Parametric Resonance in Bose-Einstein Condensates

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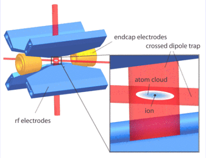
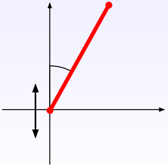
arXiv:1209.3148

Outline

Parametric resonance

- **Parametric oscillator:** harmonic oscillator with time-dependent parameters
- **Parametric resonance:** resonant behaviour of a parametric oscillator

<table>
<tr><th>Destabilization of stable equilibrium</th><th>Stabilization of unstable equilibrium</th></tr>
<tr><td>Swing
</td><td>Paul trap (Nobel Prize 1989)

</td></tr>
<tr><td colspan="2">Kapitza pendulum
</td></tr>
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Inverted pendulum with a vertically oscillated pivot

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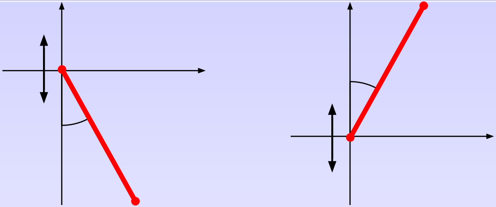
Pendulum physics



- Driving amplitude A, frequency Ω
- Equation of motion

$$\ddot{\varphi}(t) + \left(\frac{g}{l} + \frac{A\Omega^2}{l}\cos\Omega t\right)\sin\varphi(t) = 0$$

- Linearize:

$$\sin\varphi(t) \simeq \varphi(t)$$

- With definitions

$$c = \pm\tfrac{4g}{l\Omega^2} \qquad q = \mp\tfrac{2A}{l} \qquad 2t' = \Omega t \qquad x(t') = \varphi(t)$$

- Mathieu equation

$$\ddot{x}(t') + \left[c - 2q\cos 2t'\right]x(t') = 0$$

Mathieu equation

$$\ddot{x}(t') + \left[c - 2q\cos 2t'\right]x(t') = 0$$

- Floquet theory: on stability borders, $x(t')$ is π- or 2π-periodic.
- One method: Fourier series ansatz

$$x(t') = \sum_{n=0}^{\infty} A_n \cos(n\,t') + \sum_{n=1}^{\infty} B_n \sin(n\,t')$$

- Obtain decoupled systems

$$\sum_{n=0}^{\infty} A_n \Big[(c-n^2)\ \cos(n\,t') - q\ \cos\big((n-1)\,t'\big) - q\ \cos\big((n+1)\,t'\big)\Big] = 0$$

$$\sum_{n=1}^{\infty} B_n \Big[(c-n^2)\ \sin(n\,t') - q\ \sin\big((n-1)\,t'\big) - q\ \sin\big((n+1)\,t'\big)\Big] = 0$$

Mathieu equation

Continued

- Infinite matrix equations – truncate for approx. solution
- Vanishing determinants for nontrivial A_n, B_n

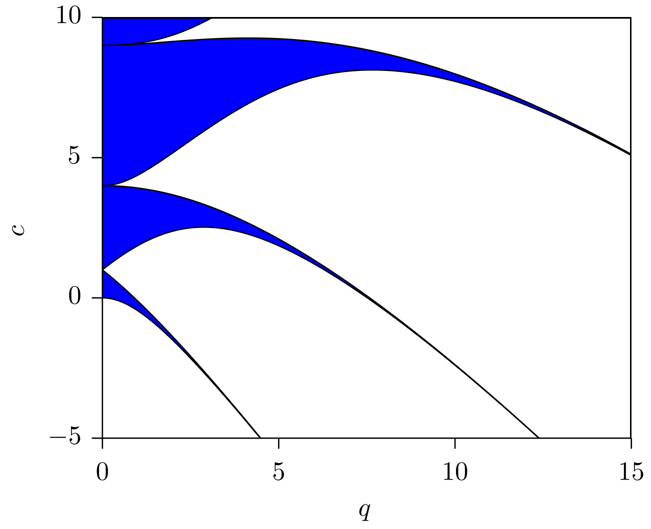
$$\begin{bmatrix} c & -q & 0 & \\ -2q & c-4 & -q & \cdots \\ 0 & -q & c-16 & \\ & \vdots & & \ddots \end{bmatrix} \begin{bmatrix} A_0 \\ A_2 \\ A_4 \\ \vdots \end{bmatrix} = \mathbf{0}, \quad \begin{bmatrix} c-4 & -q & 0 & \\ -q & c-16 & -q & \cdots \\ 0 & -q & c-36 & \\ & \vdots & & \ddots \end{bmatrix} \begin{bmatrix} B_0 \\ B_2 \\ B_4 \\ \vdots \end{bmatrix} = \mathbf{0},$$

