Collapse and Revival of Matter Waves in Bosonic Optical Lattices

Ednilson Santos, Axel Pelster

1. Theoretical description
2. Experiments
3. Effective action
4. Momentum space distribution
5. Comparison with experiments

Freie Universität Berlin

UNIVERSITÄT DUISBURG ESSEN

Landau theory: dos Santos and Pelster, PRA 79, 013614 (2009)

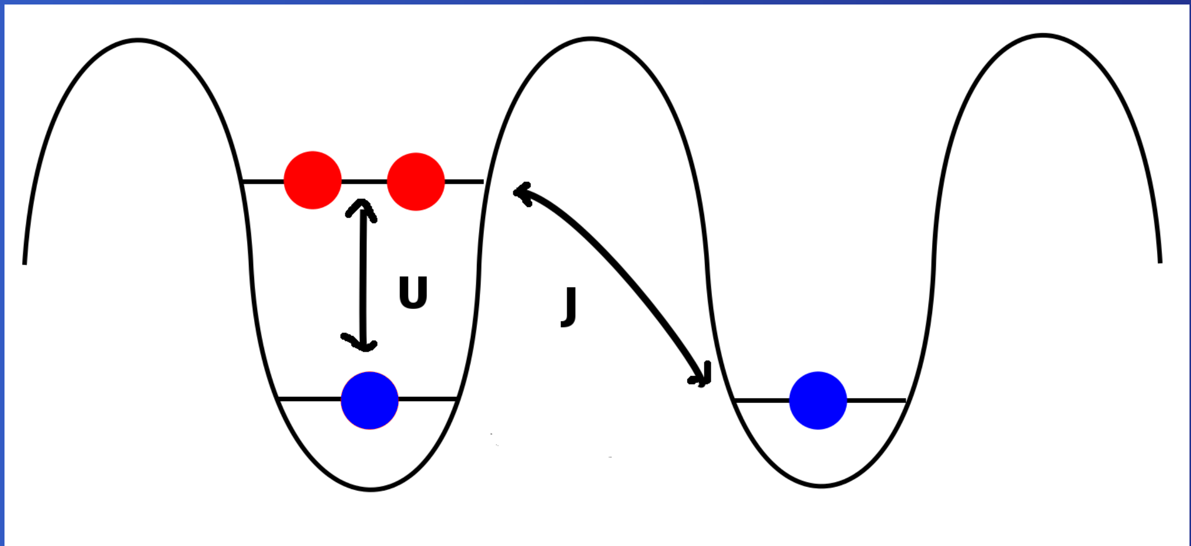
Ginzburg-Landau theory: Bradlyn, dos Santos, and Pelster, PRA 79, 013615 (2009)

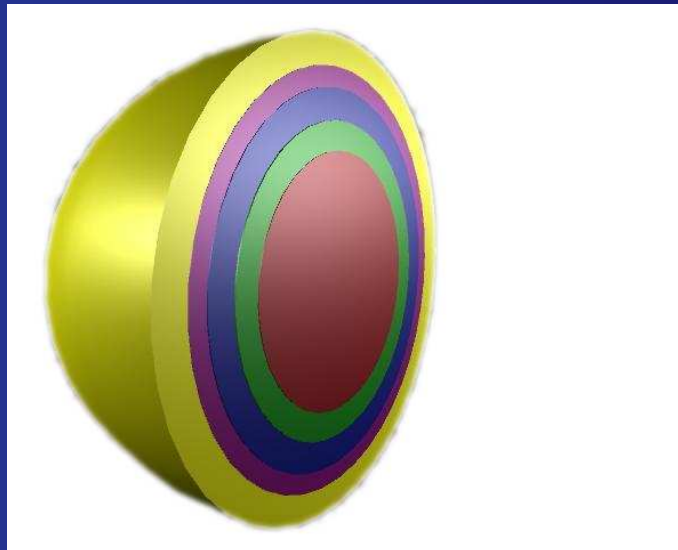
1 - Theoretical description

Inhomogeneous Bose-Hubbard Hamiltonian:

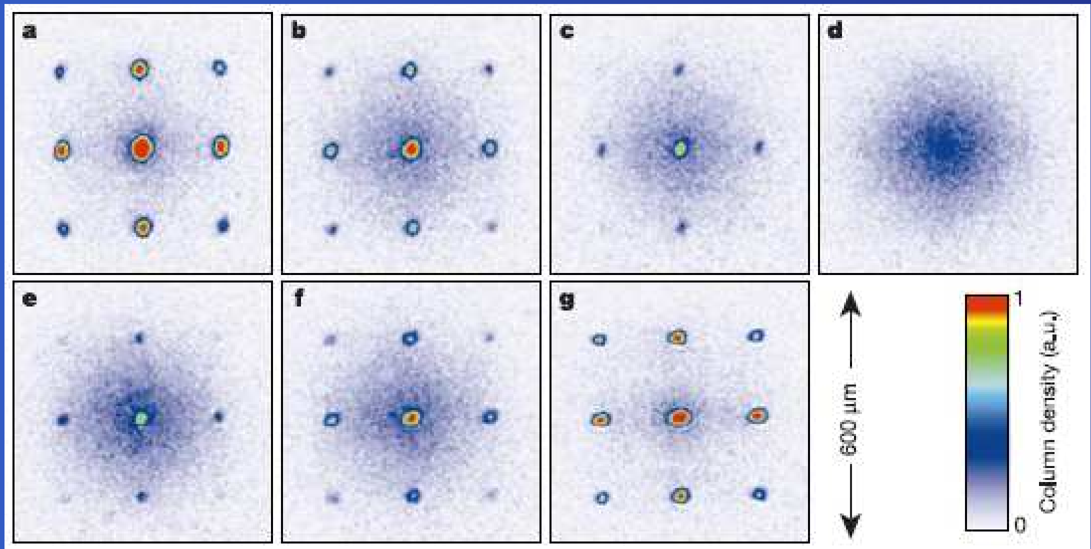
$$\hat{H}_{\mathrm{BH}} = -J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + \sum_i \left[\frac{U}{2} \hat{n}_i (\hat{n}_i - 1) - \mu_i \hat{n}_i \right],$$

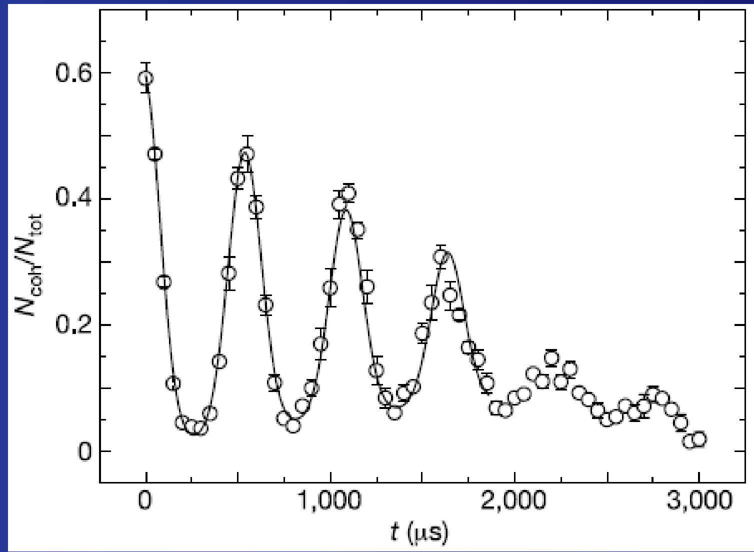
$$\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i \qquad ; \qquad \mu_i = \mu - m \frac{\omega^2}{2} \mathbf{x}_i^2$$





2 - Experiments





- Collapse and revival was observed in a sample of 2×10^5 ^{87}Rb atoms - Nature 419, 51 (2002)
- Periodic potential depth was suddenly changed from $V_A = 8\ E_r$ to $V_A = 22\ E_r$
- Condensed fraction was extracted from $130\ \mu m \times 130\ \mu m$ squares around the interference peaks

3 - Ginzburg-Landau Theory

External sources break global symmetries:

$$\hat{H}(t) = \hat{H}_{\mathrm{BH}} + \sum_i \left[j_i(t)\hat{a}_i^\dagger + j_i^*(t)\hat{a}_i \right]$$

Generating functional $(\hbar = 1)$:

$$\mathcal{Z}\left\{ j_i^*(t), j_i(t) \right\} = \langle 0|\hat{T}e^{-i\int dt\,\hat{H}(t)}|0\rangle$$

Generating functional of connected correlation functions:

$$\mathcal{F}\left\{ j_i^*(t), j_i(t) \right\} = i \ln \mathcal{Z}\left\{ j_i^*(t), j_i(t) \right\}$$

Order parameter field:

$$\psi_i(t) = \langle \hat{a}_i(t) \rangle = \frac{\delta \mathcal{F}}{\delta j_i^*(t)}$$

Effective action:

$$\Gamma\left\{ \psi^*, \psi \right\} = \mathcal{F}\left\{ j^*, j \right\} - \int dt \left[\psi_i(t) j_i^*(t) + \psi_i^*(t) j_i(t) \right]$$

3.1 - Order field dynamics

Equations of motion:

$$\frac{\delta\Gamma\left\{\psi^*,\psi\right\}}{\delta\psi_i^*(t)}=0 \qquad ; \qquad \frac{\delta\Gamma\left\{\psi^*,\psi\right\}}{\delta\psi_i(t)}=0$$

