On the Dirty Boson Problem

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1. Experimental Realizations of Dirty Bosons
2. Theoretical Description of Dirty Bosons
3. Huang-Meng Theory (T=0)
4. Shift of Condensation Temperature
5. Hartree-Fock Mean-Field Theory
6. Summary and Outlook

SFB/TR 12: Symmetries and Universality in Mesoscopic Systems

1.1 Overview of Set-Ups

- **Superfluid Helium in Porous Media:** (persistence of superfluidity)

  Reppy *et al.*, PRL **51**, 666 (1983)

- **Laser Speckles:** (controlled randomness)

  Billy *et al.*, Nature **453**, 891 (2008)

- **Wire Traps:** (undesired randomness)

  Krüger *et al.*, PRA **76**, 063621 (2007)

  Fortàgh and Zimmermann, RMP **79**, 235 (2007)

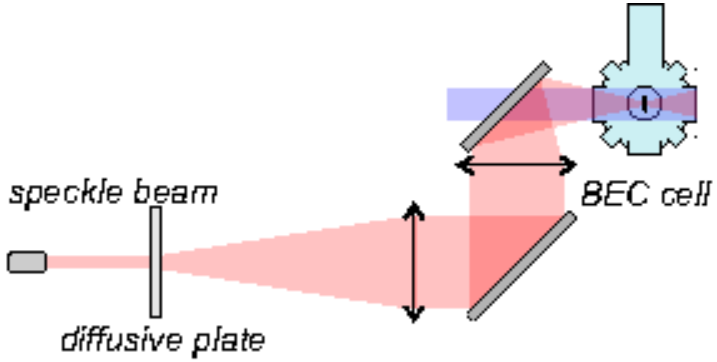
- **Localized Atomic Species:** (theoretical suggestion)

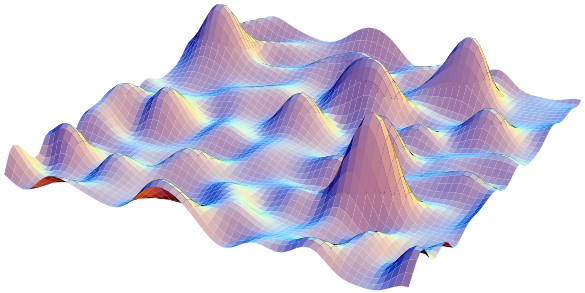
  Gavish and Castin, PRL **95**, 020401 (2005)

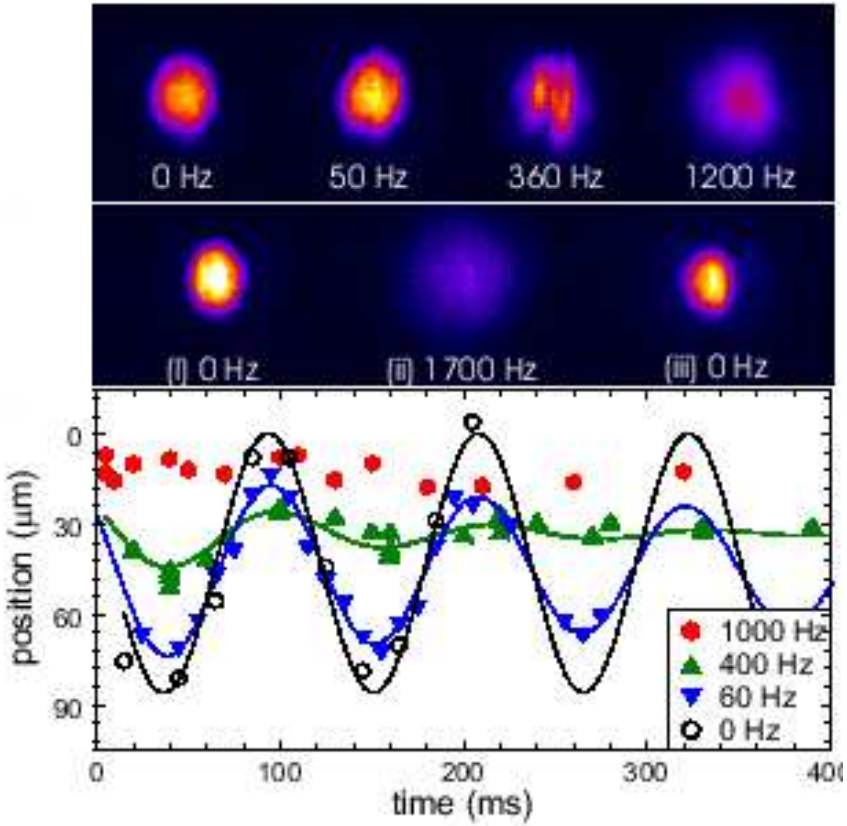
- **Incommensurate Lattices:** (quasi-randomness)