$$\begin{bmatrix} c & -q & 0 & \\ -2q & c-1 & -q & \cdots \\ 0 & -q & c-9 & \\ & \vdots & & \ddots \end{bmatrix} \begin{bmatrix} A_1 \\ A_3 \\ A_5 \\ \vdots \end{bmatrix} = \mathbf{0}, \quad \begin{bmatrix} c-1 & -q & 0 & \\ -q & c-9 & -q & \cdots \\ 0 & -q & c-25 & \\ & \vdots & & \ddots \end{bmatrix} \begin{bmatrix} B_1 \\ B_3 \\ B_5 \\ \vdots \end{bmatrix} = \mathbf{0}$$

- (q, c) for vanishing determinant gives stability borders

Mathieu equation

Stability diagram



Bose-Einstein Condensate

- Extreme Tunability of Interactions in a ^{7}Li Bose-Einstein Condensate
  S. E. Pollack et al., PRL 102, 090402 (2009)
- Tuning of scattering length by Feshbach resonance

$$a(B) = a_{\rm BG}\left(1 - \frac{\Delta}{B - B_\infty}\right)$$

- Collective excitation of a Bose-Einstein condensate by modulation of the atomic scattering length
  K. M. F. Magalhães et al., PRA 81, 053627 (2010)

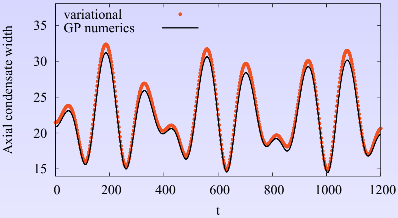
$$B(t) = B_{\rm av} + \delta_B \cos\Omega t, \qquad a = a_{\rm av} + \delta_a \cos\Omega t$$

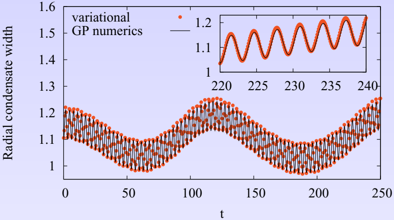
where

$$a_{\rm av} = a(B_{\rm av}), \qquad \delta_a = \frac{a_{\rm BG}\Delta\delta_B}{(B_{\rm av} - B_\infty)^2}$$

Bose-Einstein Condensate

- Analogous stability behaviour for BEC?





- Excitation of Bose-Einstein Condensates (BECs) by harmonic modulation of the scattering length
  I. Vidanović, A. Balaž, H. Al-Jibbouri, and A. Pelster, PRA 84, 013618 (2011).
- Geometric Resonances in Bose-Einstein Condensates with Two- and Three-Body Interactions
  H. Al-Jibbouri, I. Vidanović, A. Balaž, and A. Pelster, arXiv:1208.0991.
- Excellent agreement with Gross-Pitaevskii Equation

