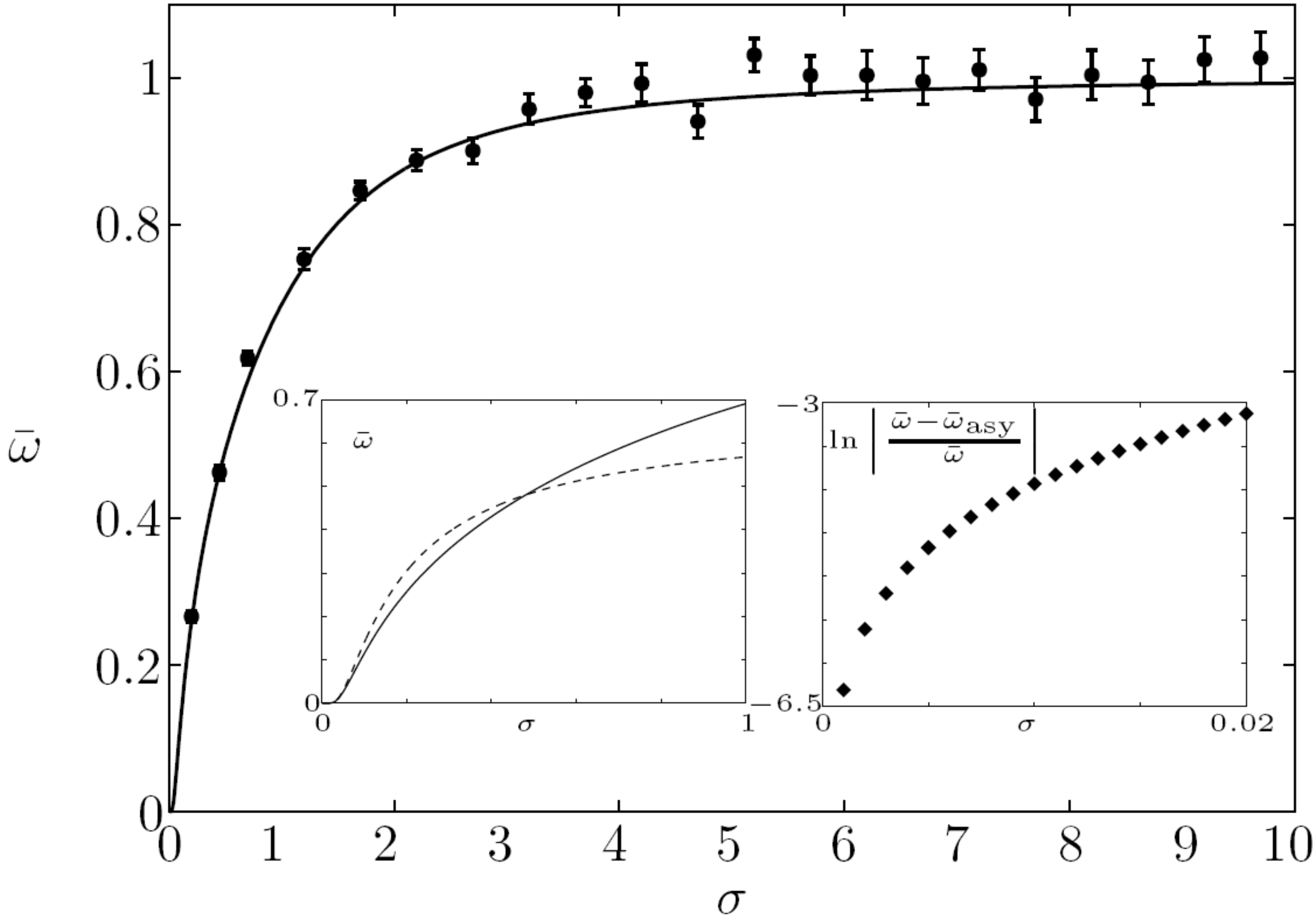
- Large-Noise Limit

$$\lim_{\sigma\to\infty} \bar{\omega} = \omega$$

- Small-Noise Limit

$$\bar{\omega}_{\mathrm{asy}} = \sqrt{a^2 - \omega^2} e^{-\frac{2}{\sigma}\left(\sqrt{a^2-\omega^2} - \omega \arccos \frac{\omega}{a}\right)}$$

