PHYSIK

TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

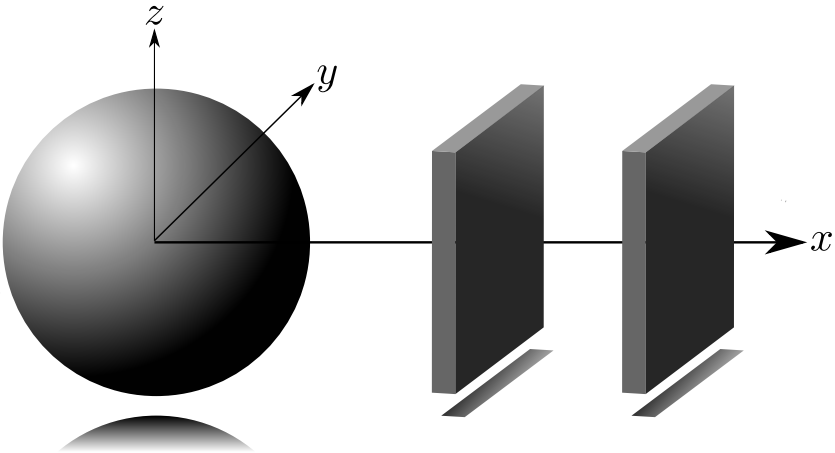
Casimir Effect in the Presence of Weak Gravity

Lukas Buß

Theoretical Quantum Optics
Department of Physics
TU Kaiserslautern

November 14, 2022

Outline



Casimir effect in Minkowski spacetime

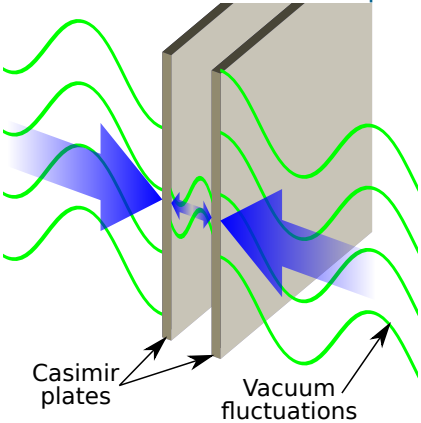


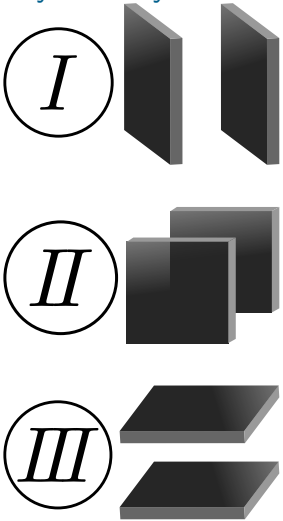
Figure 1: Illustration of the Casimir force on two parallel plates. [8]

$$E_C^{\text{flat}} = -\frac{\hbar c \pi^2}{720}\frac{A}{d^3}\,,$$

$$F_C^{\text{flat}} = -\frac{\partial E_C}{\partial d} = -\frac{\hbar c \pi^2}{240}\frac{A}{d^4}\,.$$

- Casimir force for $A = 5.1\text{cm}^2$, $d = 1\mu\text{m}$: 660nN [3]
- Electrostatic force exerted by the nucleus on the electron in a hydrogen atom: 92nN
- Force measurable with an AFM: 0.1pN [4]

Rotational symmetry



Setup I:

$$E_C^{\text{flat}} = E_C^I = -\frac{\hbar c \pi^2}{720}\frac{A}{d^3} \,,$$
$$F_C^{\text{flat}} = F_C^I = -\frac{\hbar c \pi^2}{240}\frac{A}{d^4} \,.$$

Rotational symmetry in Minkowski spacetime:

$$E_C^{\text{flat}} = E_C^I = E_C^{II} = E_C^{III} \,,$$
$$F_C^{\text{flat}} = F_C^I = F_C^{II} = F_C^{III} \,.$$