Variational approach

- Lagrangian

$$L(t) = \int \mathcal{L}(\mathbf{r},t)\, d\mathbf{r}\,,$$

- Lagrange density

$$\mathcal{L}(\mathbf{r},t) = \frac{i\hbar}{2}\left(\psi\frac{\partial\psi^*}{\partial t} - \psi^*\frac{\partial\psi}{\partial t}\right) - \frac{\hbar^2}{2m}|\nabla\psi|^2 - V(\mathbf{r})|\psi|^2 - \frac{g}{2}|\psi|^4$$

- Gaussian variational ansatz
  Phys. Rev. Lett. 77, 5320 (1996)
  Phys. Rev. A 56, 1424 (1997)

$$\psi^{\mathrm{G}}(\rho,z,t) = \mathcal{N}(t)\exp\left[-\frac{1}{2}\left(\frac{\rho^2}{\tilde{u}_\rho(t)^2} + \frac{z^2}{\tilde{u}_z(t)^2}\right) + i\left(\rho^2\phi_\rho(t) + z^2\phi_z(t)\right)\right]$$

- Time-dependent normalization

$$\mathcal{N}(t) = \frac{1}{\sqrt{\pi^{\frac{3}{2}}\tilde{u}_\rho^2(t)\tilde{u}_z(t)}}$$

Variational approach
continued

- Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0, \qquad q \in \{\tilde{u}_i, \phi_i\}$$

- Phases

$$\phi_\rho(t) = \frac{m\dot{\tilde{u}}_\rho}{2\hbar\tilde{u}_\rho}, \qquad \phi_z(t) = \frac{m\dot{\tilde{u}}_z}{2\hbar\tilde{u}_z}$$

- Dimensionless parameters:

$$\tau = \omega_\rho t, \quad u_i(\tau) = \frac{\tilde{u}_i(t)}{a_{\rm ho}}, \quad a_{\rm ho} = \sqrt{\frac{\hbar}{m\omega_\rho}}$$

- Dimensionless driving

$$p(\tau) = p_0 + p_1 \cos\left(\frac{\Omega\tau}{\omega_\rho}\right), \quad p_0 = \sqrt{\frac{2}{\pi}}\frac{N a_{\rm av}}{a_{\rm ho}}, \quad p_1 = \sqrt{\frac{2}{\pi}}\frac{N\delta_a}{a_{\rm ho}}$$

Equations of motion

- Equations of motion

$$\ddot{u}_\rho + u_\rho = \frac{1}{u_\rho^3} + \frac{p(\tau)}{u_\rho^3\, u_z}, \qquad \ddot{u}_z + \lambda^2 u_z = \frac{1}{u_z^3} + \frac{p(\tau)}{u_\rho^2\, u_z^2}$$

- Isotropic condensate: $u_\rho = u_z = u$ and $\lambda = 1$
- Reduction to one ODE:

$$\ddot{u} + u = \frac{1}{u^3} + \frac{p(\tau)}{u^4}$$

- Stationary solutions:

$$u_{\rho 0} = \frac{1}{u_{\rho 0}^3} + \frac{p_0}{u_{\rho 0}^3\, u_{z0}}, \qquad \lambda^2 u_{z0} = \frac{1}{u_{z0}^3} + \frac{p_0}{u_{\rho 0}^2\, u_{z0}^2}$$

- Isotropic case:

$$u_0 = \frac{1}{u_0^3} + \frac{p_0}{u_0^4}$$

Equations of motion
Equilibrium position continued

- Equilibrium condition: $u_0^5 - u_0 = p_0$ (isotropic condensate)

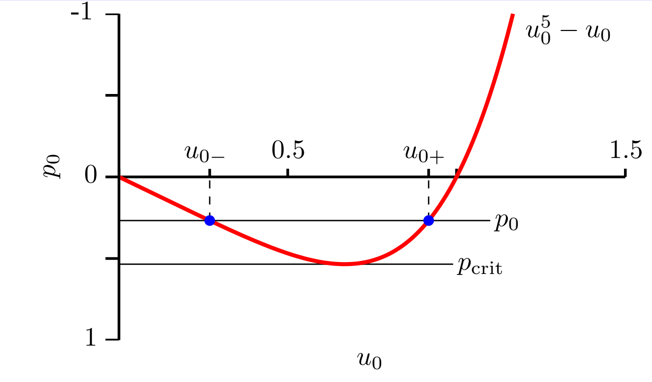


Figure: Equilibrium widths $u_{0\pm}$ of a Bose-Einstein Condensate subject to attractive interactions.