Analytical and Numerical Results



$\omega = 1\,, \qquad a = 1.2$

Deterministic Two-Rotator System

- Dynamical System: $\Phi = \dfrac{\phi_1 + \phi_2}{2}\,, \qquad \Delta = \dfrac{\phi_1 - \phi_2}{2}$

$$\dot{\Phi}(t) = \omega - a\sin\Phi(t)\cos\Delta(t) + (w_{12} - w_{21})\sin\Delta(t)\cos\Delta(t)\,,$$

$$\dot{\Delta}(t) = -a\cos\Phi(t)\sin\Delta(t) + (w_{12} + w_{21})\sin\Delta(t)\cos\Delta(t)\,.$$

- Fixed Point

$$\Phi(t) = \Phi_0 = \sin^{-1}(\omega/a)\,, \qquad \Delta(t) = 0$$

- Eigenvalues of Stability Matrix

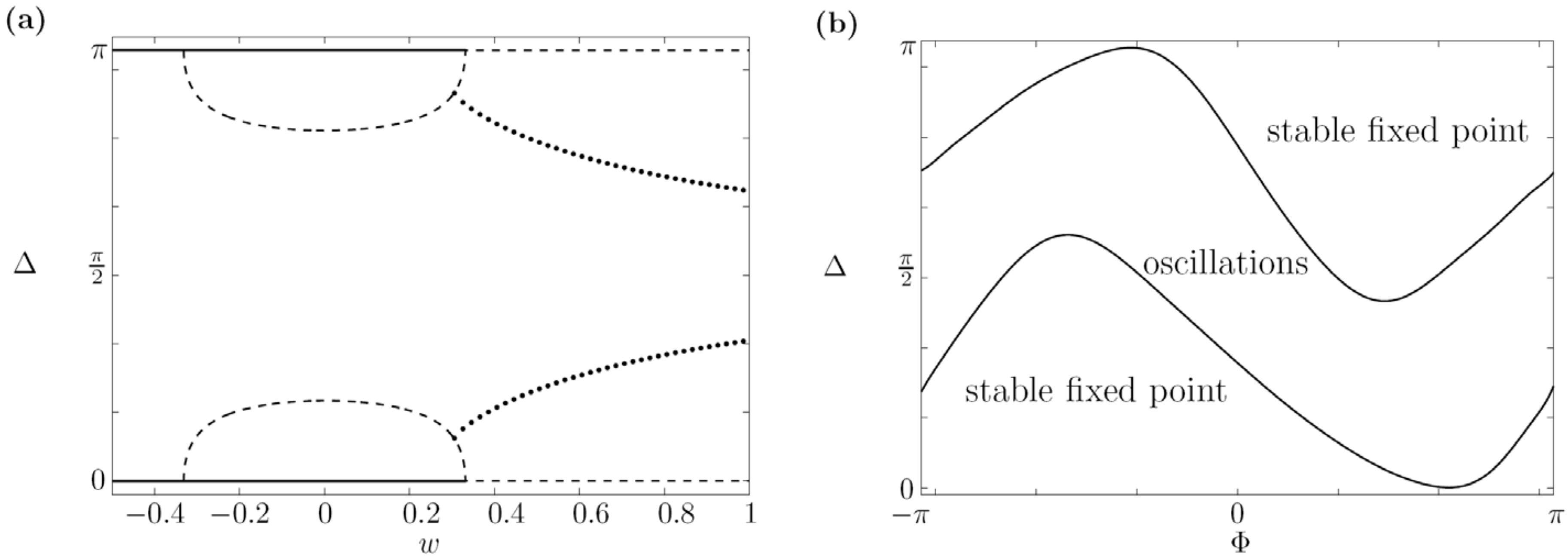
$$\lambda_1 = -\sqrt{a^2 - \omega^2}\,, \qquad \lambda_2 = w_{12} + w_{21} - \sqrt{a^2 - \omega^2}$$

