

Kinetic Description of Dipolar Fermi Gases: From Collisionless to Hydrodynamic Regime



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January 2, 2012

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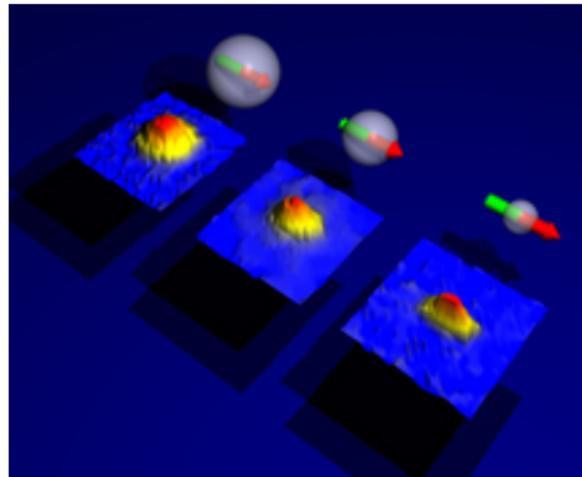
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Bose Condensed Elements

Perioden																				
I	II	IIIa	IVa	Va	VIa	VIIa	VIIia	Ia	Ib	III	IV	V	VI	VII	VIII		$\square =$ Hauptgruppen	$\square =$ Nebengruppen	$\square =$ Edelgase	Schale
1 H 1 Wasserstoff																	4 He 2 Helium			K
2 Li 3 Lithium	9 Be 4 Borium									10 B 5 Bor	12 C 6 Kohlenstoff	14 N 7 Stickstoff	15 O 8 Sauerstoff	16 F 9 Fluor	18 Ne 10 Neon				L	
12 Na 11 Magnesium	24 Mg 12 Magnesium									26 Al 13 Aluminium	28 Si 14 Silicium	30 P 15 Phosphor	32 S 16 Schwefel	36 Cl 17 Chlor	38 Ar 18 Argon				M	
19 K 18 Calcium	40 Ca 20 Calcium	44 Sc 21 Scandium	47 Ti 22 Titan	50 V 23 Vanadin	53 Cr 24 Chrom	54 Mn 25 Mangan	55 Fe 26 Eisen	58 Co 27 Kobalt	59 Ni 28 Nickel	63 Cu 29 Kupfer	65 Zn 31 Zink	69 Ga 32 Gallium	72 Ge 33 Germanium	78 As 34 Ars	79 Se 35 Brom	83 Br 36 Krypton		N		
55 Rb 37 Rubidium	87 Sr 39 Strontium	88 Y 39 Yttrium	91 Zr 40 Zirkonia	92 Nb 41 Nitinol	96 Mo 42 Molybdän	101 Tc 43 Technetium	102 Ru 44 Ruthenium	106 Rh 45 Rhodium	109 Pd 46 Palladium	112 Ag 47 Silber	114 Cd 48 Cadmium	116 In 49 Indium	121 Sn 50 Szn	127 Te 52 Tellur	131 I 53 Jod	131 Xe 54 Xenon	O			
132 Cs 55 Cäsium	137 La 58 Lanthan	138 Ce 59 Cerium	178 Hf 72 Hafnium	180 Ta 73 Tantal	183 W 74 Wolfram	186 Re 75 Rhodium	190 Os 76 Osmium	192 Ir 77 Iridium	195 Rh 78 Rhodium	196 Au 79 Gold	206 Hg 80 Quecksilber	204 Tl 81 Thallium	207 Pb 82 Blei	208 Bi 83 Bismut	209 Po 84 Polonium	222 Rn 85 Radon	P			
223 Fr 87 Francium	227 Ra 88 Radium	228 Ac 89 Aktinium	261 Rf 104 Rutherfordium	262 Ha 105 Hassium	263 Sg 106 Seesawium	262 Lr 108 Lawrencium	c.a. 365 104 105	c.a. 268 104 105	c.a. 269 105 106	c.a. 272 105 106	c.a. 277 105 106	c.a. 289 114	c.a. 239 114	c.a. 239 114	c.a. 293 114	c.a. 293 114		Q		
Aggregatzustand unter Normalbedingungen:																				
Fe fest	Hg flüssig	He gasförmig																		
* = radioaktives Element																				
Lanthanide																				
6 Ce 14 Ce	140.12 140.91	140.24 140.91	145 Pr 149 Praseodym	150.35 150.91	151.96 151.91	157.25 157.25	159.92 159.92	162.59 162.59	164.93 164.93	167.26 167.26	168.93 168.93	173.04 173.04	174.97 174.97							
6 Nd 14 Nd	140.12 140.91	140.24 140.91	145 Sm 149 Neodym	150.35 150.91	151.96 151.91	157.25 157.25	159.92 159.92	162.59 162.59	164.93 164.93	167.26 167.26	168.93 168.93	173.04 173.04	174.97 174.97							
Actinide																				
7 Th 90 Thorium	232.04 232.04	231.91 231.91	238.03 238.03	237.91 237.91	244.91 244.91	243.91 243.91	247.91 247.91	247.91 247.91	254.91 254.91	257.91 257.91	258.91 258.91	260.91 260.91	260.91 260.91							
Th Pa 91 Protactinium	U 92 Uranium	Np 93 Neptunium	Pu 94 Plutonium	Am 95 Americium	Cm 96 Curium	Bk 97 Berkelium	Cf 98 Californium	Cf 98	Es 99 Einsteinium	Fm 100 Fermium	Md 102 Mendelevium	No 103 Neptunium	Lr 103 Lawrencium							