Linearized equation for zeroth order in J:

$$\int dt'\Gamma_i^{(2)}(t-t')\psi_i(t')=0$$

Frequency space: $\Gamma_i^{(2)}(\omega)\psi_i(\omega)=0$

with: $\Gamma_i^{(2)}(\omega)=\frac{-i}{U+\mu_i+\omega}\left[\omega-(Un_i-\mu_i)\right]\left\{\omega-\left[U(n_i-1)-\mu_i\right]\right\}$

General solution:

$$\psi_i(\omega)=A_i^+\delta\left[\omega-(Un_i-\mu_i)\right]+A_i^-\delta\left\{\omega-\left[U(n_i-1)-\mu_i\right]\right\}$$

3.2 - Collapse and revival

General solution in time domain:

$$\psi_i(t) = A_i^+ e^{-it(Un_i-\mu_i)} + A_i^- e^{-it[U(n_i-1)-\mu_i]}$$

Local collapse and revival:

$$\psi_i(t)^*\psi_i(t) = |A_i^+|^2+|A_i^-|^2+2|A_i^+||A_i^-|\cos\left\{\left[U-(\theta_{A_i^+}-\theta_{A_i^-})\right]t\right\}$$

Condensed fraction for $\theta_{A_i^+} = \theta_{A_i^-}$:

$$N(t) = \sum_i |\psi_i(t)|^2 = \sum_i \left(|A_i^+|^2 + |A_i^-|^2\right) + 2\cos(Ut)\sum_i |A_i^+||A_i^-|$$

Damping occurs when A_i^+ and A_i^- are out of phase:

$$\lim_{t\to+\infty} N(t) = \sum_i \left(|A_i^+|^2 + |A_i^-|^2\right)$$

3.3 - Initial conditions

Lattice Gross-Pitaevskii equation deep in superfluid phase:

$$i\dot{\phi}_i = -J \sum_{j\in \text{ n.n. of } i} \phi_j + U\phi_i|\phi_i|^2 - \mu_i\phi_i$$

Thomas-Fermi approximation for static solution:

$$\phi_i(0) = \sqrt{\frac{\mu_{0i} + zJ}{U_0}} \qquad ; \qquad \mu_{0i} = \mu_0 - \frac{m\omega_0^2}{2}\mathbf{x}_i^2$$

Lattice Gross-Pitaevskii equation has exact solution for $J = 0$:

$$\phi_i(t) = \phi_i(0)e^{-i\left(U|\phi_i(0)|^2-\mu_i\right)t}$$

A_i^+ and A_i^- are determined from $\phi_i(0)$ and $\dot{\phi}_i(0)$:

$$A_i^+ = (\phi_i(0)^2 - n_i + 1)\phi_i(0) \quad ; \quad A_i^- = -(\phi_i(0)^2 - n_i)\phi_i(0)$$

4.1 - Dispersion in momentum space

Momentum space wave function:

$$\psi(\mathbf{k},t) = w(\mathbf{k}) \sum_i \psi_i(t) e^{i\mathbf{k}\cdot\mathbf{x}_i}$$

Contributions from different shells:

$$\psi_n^+(\mathbf{k},t) = e^{-it(Un-\mu)} w(\mathbf{k}) \chi_n^+(\mathbf{k},t);$$

$$\psi_n^-(\mathbf{k},t) = e^{-it[U(n-1)-\mu]} w(\mathbf{k}) \chi_n^-(\mathbf{k},t)$$

$$\chi_n^\pm(\mathbf{k},t) = \sum_{i \in \text{nth shell}} A_i^\pm e^{i(\mathbf{k}\cdot\mathbf{x}_i - \frac{m\omega^2}{2}\mathbf{x}_i^2 t)}$$

$\chi_n^\pm$ obey a dispersive equation: $i\frac{\partial \chi_n^\pm(\mathbf{k},t)}{\partial t} = -(\frac{2}{m\omega^2})\nabla_{\mathbf{k}}^2 \chi_n^\pm(\mathbf{k},t)$

4.2 - Long-time limit

Momentum space distribution:

$$|\psi(\mathbf{k},t)|^2 = |w(\mathbf{k})|^2 \sum_{ij} \psi_j^*(t)\psi_i(t) e^{i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}$$