Casimir setups

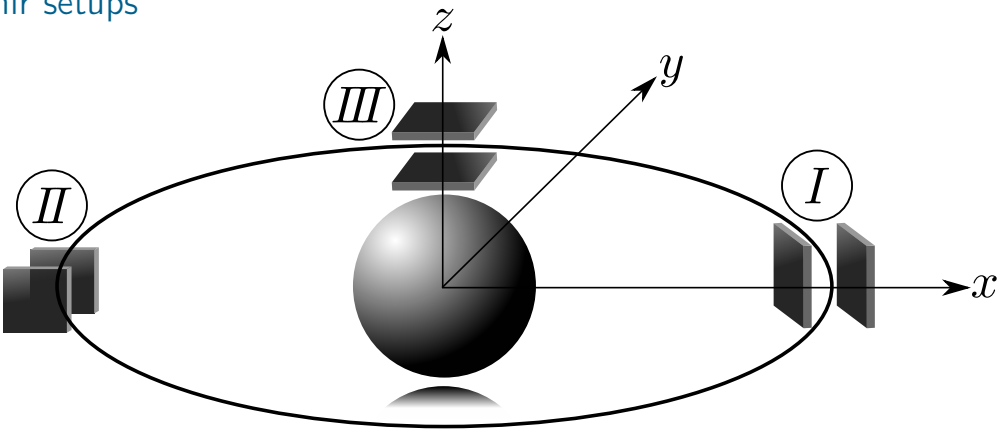


Figure 2: Three different Casimir setups on a circular orbit in the equatorial plane ($\vartheta = 0,\ z = 0$) next to a non-rotating, uncharged source of weak gravity.

Maxwell theory

Homogeneous Maxwell's equations are unaffected by gravity [2]:

$$\partial_t B^i = -\epsilon^{ijk}\partial_j E_k \ , \qquad \partial_j B^j = 0 \ .$$

Inhomogeneous Maxwell's equations read [2]

$$\partial_\mu F^{\mu\nu} + \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}\right) F^{\mu\nu} = \mu_0 j^\nu = 0 \ .$$

$\Longrightarrow$ Gauss' law is given by $\nu = 0$, Ampère's law via $\nu = i = 1, 2, 3$.

Perturbation theory

Expansion of the metric via

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} \ , \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ ,$$
$$h^{\mu\nu}, h_{\mu\nu} \propto \lambda = r_s/r \ll 1 \quad \text{with } r_s = 2GM/c^2 \ .$$

leads to the first-order correction terms

$$\sum_j \partial_j E_j^{(1)} = \frac{1}{\varepsilon_0} \rho_{\text{eff}}^{(1)}(E^{(0)}, B^{(0)}) \ ,$$
$$\frac{1}{c}\partial_0 E_p^{(1)} + \sum_m \epsilon_{kmp} \partial_m B^{k(1)} = -\mu_0 j_{p,\text{eff}}{}^{(1)}(E^{(0)}, B^{(0)}) \ .$$

Perturbation theory

If $\partial_\alpha h^{\beta\gamma} = 0$, the metric is constant in space and time, the continuity equation

$$\partial_\mu j_{\text{eff}}^{\mu(1)} = 0$$

is satisfied.
The wave equations for the first-order electromagnetic fields read

$$\Box \vec{E}^{(1)} = -\mu_0 \partial_t \vec{j}_{\text{eff}}^{(1)} - \frac{1}{\varepsilon_0} \vec{\nabla} \rho_{\text{eff}}^{(1)} \,,$$

$$\Box \vec{B}^{(1)} = \mu_0 \vec{\nabla} \times \vec{j}_{\text{eff}}^{(1)} \,.$$