Dipole-Dipole Interaction in Quantum Systems



- dipole-dipole interaction potential

$$V_{\text{int}}(\mathbf{x}) = \frac{C_{dd}}{4\pi |\mathbf{x}|^3} (1 - 3 \cos^2 \vartheta)$$

Magnetic and Electric Interactions

- **Magnetic systems:**

- Interaction strength: $C_{dd}^B = \mu_0 m^2$, with $m \sim 1$ to $10 \mu_B$
- Realized samples
 - Boson: ^{52}Cr Griesmaier *et al.*, PRL **94**, 160401 (2005)
 - Boson: ^{87}Rb Vengalattore *et al.*, PRL **100**, 170403 (2008)
 - Fermion: ^{53}Cr Chicireanu *et al.*, PRA **73**, 053406 (2006)
 - Boson: ^{164}Dy Lu *et al.*, PRL **107**, 190401 (2011)

- **Electric systems:**

- Interaction strength: $C_{dd}^E = 4\pi d^2$, with $d \sim 1$ Debye
- Realized samples
 - Fermion: $^{40}\text{K}^{87}\text{Rb}$ Ospelkaus *et al.*, Science **32**, 231 (2008)
 - Boson: $^{41}\text{K}^{87}\text{Rb}$ Aikawa *et al.*, NJP **11**, 055035 (2009)

- **Ratio:** $C_{dd}^B / C_{dd}^E \approx 10^{-4}$

Previous Theoretical State of Knowledge

- Description of limiting cases
- Contact interaction:
Guéry-Odelin, PRA **66**, 033613 (2002)
- Dipolar interaction
 - Collisionless regime via Boltzmann-Vlasov Equation and minimization of Hartree-Fock energy:
Sogo *et al.*, NJP **11**, 055017 (2009)
 - Hydrodynamic regime via action principle and common phase approximation:
Lima and Pelster, PRA **81**, 021606(R) and 063629 (2010)
- Kinetic description of all regimes with contact interaction:
Stringari *et al.*, PRA **68**, 043608 (2003)

Fermi Gas

- Hamiltonian:

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$$

$$\hat{H}_0 = \int d^3r \hat{\psi}^\dagger(\mathbf{r}) \left[-\frac{\hbar^2 \nabla^2}{2m} + U(\mathbf{r}) \right] \hat{\psi}(\mathbf{r})$$

$$\hat{H}_{\text{int}} = \frac{1}{2} \int d^3r d^3r' V_{\text{int}}(\mathbf{r} - \mathbf{r}') \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}') \hat{\psi}(\mathbf{r}') \hat{\psi}(\mathbf{r})$$

- Anticommutator relations:

$$\{\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')\} = \delta(\mathbf{r} - \mathbf{r}')$$

$$\{\hat{\psi}(\mathbf{r}), \hat{\psi}(\mathbf{r}')\} = 0$$

$$\{\hat{\psi}^\dagger(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')\} = 0$$