- Subcritical Pitchfork Bifurcation

$$w_{12} + w_{21} = \sqrt{a^2 - \omega^2}$$

Bifurcation Diagram

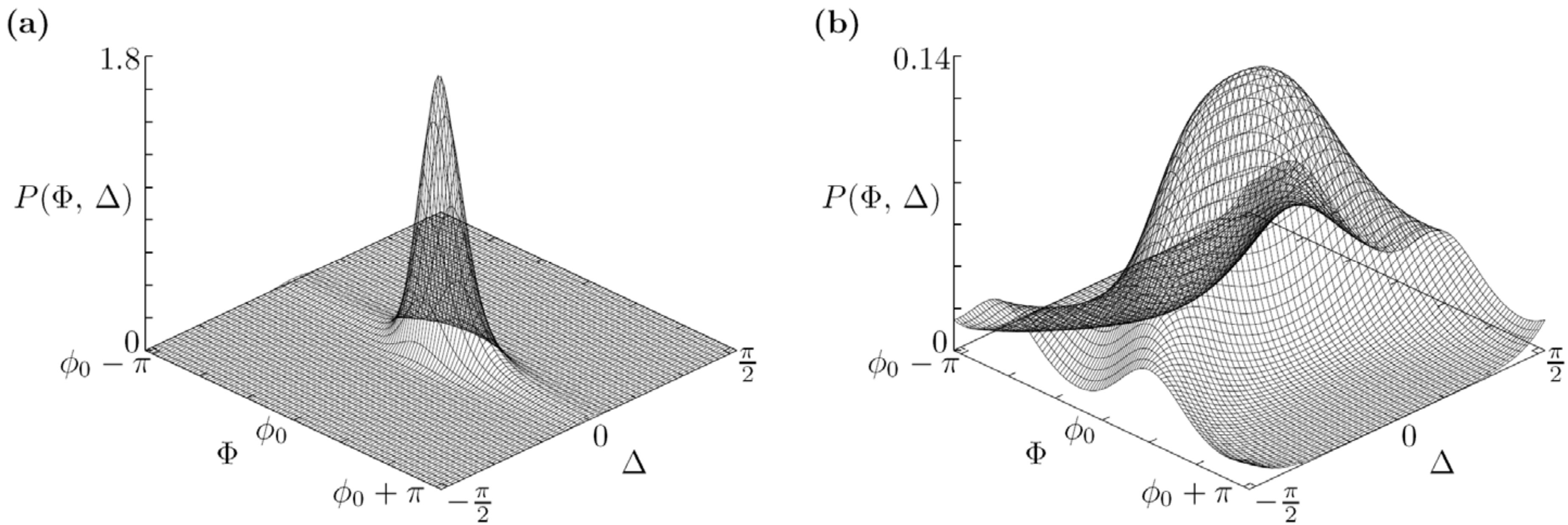
- Coexistence of Stable Fixed Point and Oscillations



$$w_{12} = w_{21} = w\,, \qquad \omega = 1\,, \qquad a = 1.2$$

Stochastic Two-Rotator System

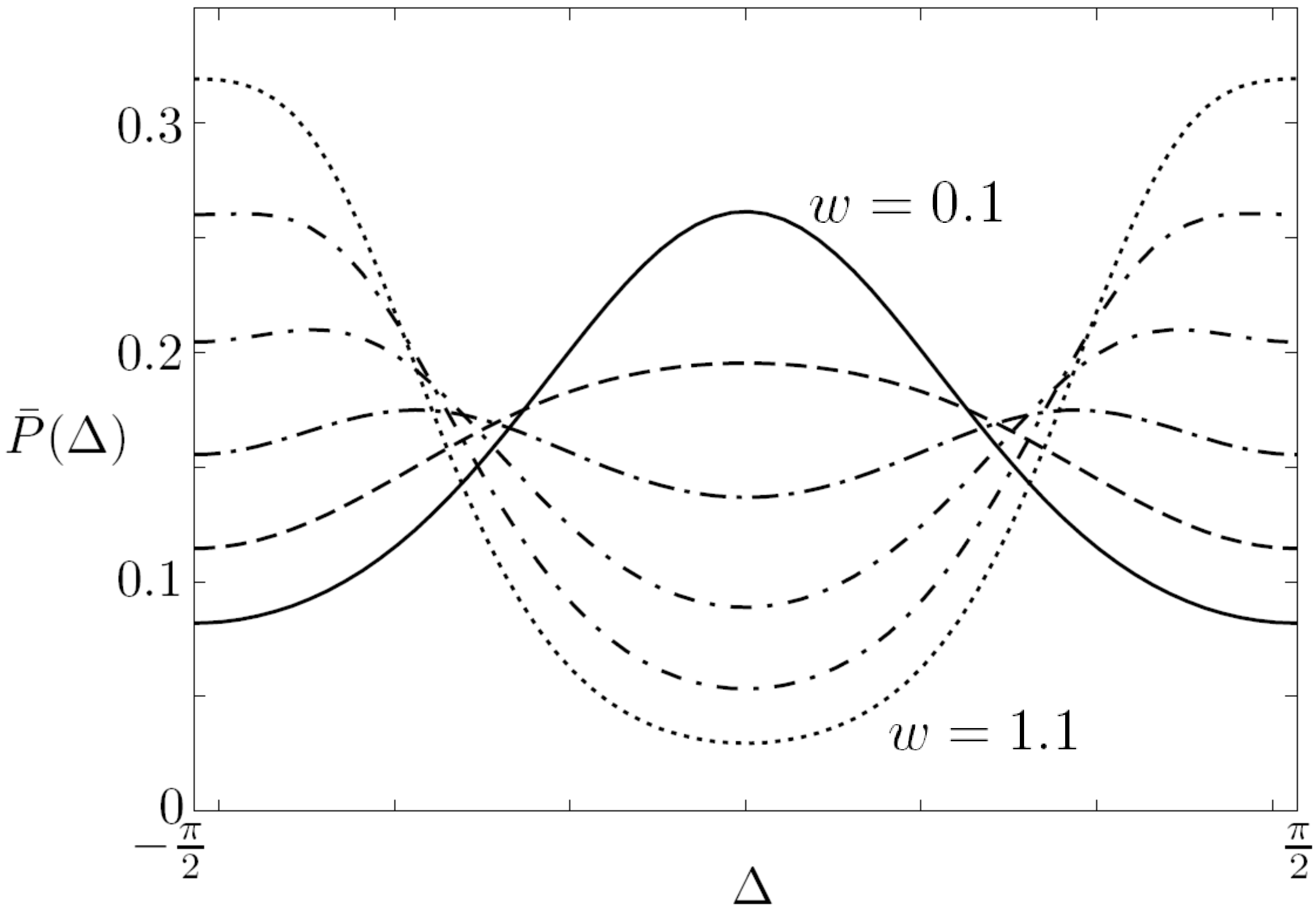
- Stationary Solution to Fokker-Planck Equation