After integrating in k_z:

$$\rho(\mathbf{k}_\perp) = |w(\mathbf{k}_\perp)|^2 \sum_{ij} G(\mathbf{r}_{\perp j};\mathbf{r}_{\perp i};t) e^{-i\frac{m\omega^2}{2}t(\mathbf{r}_{\perp i}^2-\mathbf{r}_{\perp j}^2)} e^{i\mathbf{k}_\perp\cdot(\mathbf{r}_{\perp i}-\mathbf{r}_{\perp j})}$$

with:

$$G(\mathbf{r}_{\perp j};\mathbf{r}_{\perp i};t) = \sum_{z_l} \{A^+(\mathbf{r}_{\perp i},z_l)A^+(\mathbf{r}_{\perp j},z_l) + A^-(\mathbf{r}_{\perp i},z_l)A^-(\mathbf{r}_{\perp j},z_l)$$
$$+\cos(Ut)\left[A^+(\mathbf{r}_{\perp i},z_l)A^-(\mathbf{r}_{\perp j},z_l) + A^-(\mathbf{r}_{\perp i},z_l)A^+(\mathbf{r}_{\perp j},z_l)\right]\}$$

Long-time limit $t \to \infty$:

$$\rho(\mathbf{k}_\perp) \approx |w(\mathbf{k}_\perp)|^2 G(0;0;t)\left(\frac{2\pi}{m\omega^2 a^2 t}\right)^2$$

5.1 - Condensate fraction

Experiments observe a small region in $\mathbf{k}_\perp$ space:

$$N_c \approx \rho(\mathbf{k_r} = 0)(\delta k)^2 = (\delta k)^2 |w(0)|^2 G(0;0;t) \left(\frac{2\pi}{m\omega^2 a^2 t} \right)^2$$

δk relates to time-of-flight measurement:

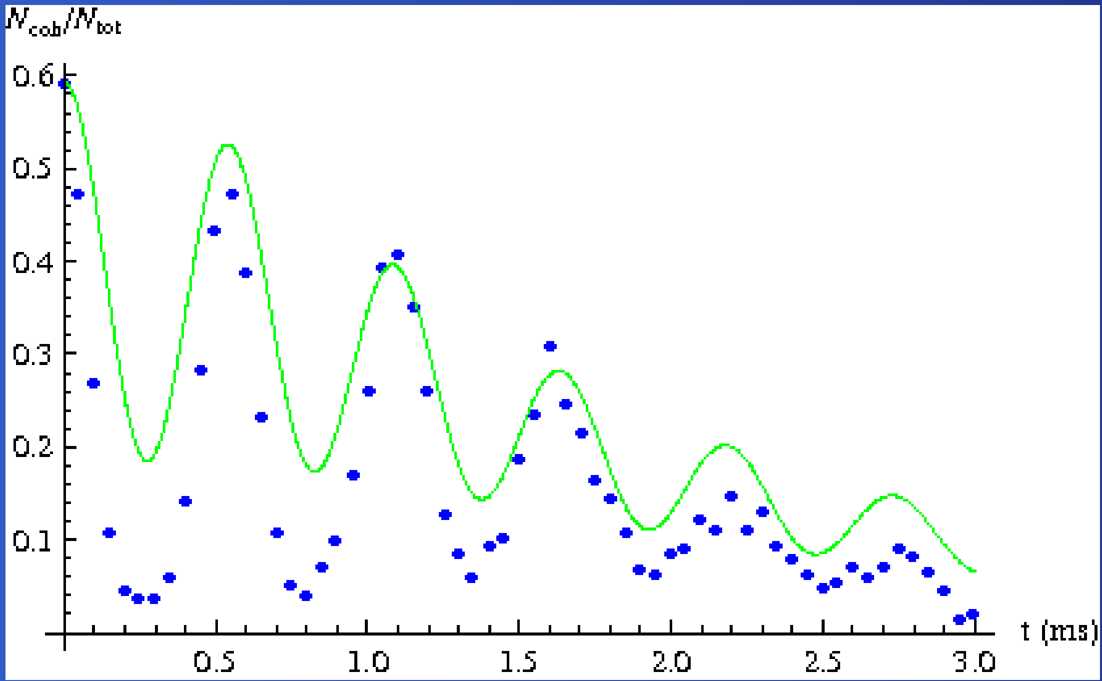
$$\delta k = \frac{m\delta x}{\hbar t_f}$$

Wannier function in harmonic approximation:

$$w(k_x) = \sqrt{\frac{a}{\pi^2}} \left(\frac{\pi^2}{V_0/E_r} \right)^{1/8} \exp\left(-\frac{a^2 k_x^2}{2\pi^2 \sqrt{V_0/E_r}} \right)$$

Final result: $N_c \approx \frac{4\delta k^2}{\pi a^2 m^2 \omega^4 \sqrt{V_0/E_r}} \frac{G(0;0;t)}{t^2}$

5.2 - Comparison with experiments



Blue dots: experimental results

Solid green: Initially deep in superfluid phase

- The revival time $h/U = 0.55\ \mathrm{ms}$ was calculated numerically.
- Interpolation between large- and small-time asymptotic limits was obtained from first Padé approximant.