  Roati *et al.*, Nature **453**, 895 (2008)

1.2 Laser Speckles



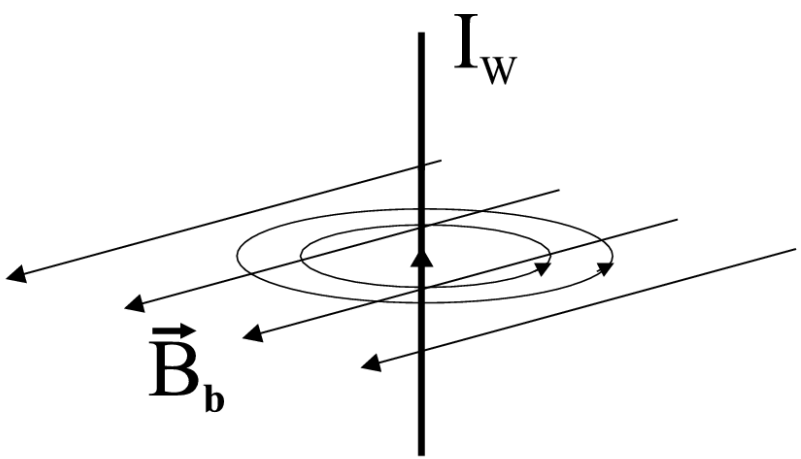


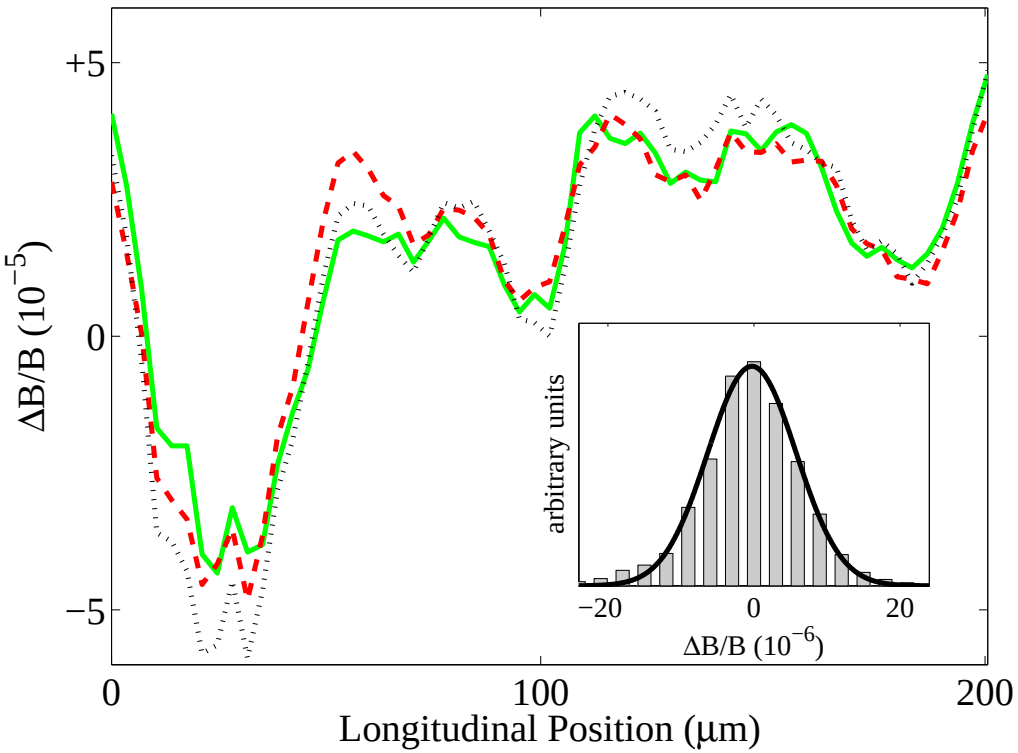


Lye *et al.*, PRL **95**, 070401 (2005)

global condensate vanishes

1.3 Wire Trap





Distance: $d = 10\ \mu$m

Wire Width: $100\ \mu$m

Magnetic Field: 10 G, 20 G, 30 G

Deviation: $\Delta B/B \approx 10^{-4}$

Krüger *et al.*, PRA **76**, 063621 (2007)

Fortàgh and Zimmermann, RMP **79**, 235 (2007)

2.1 Model System

Action of a Bose Gas:

$$\mathcal{A} = \int_0^{\hbar\beta} d\tau \int d^3x \left\{ \psi^* \left[\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2M} \boldsymbol{\Delta} + \boldsymbol{U}(\mathbf{x}) + \boldsymbol{V}(\mathbf{x}) - \boldsymbol{\mu} \right] \psi + \frac{\boldsymbol{g}}{2} \psi^{*2} \psi^2 \right\}$$