$w = 0.3$ $\qquad\qquad$ $w = 0.4$

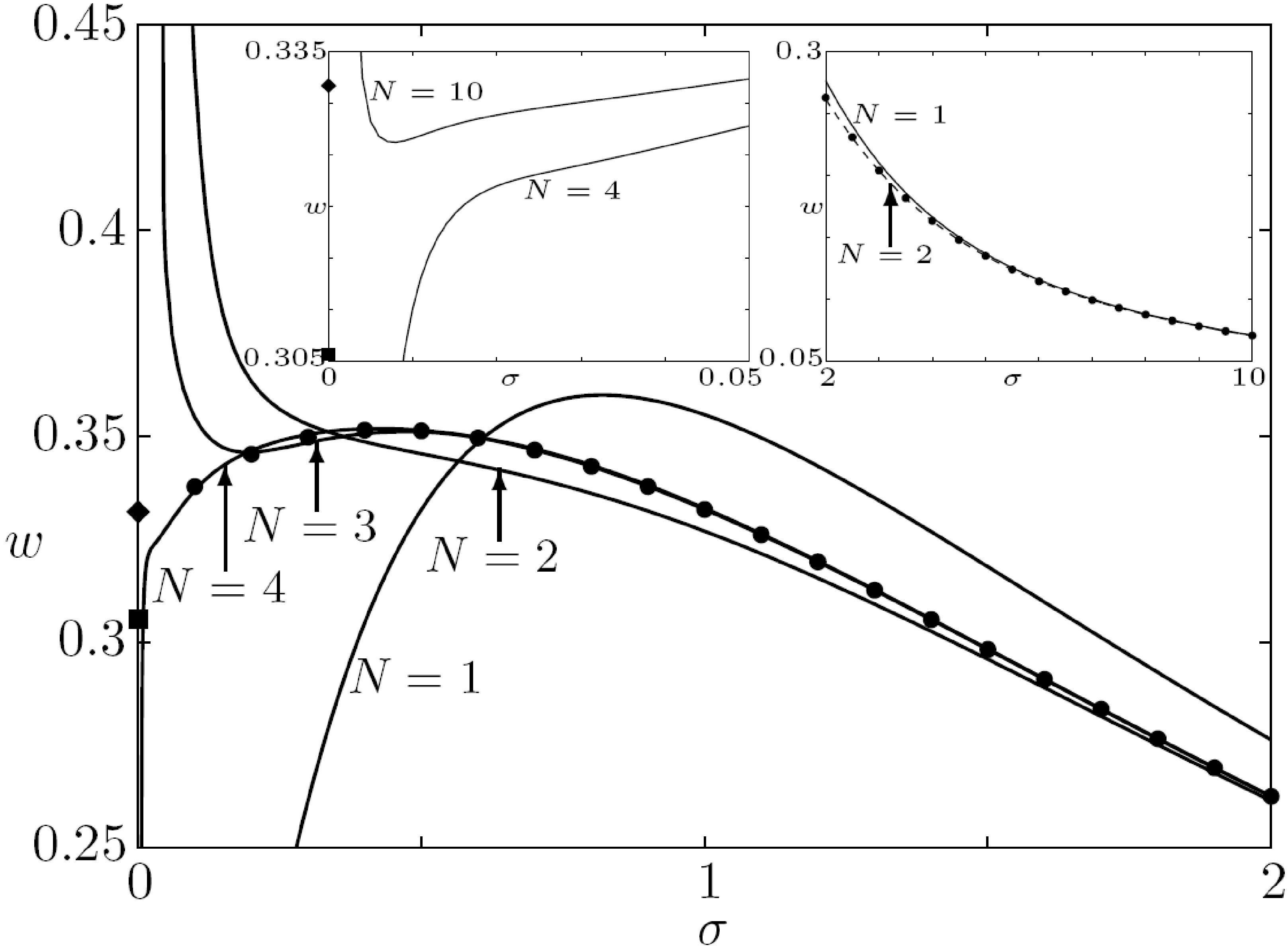
$$w_{12} = w_{21} = w\,, \qquad \omega = 1\,, \qquad a = 1.2\,, \qquad \sigma = 0.4$$