Metric & Restrictions

Schwarzschild metric

$$\mathrm{d}s^2_{\mathrm{SSM}} = c^2\left(1-\frac{r_s}{r}\right)\mathrm{d}t^2 - \frac{1}{1-\frac{r_s}{r}}\,\mathrm{d}r^2 - r^2\left[\mathrm{d}\vartheta^2 + \sin^2(\vartheta)\,\mathrm{d}\varphi^2\right] \ .$$

First-order correction term of the Schwarzschild metric with respect to the Minkowski metric

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} \Longrightarrow h^{\mu\nu} = \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{x^2}{r^2} & \frac{xy}{r^2} & \frac{xz}{r^2} \\ 0 & \frac{xy}{r^2} & \frac{y^2}{r^2} & \frac{yz}{r^2} \\ 0 & \frac{xz}{r^2} & \frac{yz}{r^2} & \frac{z^2}{r^2} \end{pmatrix} \ , \quad \text{where } \lambda = \frac{r_s}{r} = \frac{2GM}{rc^2} \ .$$

Fixed location for all three setups:

$$\vec{r} = \begin{pmatrix} R \\ 0 \\ 0 \end{pmatrix} \quad \text{for const} = R \gg r_s \ .$$

Generalized metric:

$$h^{\mu\nu} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ .$$

Secular terms & Renormalization

Solution of first-order wave equations

$$\vec{E}^{(1)} \propto \sigma \sin(\sigma),\ \sigma \cos(\sigma) \quad \text{diverges in } \sigma \equiv \vec{k}_\perp \cdot \vec{r}_\perp - \omega t\ .$$

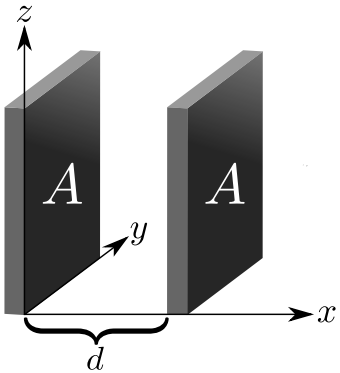
This is called a secular term, since it violates energy conservation law.
$\Longrightarrow$ Renormalization necessary:

$$\vec{E}^{\text{curved}} = \vec{E}^{(0)} + \vec{E}^{(1)} \propto \sin\left[\sigma\left(1-\tilde{\lambda}\right)\right],\ \cos\left[\sigma\left(1-\tilde{\lambda}\right)\right]\ ,$$

where

$$\tilde{\lambda} \equiv \lambda \cdot \frac{\frac{\omega^2}{c^2} + k_x^2}{2k_\parallel^2}\ ,$$

$$\left|\vec{k}\right|^2 \equiv k_\parallel^2 + k_\perp^2\ .$$



Secular terms & Renormalization

Renormalized vector elements:

$$k'_{\parallel} = k_{\parallel} \ , \quad k'_{\perp} = k_{\perp} + \Delta k_{\perp} = k_{\perp}(1-\tilde{\lambda}) \ , \quad \omega' = \omega + \Delta\omega = \omega(1-\tilde{\lambda}) \ .$$

Dispersion relation:

$$\omega(\vec{k}) = c|\vec{k}| = c\sqrt{k_{\parallel}^2 + k_{\perp}^2} = \frac{\omega'}{1-\tilde{\lambda}} = c\sqrt{{k'_{\parallel}}^2 + \frac{{k'_{\perp}}^2}{\left(1-\tilde{\lambda}\right)^2}} \ ,$$

$$\omega'(\vec{k'}) = c\sqrt{{k'_{\parallel}}^2\left(1-\tilde{\lambda}\right)^2 + {k'_{\perp}}^2} \ .$$