Wigner Representation

- Wigner function:

$$f(\mathbf{r}, \mathbf{p}, t) = \int d^3 s e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}} \text{Tr} \left[\hat{\rho}(t) \hat{\psi}^\dagger \left(\mathbf{r} + \frac{\mathbf{s}}{2} \right) \hat{\psi} \left(\mathbf{r} - \frac{\mathbf{s}}{2} \right) \right]$$

- Contains the whole quantum mechanical information:

$$n(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi\hbar)^3} f(\mathbf{x}, \mathbf{p}, t)$$

$$\langle \psi | \mathbf{p} | \psi \rangle = \int d^3 x \int \frac{d^3 p}{(2\pi\hbar)^3} \mathbf{p} f(\mathbf{x}, \mathbf{p}, t)$$

Equation of Motion for the Distribution function

- Time evolution of the density matrix: von Neumann equation

$$i\hbar \frac{\partial \hat{\rho}}{\partial t}(t) = [\hat{H}, \hat{\rho}(t)]$$

- With weakly varying trapping potential:

$$\begin{aligned} & \frac{\partial f}{\partial t}(\mathbf{r}, \mathbf{p}, t) + \frac{\mathbf{p}}{m} \cdot \frac{\partial f(\mathbf{r}, \mathbf{p}, t)}{\partial \mathbf{r}} - \frac{\partial U(\mathbf{r})}{\partial \mathbf{r}} \cdot \frac{\partial f(\mathbf{r}, \mathbf{p}, t)}{\partial \mathbf{p}} \\ &= \frac{1}{i\hbar} \int d^3 s \int d^3 x e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}} \left[V_{\text{int}} \left(\mathbf{x} - \mathbf{r} + \frac{\mathbf{s}}{2} \right) - V_{\text{int}} \left(\mathbf{x} - \mathbf{r} - \frac{\mathbf{s}}{2} \right) \right] \\ & \quad \times \text{Tr} \left[\hat{\rho}(t) \hat{\psi}^\dagger \left(\mathbf{r} + \frac{\mathbf{s}}{2} \right) \hat{\psi}^\dagger (\mathbf{x}) \hat{\psi} (\mathbf{x}) \hat{\psi} \left(\mathbf{r} - \frac{\mathbf{s}}{2} \right) \right] \end{aligned}$$

Hierarchical Strucutre & Perturbation Theory

- Perturbation theory up to second order in interaction
- Calculate n-field operator averages with Wick contraction for weak spatial variation of Wigner function:
Boltzmann-Vlasov Equation

$$\left\{ \frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{r}} - \frac{\partial [U(\mathbf{r}) + U_{\text{mf}}(\mathbf{r}, \mathbf{p})]}{\partial \mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{p}} + \frac{\partial U_{\text{mf}}(\mathbf{r}, \mathbf{p})}{\partial \mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{r}} \right\} f(\mathbf{r}, \mathbf{p}, t) = I_{\text{coll}}[f](\mathbf{r}, \mathbf{p}, t)$$

$$U_{\text{mf}}(\mathbf{r}, \mathbf{p}) = \int d^3x V_{\text{int}}(\mathbf{r} - \mathbf{x}) n(\mathbf{x}, t) - \int \frac{d^3 p'}{(2\pi\hbar)^3} f(\mathbf{r}, \mathbf{p}', t) \tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{p}')$$

- terms of first order in the interaction: mean-field terms
- terms of second order in the interaction: collision integral

Properties of Collision Integral

$$\begin{aligned}
 I_{\text{coll}}[f] = & \frac{\pi}{\hbar(2\pi\hbar)^6} \int d^3q_2 \int d^3q_3 \int d^3q_4 [(1-f)(1-f_2)f_3f_4 \\
 & - ff_2(1-f_3)(1-f_4)] \delta(E_{\mathbf{p}} + E_{\mathbf{q}_2} - E_{\mathbf{q}_3} - E_{\mathbf{q}_4}) \\
 & \times \delta(\mathbf{p} + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4) \left[\tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{q}_3) - \tilde{V}_{\text{int}}(\mathbf{p} - \mathbf{q}_4) \right]^2 \\
 f \equiv & f(\mathbf{r}, \mathbf{p}, t), \quad f_i \equiv f(\mathbf{r}, \mathbf{q}_i, t)
 \end{aligned}$$