Equations of motion

Mathieu equation

- Linearize about equilibrium position u_0

$$u(\tau) = u_0 + \delta u(\tau)$$

- Taylor expand nonlinear terms to first order in δu

$$\frac{1}{(u_0+\delta u)^3} = \frac{1}{u_0^3} - 3\,\frac{\delta u}{u_0^4} + \ldots, \qquad \frac{1}{(u_0+\delta u)^4} = \frac{1}{u_0^4} - 4\,\frac{\delta u}{u_0^5} + \ldots$$

- With definitions

$$q = -\frac{8\,p_1}{u_0^5}\left(\frac{\omega}{\Omega}\right)^2 \qquad 2t' = \frac{\Omega\,\tau}{\omega_\rho}$$

$$c = 4\left(\frac{\omega}{\Omega}\right)^2\left(5 - \frac{1}{u_0^4}\right) \qquad x(t') = \delta u(\tau)$$

- Obtain an inhomogeneous Mathieu equation

$$\ddot{x}(t') + \left[c - 2\,q\cos(2\,t')\right]x(t') = -\frac{u_0}{2}\,q\cos(2\,t')$$

Isotropic stability

Non-homogeneous term

- Stability unaffected by non-homogeneous term

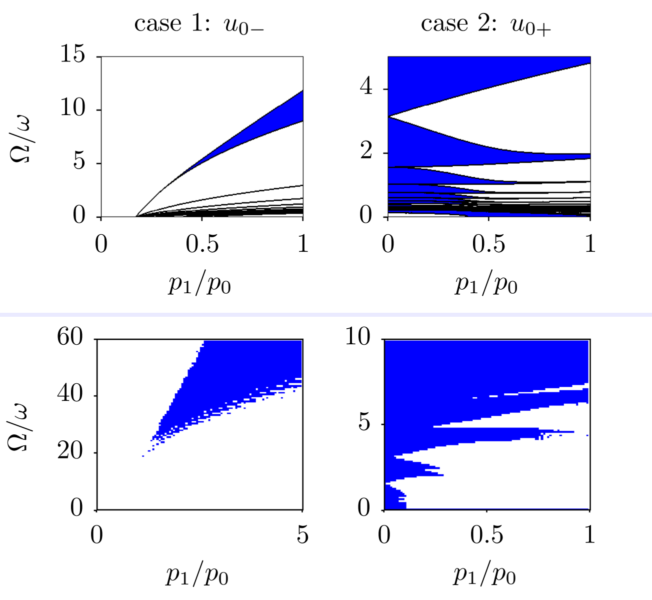
$$\ddot{x}(t') + \left[c - 2q\cos 2t'\right] x(t') = -\frac{u_0}{2} q \cos 2t'$$

- Infinite determinant method:

$$\underbrace{\begin{bmatrix} c & -q & 0 & \\ -2\,q & c-4 & -q & \cdots \\ 0 & -q & c-16 & \\ & \vdots & & \ddots \end{bmatrix}}_{M} \begin{bmatrix} A_0 \\ A_2 \\ A_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{u_0}{2} q \\ 0 \\ \vdots \end{bmatrix}$$

- Coefficients: $A_n \sim (\det M)^{-1}$
- Stability borders $\iff$ coefficients diverge
- Transform diagram for relevant parameters