Spaghettification

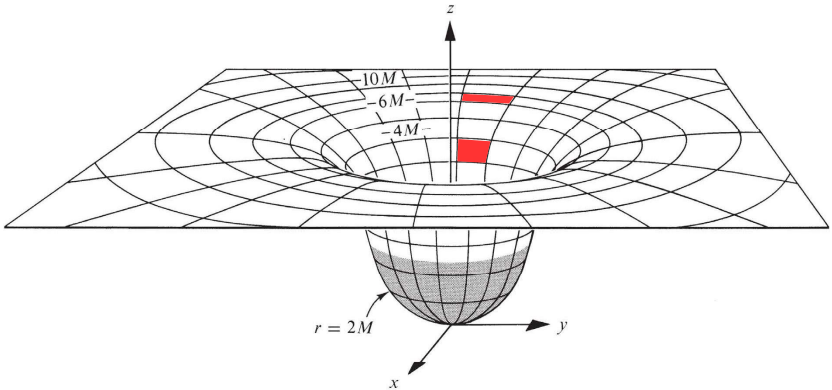


Figure 3: Spaghettification in Schwarzschild spacetime [5]. Reciprocal for $\omega' = c\sqrt{k_{\parallel}'^{2}\left(1-\tilde{\lambda}\right)^{2} + k_{\perp}'^{2}}$.

Secular terms & Renormalization

Dispersion relation:

$$\omega'(\vec{k}') = c\sqrt{k_x'^2 + k_y'^2 + k_z'^2} - \frac{c k_x'^2 \tilde{\lambda}}{\sqrt{k_x'^2 + k_y'^2 + k_z'^2}} + \mathcal{O}\left(\tilde{\lambda}^2\right) ,$$

$$\omega'(\vec{k}') = c|\vec{k}'| \left(1 - \frac{k_x'^2 \tilde{\lambda}}{|\vec{k}'|^2}\right) + \mathcal{O}\left(\tilde{\lambda}^2\right) .$$

Notation: Drop the $'$ signs. Inserting $\tilde{\lambda}$ yields

$$\omega(\vec{k}) = c|\vec{k}| \left[1 - \frac{\lambda}{2}\left(1 + \frac{k_x^2}{|\vec{k}|^2}\right)\right] .$$

Proper Casimir lengths

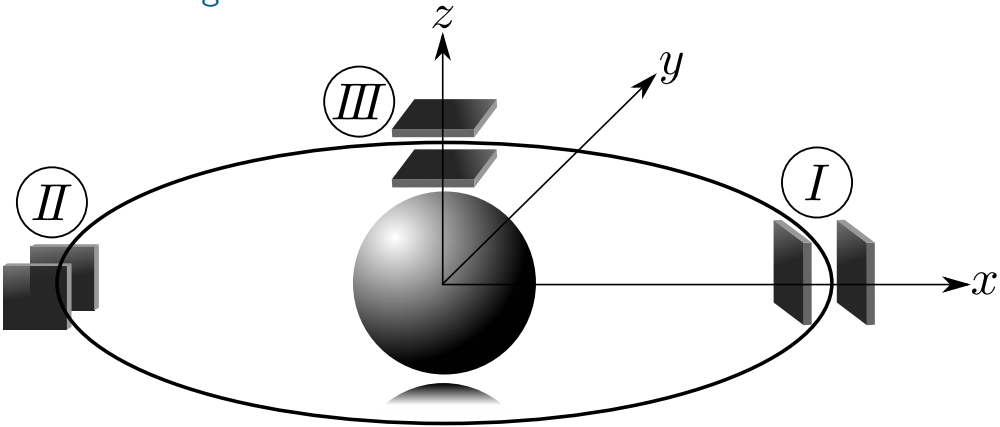
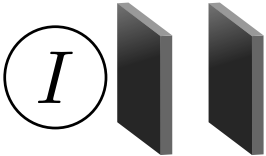


Figure 4: $d_p = \sqrt{g_{11}}d = \sqrt{\eta_{11} + h_{11}}d$

Table 1: Casimir forces for setups *I*–*III* in the presence of weak gravity.