- 5 fundamental conserved quantities: kinetic energy (p^2), 3 momentum coordinates (p^1), particle number (p^0)
- Collisions are assumed to be local

Relaxation Time Approximation

- Limiting function of relaxation: local equilibrium

$$I_{\text{coll}} \left[f^{\text{le}} \right] = 0$$

- Linearization of collision integral:

$$f(\mathbf{r}, \mathbf{p}, t) \approx f^{\text{le}}(\mathbf{r}, \mathbf{p}, t) + \delta f(\mathbf{r}, \mathbf{p}, t)$$

$$I_{\text{coll}} \approx -\hat{C} [\delta f]$$

- Ad hoc approximation:

$$I_{\text{coll}} = -\frac{\delta f}{\tau}$$

Scaling Ansatz

- Boltzmann-Vlasov equation (BVE) in equilibrium:

$$\left\{ \frac{\hbar \mathbf{q}}{m} + \frac{1}{\hbar} \frac{\partial [U + U_{\text{mf}}(\mathbf{x}, \mathbf{q})]}{\partial \mathbf{q}} \right\} \frac{\partial f^0}{\partial \mathbf{x}} - \frac{1}{\hbar} \frac{\partial [U + U_{\text{mf}}(\mathbf{x}, \mathbf{q})]}{\partial \mathbf{x}} \frac{\partial f^0}{\partial \mathbf{q}} = 0$$

- Stringari *et al.*, PRA **68**, 043608 (2003): scaling ansatz

$$f(\mathbf{x}, \mathbf{q}, t) \rightarrow \Gamma f^0(\mathbf{r}(t), \mathbf{k}(t))$$

$$r_i = \frac{x_i}{b_i(t)}$$

$$k_i = \frac{1}{\Theta_i^{\frac{1}{2}}(t)} \left[q_i - \frac{m \dot{b}_i(t) x_i}{\hbar b_i(t)} \right]$$

$$\Gamma = \frac{1}{\prod_i b_i \Theta_i^{\frac{1}{2}}}$$

Equations for Scaling Parameters

- Ordinary differential equations by taking moments
- $r_i k_i \rightarrow$ equation for scaling parameter b_i ;
- $k_i^2 \rightarrow$ equation for scaling parameter Θ_i

$$\frac{\dot{\Theta}_i}{\Theta_i} + 2 \frac{\dot{b}_i}{b_i} = \frac{1}{\Gamma \langle k_i^2 \rangle^0} \int \frac{d^3 r d^3 k}{(2\pi)^3} k_i^2 I_{\text{coll}}[f]$$

$$\begin{aligned} \ddot{b}_i + \omega_i^2 b_i - \frac{\hbar^2 \langle k_i^2 \rangle^0 \Theta_i}{m^2 b_i \langle r_i^2 \rangle^0} + \frac{1}{2mb_i \langle r_i^2 \rangle^0} & \left[\int \frac{d^3 k}{(2\pi)^3} \tilde{W}_i(b, \mathbf{k}) \tilde{n}^0(\mathbf{k}) \right. \\ & \times \tilde{n}^0(-\mathbf{k}) - \frac{1}{\prod_j b_j} \int \frac{d^3 r d^3 k d^3 k'}{(2\pi)^6} f^0(\mathbf{r}, \mathbf{k}) f^0(\mathbf{r}, \mathbf{k}') \tilde{W}_i(\Theta, \mathbf{k} - \mathbf{k}') \Big] \\ & = 0 \end{aligned}$$

Static Solution

- Ansatz:

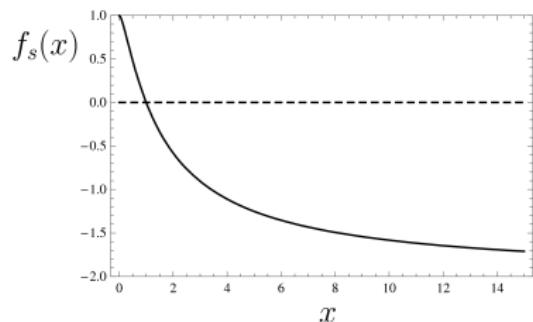
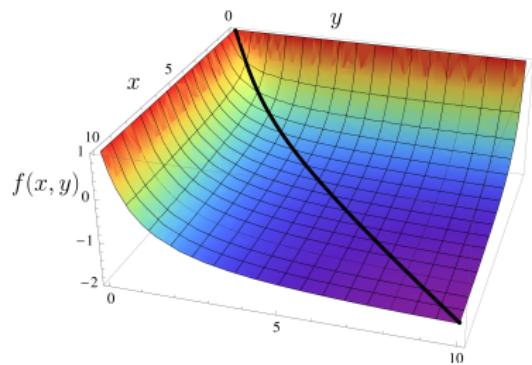
$$f^0(\mathbf{r}, \mathbf{k}) = \Theta \left(1 - \sum_j \frac{r_j^2}{R_j^2} - \sum_j \frac{k_j^2}{K_j^2} \right)$$