- Marginal Probability Density: $\bar{P}(\Delta) = \int_{\Phi_0-\pi}^{\Phi_0+\pi} d\Phi P(\Phi, \Delta)$



$w_{12} = w_{21} = w\,, \qquad \omega = 1\,, \qquad a = 1.2\,, \qquad \sigma = 0.4$

Analytical and Numerical Results



$w_{12} = w_{21} = w\,, \qquad \omega = 1\,, \qquad a = 1.2$

Fourier Expansion

- Ansatz

$$P(\phi_1,\phi_2) = \sum_{k_1,k_2} C(k_1,k_2)e^{i(k_1\phi_1+k_2\phi_2)}$$

- Infinite System of Algebraic Equations

$$0 = \sum_{k_1,k_2} e^{i(k_1\phi_1+k_2\phi_2)} \sum_{|l_1|<1,|l_2|<1} C(k_1-l_1,k_2-l_2)\tilde{C}(l_1,k_1-l_1,l_2,k_2-l_2)$$

$$\begin{aligned}
\tilde{C}(\pm 1,k_1,0,k_2) &= \frac{a}{2}(1\pm k_1)\\
\tilde{C}(0,k_1,\pm 1,k_2) &= \frac{a}{2}(1\pm k_2)\\
\tilde{C}(1,k_1,-1,k_2) &= -\frac{1+k_1}{2}w_{12} - \frac{1-k_2}{2}w_{21}\\
\tilde{C}(-1,k_1,1,k_2) &= -\frac{1-k_1}{2}w_{12} - \frac{1+k_2}{2}w_{21}\\
\tilde{C}(\pm 1,k_1,\pm 1,k_2) &= 0
\end{aligned}$$

First-Order Expansion Result

- Probability Density

$$P_1(\phi_1,\phi_2)=\frac{1}{4\pi^2}+\alpha\{2a\beta\gamma\sigma(\cos\phi_1+\cos\phi_2)+4a\beta\sigma\omega(\sin\phi_1+\sin\phi_2)+(a^2\gamma\sigma^2-2a^2\sigma\omega^2)$$
$$\times\cos(\phi_1+\phi_2)+[a^2\beta\gamma-w\beta(\gamma^2+4\omega^2)]\cos(\phi_1-\phi_2)+a^2\sigma(w\omega+4\sigma\omega)\sin(\phi_1+\phi_2)\}$$

- Marginal Probability Density

$$\bar{P}_1(\Delta)=\frac{1}{2\pi}+2\pi\alpha\beta(a^2\gamma-w\gamma^2-4w\omega^2)\cos(2\Delta)$$

- Phase Transition

$$\sigma=\frac{a^2-2w^2\pm\sqrt{a^4-16w^2\omega^2}}{4w}$$