Quantities			
F_C^{flat}	$-\frac{\pi^2 \hbar c A}{240 d^4}$	$-\frac{\pi^2 \hbar c A}{240 d^4}$	$-\frac{\pi^2 \hbar c A}{240 d^4}$
F_C^{curved}	$F_C^{\text{flat}}(1-2\lambda) = F_C^{\text{flat}}(d_p, A_p)$	$F_C^{\text{flat}}\left(1+\frac{\lambda}{2}\right) = F_C^{\text{flat}}(d_p, A_p)$	$F_C^{\text{flat}}\left(1+\frac{\lambda}{2}\right) = F_C^{\text{flat}}(d_p, A_p)$

Casimir force of setup *I*–*III*

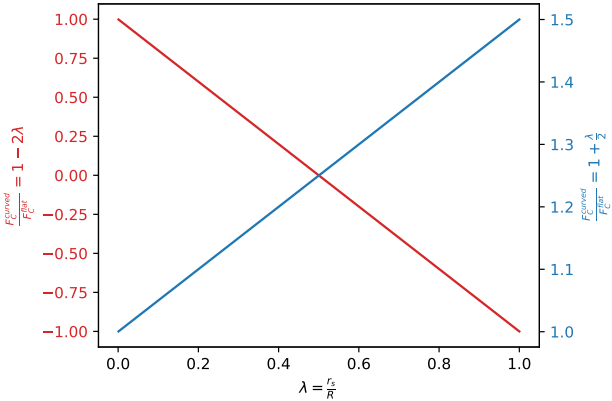


Figure 5: Ratio of the perturbed Casimir force in the presence of weak gravity to the Casimir force in a Minkowskian spacetime for setup *I* (red) and setups *II* and *III* (blue).

Energy splitting

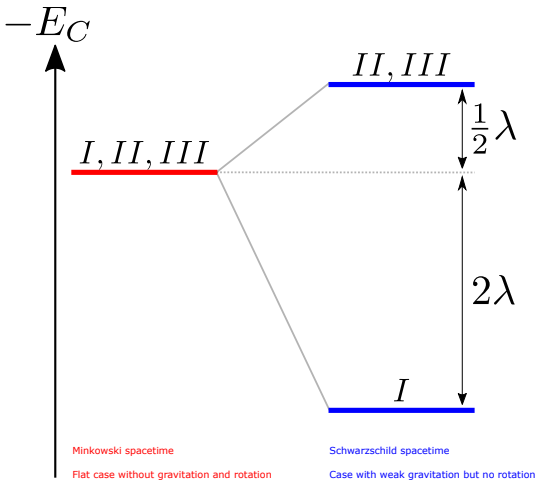


Figure 6: Energy splitting of the three Casimir setups in the presence of weak gravity.

Solar system

Table 2: The first-order gravitational perturbation parameter $\lambda = r_s/r$ with respect to the Minkowski metric for several objects in our solar system. Values were calculated using [6, 7].

Object	r_s/R_{surface}	$r_{s\ \text{Sun}}/R_{\text{mean Sun}}$
Sun	$4.25 \cdot 10^{-6}$	not defined
Mercury	$2.01 \cdot 10^{-10}$	$5.10 \cdot 10^{-5}$
Venus	$1.19 \cdot 10^{-9}$	$2.73 \cdot 10^{-5}$
Earth	$1.39 \cdot 10^{-9}$	$1.97 \cdot 10^{-5}$
PSP	$8.48 \cdot 10^{-25}$	$4.80 \cdot 10^{-4}$

Comparing different setups

Assumptions:

- Schwarzschild spacetime
- $\lambda = r_s/r \approx 10^{-5} \ll 1 \implies$ weak gravity
- First-order corrections are sufficient
- No Rotations
- Casimir setups do not move
- Setup located in vacuum
- Perfect metal plates
- $T = 0$

Setup *II* and *III*

$$\frac{F_C^{II}}{F_C^{III}} - 1 = 0 \, .$$

Test measurement.