Properties:

- harmonic trap potential: $\boldsymbol{U}(\mathbf{x}) = \frac{M}{2}\omega^2 \mathbf{x}^2$
- disorder potential: $\boldsymbol{V}(\mathbf{x})$; bounded from below, i.e. $V(\mathbf{x}) \geq V_0$

$$\overline{V(\mathbf{x}_1)} = 0\,, \qquad \overline{V(\mathbf{x}_1)V(\mathbf{x}_2)} = R(\mathbf{x}_1 - \mathbf{x}_2)\,, \qquad \ldots$$

- chemical potential: $\boldsymbol{\mu}$
- repulsive interaction: $\boldsymbol{g} = \frac{4\pi\hbar^2 a}{M}$
- periodic Bose fields: $\psi(\mathbf{x}, \tau + \hbar\beta) = \psi(\mathbf{x}, \tau)$

2.2 Random Potential

Disorder Ensemble Average:

$$\overline{\bullet} = \int \mathcal{D}V \bullet P[V]\,, \qquad \int \mathcal{D}V\, P[V] = 1\,, \qquad P\left[V < V_0\right] = 0$$

Assumption:

$$\overline{V(\mathbf{x}_1)} = 0\,, \qquad \overline{V(\mathbf{x}_1)V(\mathbf{x}_2)} = R^{(2)}(\mathbf{x}_1 - \mathbf{x}_2) = \frac{R}{(2\pi\xi^2)^{3/2}}\, e^{-\frac{(\mathbf{x}_1-\mathbf{x}_2)^2}{2\xi^2}}$$

Characteristic Functional:

$$\overline{\exp\left\{ i \int d^D x\, j(\mathbf{x}) V(\mathbf{x}) \right\}}$$

$$= \exp\left\{ \sum_{n=2}^{\infty} \frac{i^n}{n!} \int d^D x_1 \cdots \int d^D x_n\, R^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n)\, j(\mathbf{x}_1) \cdots j(\mathbf{x}_n) \right\}$$

2.3 Grand-Canonical Potential

Aim:

$$\begin{aligned}\Omega &= -\frac{1}{\beta}\overline{\ln \mathcal{Z}} \\ \mathcal{Z} &= \oint D\psi D\psi^* \, e^{-\mathcal{A}[\psi^*,\psi]/\hbar}\end{aligned}$$

Problem:

$$\overline{\ln \mathcal{Z}} \neq \ln \overline{\mathcal{Z}}$$

Solution: Replica Trick

$$\Omega = -\frac{1}{\beta}\lim_{N\to 0}\frac{\overline{\mathcal{Z}^N}-1}{N}$$

2.4 Replica Trick

Disorder Averaged Partition Function:

$$\overline{\mathcal{Z}^N} = \overline{\oint \left\{ \prod_{\alpha'=1}^{N} D^2\psi_{\alpha'} \right\} e^{-\sum_{\alpha=1}^{N} \mathcal{A}([\psi_\alpha^*,\psi_\alpha])/\hbar}} = \oint \left\{ \prod_{\alpha=1}^{N} D^2\psi_\alpha \right\} e^{-\mathcal{A}^{(N)}/\hbar}$$

Replicated Action:

$$\mathcal{A}^{(N)} = \int_0^{\hbar\beta} d\tau \int d^D x \sum_{\alpha=1}^{N} \left\{ \psi_\alpha^*(\mathbf{x},\tau) \left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2M}\mathbf{\Delta} + U(\mathbf{x}) - \mu \right] \psi_\alpha(\mathbf{x},\tau) \right.$$

$$\left. + \frac{g}{2} \left|\psi_\alpha(\mathbf{x},\tau)\right|^4 \right\} + \sum_{n=2}^{\infty} \frac{1}{n!} \left(\frac{-1}{\hbar}\right)^{n-1} \int_0^{\hbar\beta} d\tau_1 \cdots \int_0^{\hbar\beta} d\tau_n \int d^D x_1 \cdots \int d^D x_n$$

$$\times \sum_{\alpha_1=1}^{N} \cdots \sum_{\alpha_n=1}^{N} R^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n) \left|\psi_{\alpha_1}(\mathbf{x}_1,\tau_1)\right|^2 \cdots \left|\psi_{\alpha_n}(\mathbf{x}_n,\tau_n)\right|^2$$

$\Longrightarrow$ Disorder amounts to attractive interaction for $n = 2$

$\Longrightarrow$ Higher-order disorder cumulants negligible in replica limit $N \to 0$