Isotropic stability
Results



Anisotropic stability

Coupled Mathieu equations

- Equations of motion

$$\ddot{u}_\rho + u_\rho = \frac{1}{u_\rho^3} + \frac{p(\tau)}{u_\rho^3\, u_z}, \quad \ddot{u}_z + \lambda^2 u_z = \frac{1}{u_z^3} + \frac{p(\tau)}{u_\rho^2\, u_z^2}$$

- Linearize: $u_i = u_{i0} + \delta u_i$
- Definitions:

$$2t' = \frac{\Omega\tau}{\omega_\rho}, \qquad q = p_1,$$

$$\mathbf{x}(t') = \begin{pmatrix} \delta u_\rho(\tau) \\ \delta u_z(\tau) \end{pmatrix}, \qquad \mathbf{A} = 4\left(\frac{\omega_\rho}{\Omega}\right)^2 \begin{pmatrix} 4 & \frac{p_0}{u_{\rho 0}^3 u_{z0}^2} \\ \frac{2p_0}{u_{\rho 0}^3 u_{z0}^2} & 3\lambda^2 + \frac{1}{u_{z0}^4} \end{pmatrix},$$

$$\mathbf{f} = 4\left(\frac{\omega_\rho}{\Omega}\right)^2 \begin{pmatrix} \frac{p_1}{u_{\rho 0}^3 u_{z0}} \\ \frac{p_1}{u_{\rho 0}^2 u_{z0}^2} \end{pmatrix}, \qquad \mathbf{Q} = -2\left(\frac{\omega_\rho}{\Omega}\right)^2 \begin{pmatrix} \frac{3}{u_{\rho 0}^4 u_{z0}} & \frac{1}{u_{\rho 0}^3 u_{z0}^2} \\ \frac{2}{u_{\rho 0}^3 u_{z0}^2} & \frac{2}{u_{\rho 0}^2 u_{z0}^3} \end{pmatrix}$$

- Coupled, inhomogeneous Mathieu equations:

$$\ddot{\mathbf{x}}(t') + [\mathbf{A} - 2q\,\mathbf{Q}\cos(2t')]\,\mathbf{x}(t') = \mathbf{f}\cos(2t')$$

Anisotropic stability

Coupled Mathieu equations continued

- Non-homogeneity does not affect stability – **J. Slane et al., J. Nonlinear Dynamics and Systems Theory, 11 (2) (2011).**
- Floquet ansatz:

$$\mathbf{x}(t') = \sum_{n=-\infty}^{\infty} \mathbf{u}_{2n} e^{(\beta+2in)t'}$$

- Recursion relation

$$\left[\mathbf{A} + (\beta + 2in)^2 \mathbf{I}\right] \mathbf{u}_{2n} - q\mathbf{Q}\left(\mathbf{u}_{2n+2} + \mathbf{u}_{2n-2}\right) = \mathbf{0}$$

- Ladder operators

$$\mathbf{S}_{2n}^{\pm} = \left\{\mathbf{A} + \left[\beta + 2i\left(n+1\right)\right]^2 \mathbf{I} - q\mathbf{Q}\mathbf{S}_{2n\pm2}^{\pm}\right\}^{-1} q\mathbf{Q}$$