Setup *I* and *III*

$$\frac{F_C^{I}}{F_C^{III}} - 1 = -\frac{5}{2}\lambda \, .$$

$\implies$ Use experiments on Earth and control the orientation towards the sun precisely.

Measurability

Metrological framework:

- AFM precision approximately 0.1pN
- Gravity correction $\propto \lambda \approx 10^{-5}$
- Casimir force for Lamoureux experiment (1997) [3] approximately 660nN with distances about 1μm
- $\Delta F_C^{\text{gravity}} \approx 6.6\text{pN} > 0.1\text{pN} \implies$ measurable today

Problems:

- Classical effects like roughness, finite conductivity, finite temperature
  $\implies$ Use the same plates on a pivoted stand at constant temperature

Slow rotation approximation

First extension $\Longrightarrow$ Slow rotation $\Longrightarrow$ Approximate the Kerr metric in first-order via

$$\mathrm{d}s^2_{\mathrm{SRA}} = c^2\left(1-\frac{r_s}{r}\right)\mathrm{d}t^2 + 2\frac{r_s}{r}a\sin^2(\vartheta)c\,\mathrm{d}t\,\mathrm{d}\varphi - \frac{1}{1-\frac{r_s}{r}}\mathrm{d}r^2 - r^2\left[\mathrm{d}\vartheta^2+\sin^2(\vartheta)\,\mathrm{d}\varphi^2\right] .$$

Ending up with a metric tensor $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$ and

$$h^{\mu\nu} = \lambda\begin{pmatrix} 1 & a\frac{y}{r^2-z^2} & -a\frac{x}{r^2-z^2} & 0 \\ a\frac{y}{r^2-z^2} & \frac{x^2}{r^2} & \frac{xy}{r^2} & \frac{xz}{r^2} \\ -a\frac{x}{r^2-z^2} & \frac{xy}{r^2} & \frac{y^2}{r^2} & \frac{yz}{r^2} \\ 0 & \frac{xz}{r^2} & \frac{yz}{r^2} & \frac{z^2}{r^2} \end{pmatrix}, \qquad \text{with } a = \frac{J}{Mc}$$

similar to Schwarzschild metric with $h^{10} = h^{01}$ and $h^{20} = h^{02}$ as new entries.

Results for SRA

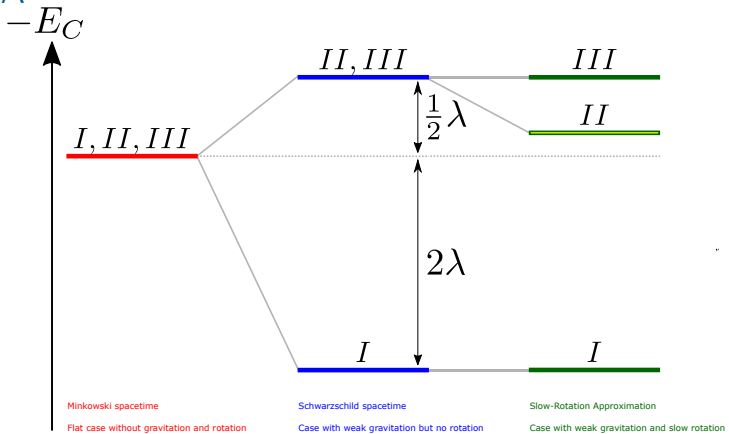
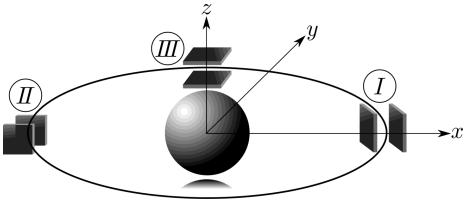
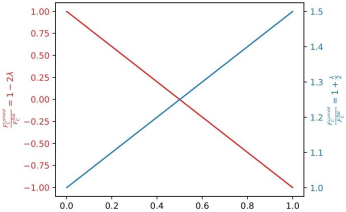