3.1 Condensate Density

Assumptions:

homogeneous Bose gas: $U(\mathbf{x}) = 0$

δ-correlated disorder: $R(\mathbf{x}) = R\,\delta(\mathbf{x})$

Bogoliubov Theory:

background method: $\psi_\alpha(\mathbf{x},\tau) = \Psi_\alpha + \delta\psi_\alpha(\mathbf{x},\tau)$

replica symmetry: $\Psi_\alpha = \sqrt{n_0}$

Result:

$$n_0 = n - \frac{8}{3\sqrt{\pi}}\sqrt{a\,n_0}^{\,3} - \frac{M^2 R}{8\pi^{3/2}\hbar^4}\sqrt{\frac{n_0}{a}}$$

Huang and Meng, PRL **69**, 644 (1992)

Falco, Pelster, and Graham, PRA **75**, 063619 (2007)

3.2 Superfluid Density

Galilei Boost:

$$\Delta\mathcal{A} = \int_0^{\hbar\beta} d\tau \int d^3x\, \psi^*(\mathbf{x},\tau)\, \mathbf{u}\, \frac{\hbar}{i}\, \boldsymbol{\nabla}\, \psi(\mathbf{x},\tau)$$

$$d\Omega = -S\, dT - p\, dV - N\, d\mu - \mathbf{p}\, d\mathbf{u}$$

$$\mathbf{p} = -\left.\frac{\partial\Omega(T,V,\mu,\mathbf{u})}{\partial\mathbf{u}}\right|_{T,V,\mu} = MV\, n_n\, \mathbf{u} + \ldots$$

Result:

$$n_s = n - n_n = n - \frac{4}{3}\frac{M^2 R}{8\pi^{3/2}\hbar^4}\sqrt{\frac{n_0}{a}}$$

Huang and Meng, PRL **69**, 644 (1992)

Falco, Pelster, and Graham, PRA **75**, 063619 (2007)

3.3 Collective Excitations

Hydrodynamic Equation in Trap With Disorder:

$$m\frac{\partial^2}{\partial t^2}\delta n(\mathbf{x},t) - \boldsymbol{\nabla}\Big[g n_{\mathrm{TF}}(\mathbf{x})\boldsymbol{\nabla}\delta n(\mathbf{x},t)\Big]$$
$$= -\boldsymbol{\nabla}^2\Big[3g n_R(\mathbf{x})\delta n(\mathbf{x},t)\Big] - \boldsymbol{\nabla}\left[\frac{4g}{3}n_R(\mathbf{x})\boldsymbol{\nabla}\delta n(\mathbf{x},t)\right]$$

$n_R(\mathbf{x})$: Huang-Meng depletion in trap

$n_{\mathrm{TF}}(\mathbf{x}) = [\mu - V(\mathbf{x})]/g$: Thomas-Fermi density

Violation of Kohn Theorem:

Surface dipole mode ($n = 0\,, l = 1$):

$$\frac{\delta\omega_{\mathrm{dip}}(\xi = 0)}{\omega_{\mathrm{dip}}} = -\frac{5\pi}{16}\frac{M^2 R}{8\pi^{3/2}\hbar^4\sqrt{n_{\mathrm{TF}}(\mathbf{0})a}}$$

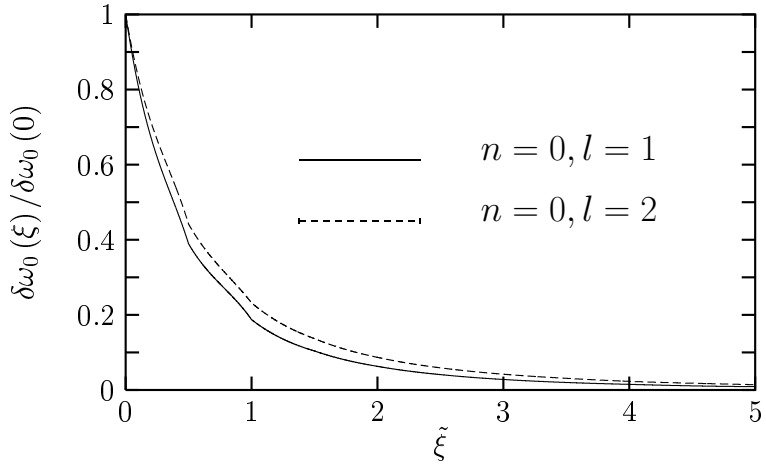
Falco, Pelster, and Graham, PRA **76**, 013624 (2007)

3.4 Comparison With Experiment

Typical Values:

Inguscio *et al.*, PRL **95**, 070401 (2005)

$$\left.\begin{array}{rcl} \xi & = & 10\ \mu\text{m} \\ R_{\text{TF}} & = & 100\ \mu\text{m} \\ l_{\text{HO}} & = & 10\ \mu\text{m} \end{array}\right\} \quad \tilde{\xi} = \frac{\xi R_{\text{TF}}}{l_{\text{HO}}^2\sqrt{2}} \approx 7$$



$\Longrightarrow$ **Disorder effect vanishes in laser speckle experiment**

Improvement:

laser speckle setup with correlation length $\xi = 1\ \mu$m

Aspect *et al.*, NJP **8**, 165 (2006)

$\Longrightarrow$ **Disorder effect should be measurable**

Falco, Pelster, and Graham, PRA **76**, 013624 (2007)

3.5 Rederivation of Huang-Meng Depletion

Gross-Pitaevskii Equation:

$$\left\{ -\frac{\hbar^2}{2M} \mathbf{\Delta} + V(\mathbf{x}) + g \left| \Psi(\mathbf{x}) \right|^2 \right\} \Psi(\mathbf{x}) = \mu \, \Psi(\mathbf{x})$$

Perturbative Expansion:

$$\Psi(\mathbf{x}) = \sqrt{\frac{\mu}{g}} + \Psi_1(\mathbf{x}) + \Psi_2(\mathbf{x}) + \ldots$$

$\Longrightarrow$ Condensate density: $n_0(\mu) = |\overline{\Psi(\mathbf{x})}|^2$

$\Longrightarrow$ Particle density: $n(\mu) = \overline{|\Psi(\mathbf{x})|^2}$

Disorder-Induced Depletion:

$$n_0 = n - n \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})}{\left(\frac{\hbar^2 \mathbf{k}^2}{2M} + 2gn \right)^2} + \ldots$$

$R(\mathbf{k}) = 1 \quad \Longrightarrow$ Huang-Meng depletion

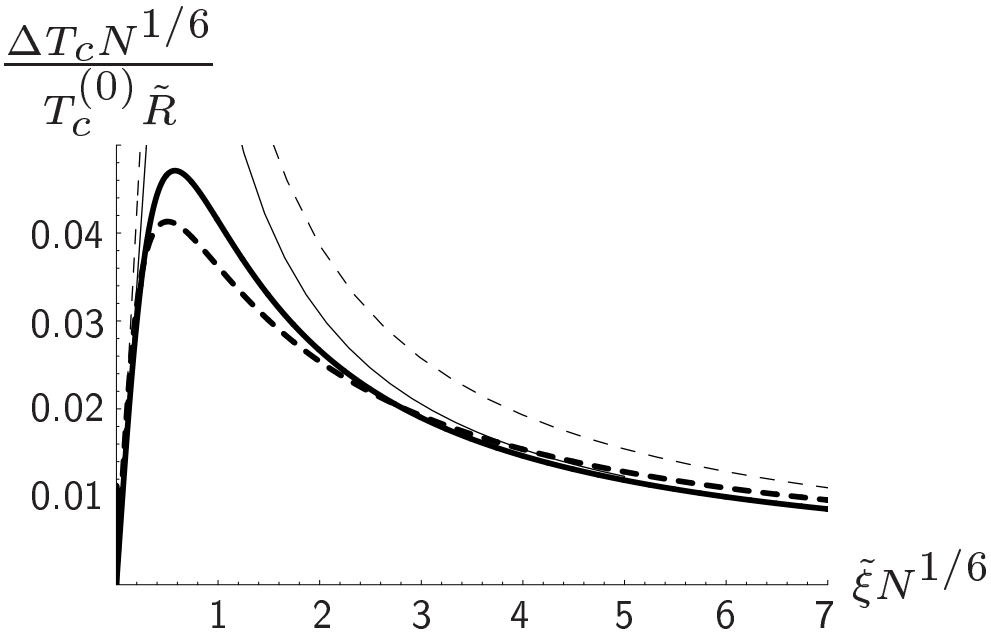
Krumnow, von Hase, and Pelster (to be published)

4.1 Earlier Results

trapped Bose gas	homogeneous Bose gas
$T_c^{(0)} = \frac{\hbar\omega_g}{k_B}\left[\frac{N}{\zeta(3)}\right]^{1/3}$	$T_c^{(0)} = \frac{2\pi\hbar^2}{k_B M}\left[\frac{n}{\zeta(3/2)}\right]^{2/3}$
$\frac{\Delta T_c}{T_c^{(0)}} = -3.426\,\frac{a}{\lambda_c^{(0)}}$ Giorgini et al., PRA **54**, R4633 (1996) Gerbier et al., PRL **92**, 030405 (2004)	$\frac{\Delta T_c}{T_c^{(0)}} = 1.3\,an^{1/3}$ Kleinert, MPLB **17**, 1011 (2003) Kastening, PRA **69**, 043613 (2004)
$R(\mathbf{x}) = ?$ $\frac{\Delta T_c}{T_c^{(0)}} = ?$	$R(\mathbf{x}) = R\,\delta(\mathbf{x})$ $\frac{\Delta T_c}{T_c^{(0)}} = -\frac{M^2 R}{3\pi[\zeta(3/2)]^{2/3}\hbar^2 n^{1/3}}$ Lopatin and Vinokur, PRL **88**, 235503 (2002)