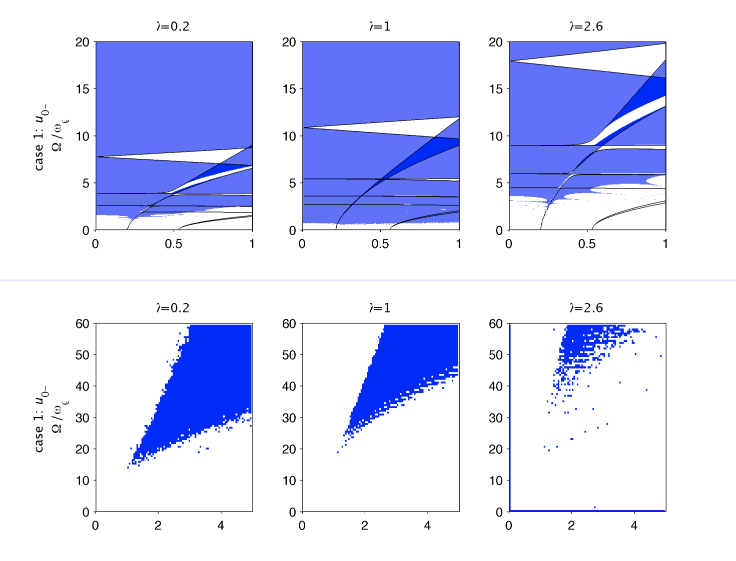
- Continued matrix inversion

$$\left(\mathbf{A} + \beta^2\mathbf{I} - q^2\mathbf{Q}\left\{\left[\mathbf{A} + (\beta+2i)^2 - \dots\right]^{-1} + \left[\mathbf{A} + (\beta-2i)^2 - \dots\right]^{-1}\right\}\mathbf{Q}\right)\mathbf{u}_0 = \mathbf{0}$$

- Vanishing determinant for stability borders

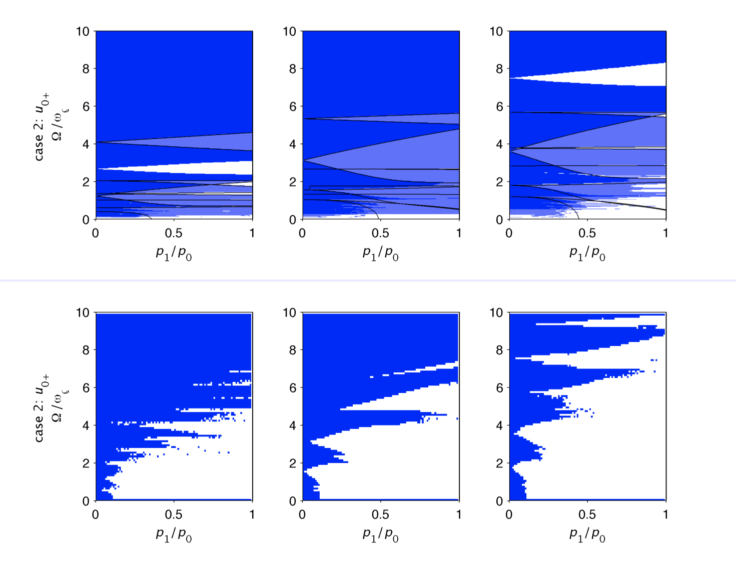
Anisotropic stability

Results, case 1: $\mathbf{u}_{0-}$



Anisotropic stability

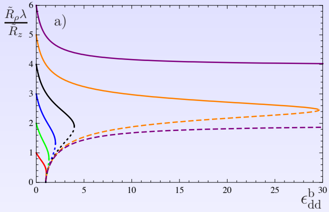
Results, case 2: $\mathbf{u}_{0+}$



Conclusions and Outlook

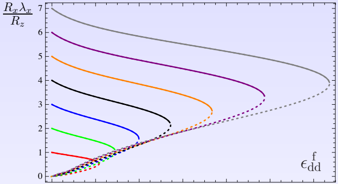
- Analogous physics: BEC and pendulum
- Stabilize unstable equilibrium
- Experimental possibilities

Dipolar BEC



$$\epsilon_{dd}^{b} = \frac{C_{dd}}{3g},$$

Dipolar Fermi Gas



$$\epsilon_{dd}^{f} = \frac{C_{dd}}{4\pi} \left(\frac{M^3 \bar{\omega}^3}{\hbar^5} \right)^{1/2} N^{1/6}$$

- **A.R.P. Lima and A. Pelster, PRA 81, 021606(R)/1-4 (2010)** and **PRA 84, 041604(R)/1-4 (2011)**

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