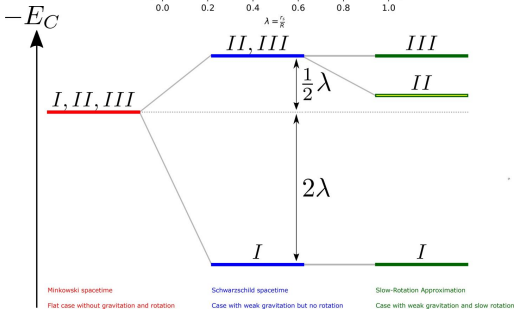
Figure 7: Energy splitting in SRA. Setup *I* in accordance with [1].

Conclusion





Object	r_s/R_{surface}	$r_{s\ \text{Sun}}/R_{\text{mean Sun}}$
Sun	$4.25 \cdot 10^{-6}$	not defined
Mercury	$2.01 \cdot 10^{-10}$	$5.10 \cdot 10^{-5}$
Venus	$1.19 \cdot 10^{-9}$	$2.73 \cdot 10^{-5}$
Earth	$1.39 \cdot 10^{-9}$	$1.97 \cdot 10^{-5}$
PSP	$8.48 \cdot 10^{-25}$	$4.80 \cdot 10^{-4}$



Compare different setups at the same location
→ The effects of gravity are measurable today

The orientation of the plates with respect to the source of gravity leads to fundamental differences!

[1] V. B. Bezerra, H. F. Mota, and C. R. Muniz. "Casimir effect due to a slowly rotating source in the weak-field approximation." In: *Physical Review D* 89.4 (2014). ISSN: 1550-7998. DOI: 10.1103/PhysRevD.89.044015.

[2] F. Cabral and F. S. N. Lobo. "Electrodynamics and Spacetime Geometry: Foundations." In: *Foundations of Physics* 47.2 (2017), pp. 208–228. ISSN: 0015-9018. DOI: 10.1007/s10701-016-0051-6.

[3] S. K. Lamoreaux. "Demonstration of the Casimir Force in the 0.6 to 6µm Range." In: *Physical Review Letters* 78.1 (1997), pp. 5–8. ISSN: 0031-9007. DOI: 10.1103/PhysRevLett.78.5.

[4] A. Liebig, A. Peronio, D. Meuer, A. J. Weymouth, and F. J. Giessibl. "High-precision atomic force microscopy with atomically-characterized tips." In: *New Journal of Physics* 22.6 (2020), p. 063040. DOI: 10.1088/1367-2630/ab8efd.

[5] C. W. Misner, K. S. Thorne, J. A. Wheeler, and D. I. Kaiser. *Gravitation*. First Princeton University Press edition. Princeton and Oxford: Princeton University Press, 2017. ISBN: 978-0-691-17779-3.

[6] S. Pieth and U. Köhler. *Our Solar System: A Short Introduction to the Bodies of our Solar System and their Exploration*. Ed. by German Aerospace Center. 2013. URL: http://europlanet.dlr.de/fileadmin/brochure_Solarsystem_2013_en_web.pdf (visited on July 21, 2022).

[7] The Johns Hopkins University Applied Physics Laboratory, ed. *THE MISSION: NASA's Parker Solar Probe mission will revolutionize our understanding of the Sun*. 2019. URL: http://parkersolarprobe.jhuapl.edu/The-Mission/index.php#Journey-to-the-Sun (visited on July 21, 2022).

[8] Wikipedia. *Illustration of the Casimir effect*. Ed. by Wikipedia. 2008. URL: https://commons.wikimedia.org/wiki/File:Casimir_plates.svg#/media/File:Casimir_plates.svg (visited on Nov. 21, 2021).