Procedure: $n = n(\mu), \quad \mu \nearrow \mu_c \quad \Rightarrow \quad T_c$

4.2 Our Results



solid: Gaussian

dashed: Lorentzian

Length Scale: $$l_{\mathrm{HO}} = \sqrt{\frac{\hbar}{M\omega_{\mathrm{g}}}} \quad , \qquad \omega_{\mathrm{g}} = (\omega_1\omega_2\omega_3)^{1/3}$$

Dimensionless Units: $$\tilde{\xi} = \frac{\xi}{l_{\mathrm{HO}}} \quad , \qquad \tilde{R} = \frac{R}{\left(\frac{\hbar^2}{M l_{\mathrm{HO}}^2}\right)^2 l_{\mathrm{HO}}^3} = \frac{M^{3/2} R}{\hbar^{7/2} \omega_{\mathrm{g}}^{1/2}}$$

Timmer, Pelster, and Graham, EPL **76**, 760 (2006)

5.1 Order Parameters

Definition:

$$\lim_{|\mathbf{x}-\mathbf{x}'|\to\infty} \overline{\langle\psi(\mathbf{x},\tau)\psi^*(\mathbf{x}',\tau)\rangle} = n_0$$

$$\lim_{|\mathbf{x}-\mathbf{x}'|\to\infty} \overline{|\langle\psi(\mathbf{x},\tau)\psi^*(\mathbf{x}',\tau)\rangle|^2} = (n_0+q)^2$$

Note:

q is similar to Edwards-Anderson order parameter of spin-glass theory

Hartree-Fock Mean-Field Theory:

Self-consistent determination of n_0 and q for $R(\mathbf{x}-\mathbf{x}') = R\,\delta(\mathbf{x}-\mathbf{x}')$

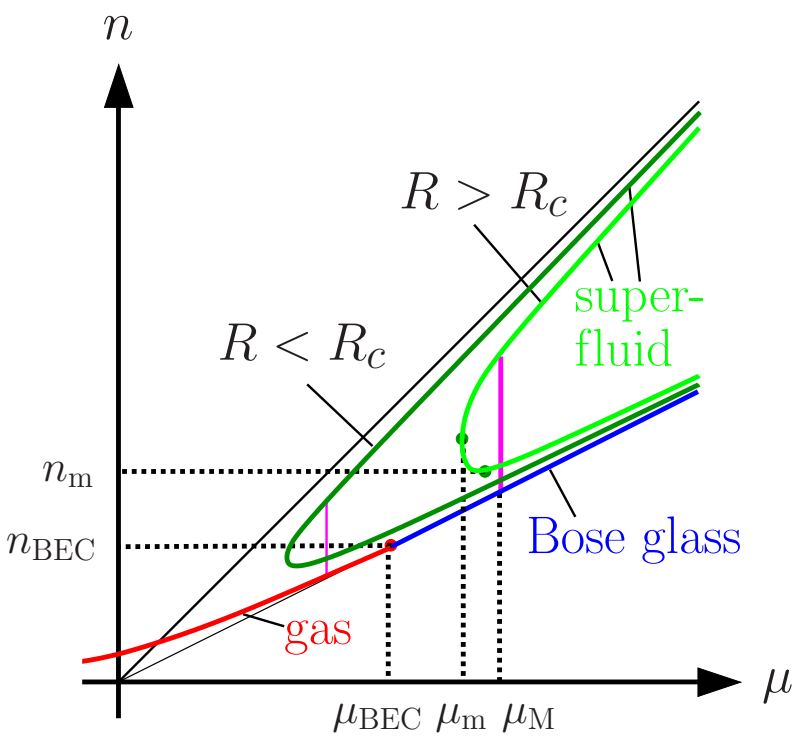
Phase Classification:

gas	Bose glass	superfluid
$q = n_0 = 0$	$q > 0, n_0 = 0$	$q > 0, n_0 > 0$

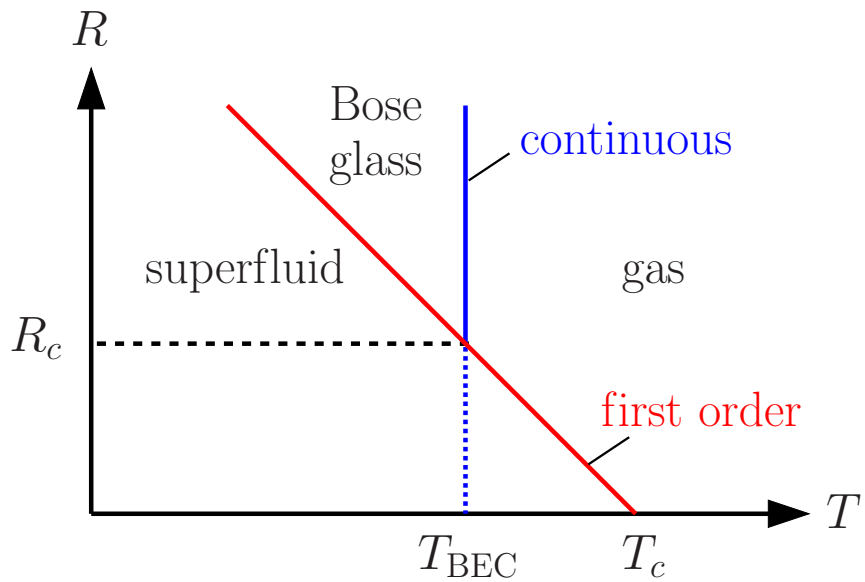
5.2 Hartree-Fock Results

Isotherm: $T = \text{const.}$

disorder strength R = const.



Phase Diagram: $\mu = \text{const.}$



Graham and Pelster, IJBC **19**, 2745 (2009)

5.3 Properties of Bose Glass

- **Absence of ODLRO and phase coherence** ($n_0 = 0$), **presence of long-range glassy order** ($q > 0$)

- **Localization length of excitations:** $$\xi = \frac{2\pi\hbar^4}{M^2 R}$$

- **Life time (width) of excitations:** $$\frac{M\xi}{\hbar|\mathbf{k}|}$$

- **Superfluid density:** Localized part of condensate q contributes to normal but not to superfluid density

- **Note:** Localized condensation ($q > 0$) without global condensate, but not global condensation $n_0 > 0$ without localized part

6.1 Summary and Outlook

- **Frozen Disorder Potential**:

  arises both artificially (laser speckles) or naturally (wire trap)

- **Bosons:**

  local condensates in minima + global condensate + thermally excited

- **Localization Versus Transport:**

  disorder reduces superfluidity

- **Phase Diagram:**

  yet unknown for strong disorder

  Navez, Pelster, and Graham, APB **86**, 395 (2007)

- **Disordered Bosons in Lattice:**

  Bose Glass versus Mott phase

6.2 Quantum Phase Transition

Gross-Pitaevskii Equation:

$$\left\{-\frac{\hbar^2}{2M}\,\mathbf{\Delta} + V(\mathbf{x}) + g\Psi^2(\mathbf{x})\right\}\Psi(\mathbf{x}) = \mu\,\Psi(\mathbf{x})$$

Nonperturbative Approach:

Gaussian approximation for random fields $V(\mathbf{x})$ and $\Psi(\mathbf{x})$

$$\overline{V(\mathbf{x})} = 0\ , \qquad \overline{V(\mathbf{x})V(\mathbf{x}')} = R(\mathbf{x}-\mathbf{x}')\ , \ \overline{\Psi(\mathbf{x})} = \sqrt{n_0}$$

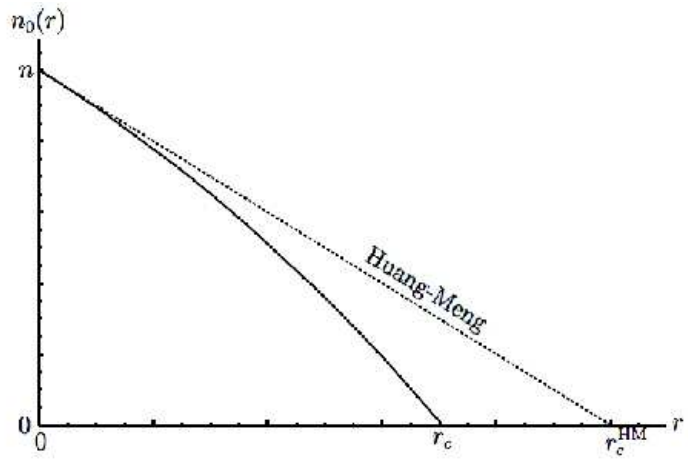
$$\overline{\Psi(\mathbf{x})\Psi(\mathbf{x}')} = G_{\Psi\Psi}(\mathbf{x}-\mathbf{x}')\ , \qquad \overline{V(\mathbf{x})\Psi(\mathbf{x}')} = G_{V\Psi}(\mathbf{x}-\mathbf{x}')$$

$\Longrightarrow$ Self-consistency equations

Result:

$$n_0(r) = n\left(1 - \frac{r^2}{2} - \frac{r}{2}\sqrt{r^2+4}\right)$$

$$r_c = \frac{1}{\sqrt{2}},\ r = \sqrt{\frac{\pi}{2gn}}R\left(\frac{M}{2\pi\hbar^2}\right)^{3/2}$$



Krumnow, von Hase, and Pelster (to be published)

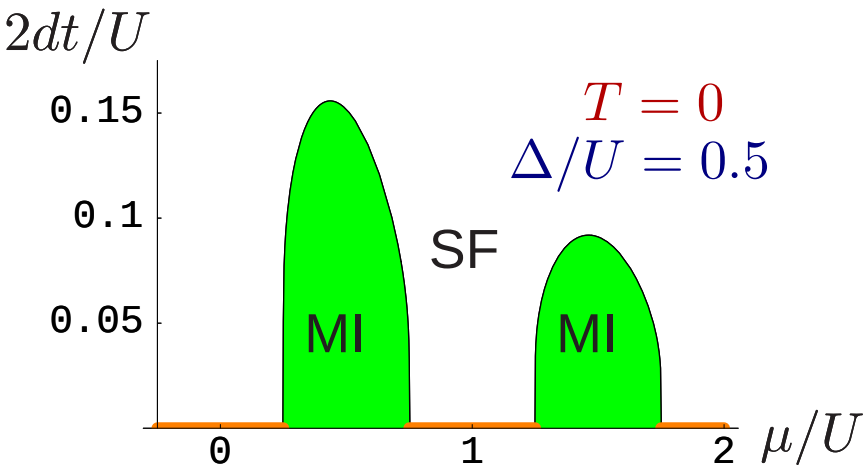
6.3 Disordered Bosons in Lattice

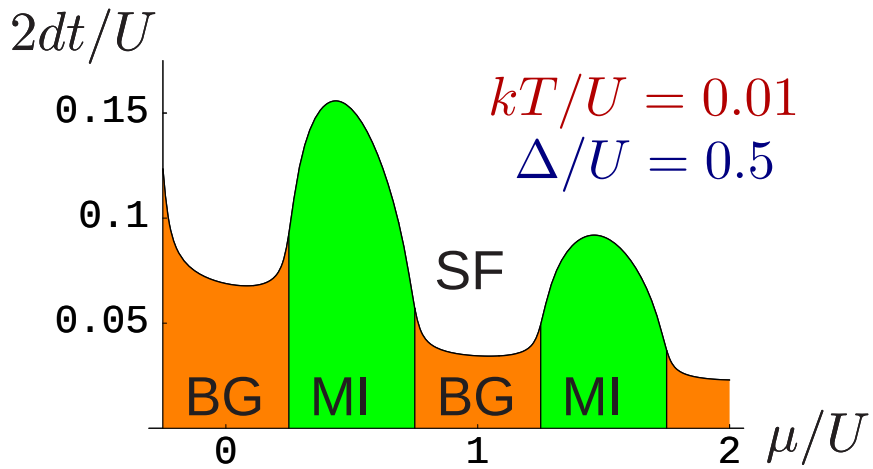
Bose-Hubbard Hamilton Operator:

$$\hat{H}_{\mathrm{BH}} = -t \sum_{<\mathbf{i},\mathbf{j}>} \hat{a}_{\mathbf{i}}^{\dagger} \hat{a}_{\mathbf{j}} + \sum_{\mathbf{i}} \left[\frac{U}{2} \hat{n}_{\mathbf{i}} (\hat{n}_{\mathbf{i}} - 1) + (\epsilon_{\mathbf{i}} - \mu)\, \hat{n}_{\mathbf{i}} \right]$$

$$\overline{\bullet} = \prod_{\mathbf{i}} \int_{-\infty}^{+\infty} \bullet \; p(\epsilon_{\mathbf{i}})\, d\epsilon_{\mathbf{i}} \;, \qquad p(\epsilon_{\mathbf{i}}) = \begin{cases} 1/\Delta & ; \; \epsilon_{\mathbf{i}} \in [-\Delta/2, +\Delta/2] \\ 0 & ; \; \text{otherwise} \end{cases}$$

Mean-Field Phase Diagram:





Krutitsky, Pelster, and Graham, NJP **8**, 187 (2006)

Stochastic Mean-Field Theory: Bissbort and Hofstetter, EPL **86**, 50007 (2009)