- Thomas-Fermi radius R_i , Thomas-Fermi momentum K_i
- Total energy in equilibrium

$$E_{\text{tot}} = \frac{N}{8} \sum_j \frac{\hbar^2 K_j^2}{2m} + \frac{N m}{8} \frac{1}{2} \sum_j \omega_j^2 R_j^2 - \frac{6N^2 c_0}{\overline{R}^3} f \left(\frac{R_x}{R_z}, \frac{R_y}{R_z} \right) + \frac{6N^2 c_0}{\overline{R}^3} f \left(\frac{K_z}{K_x}, \frac{K_z}{K_y} \right)$$

$$\overline{R} = (R_x R_y R_z)^{\frac{1}{3}}, \quad c_0 = \frac{2^{10} C_{dd}}{(3^4 \cdot 5 \cdot 7 \cdot \pi^3)}$$

Anisotropy-Function



Static Equations

- Symmetry in momentum space depends on Fock term
- Trap does not influence the Fock term
 \rightarrow cylinder symmetric momentum distribution

$$N = \frac{1}{48} \overline{R}^3 \overline{K}^3$$

$$K_x = K_y$$

$$\frac{\hbar^2 K_z^2}{2m} - \frac{\hbar^2 K_x^2}{2m} = -\frac{72 N^2 c_0}{\prod_j R_j} \left[-1 + \frac{(2K_x^2 + K_z^2) f_s \left(\frac{K_z}{K_x} \right)}{2(K_x^2 - K_z^2)} \right]$$

$$\omega_x^2 R_x^2 - \frac{1}{3} \sum_j \frac{\hbar^2 K_j^2}{m^2} + \frac{48 N c_0}{m \bar{R}^3} \left[f \left(\frac{R_x}{R_z}, \frac{R_y}{R_z} \right) \right.$$

$$\left. - f \left(\frac{K_z}{K_x}, \frac{K_z}{K_y} \right) - \frac{R_x}{R_z} f_1 \left(\frac{R_x}{R_z}, \frac{R_y}{R_z} \right) \right] = 0$$

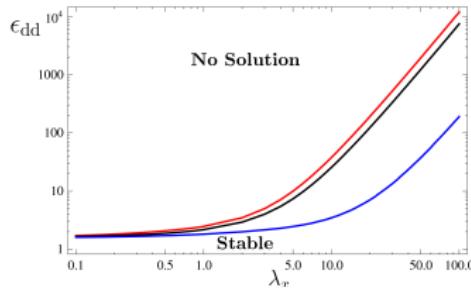
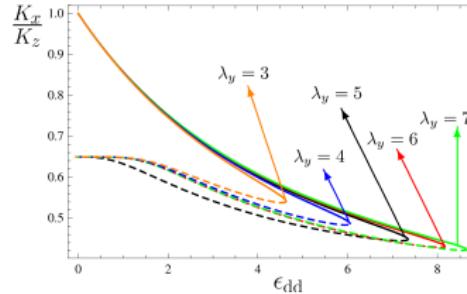
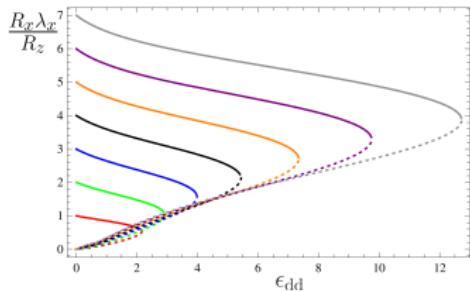
$$\omega_y^2 R_y^2 - \frac{1}{3} \sum_j \frac{\hbar^2 K_j^2}{m^2} + \frac{48 N c_0}{m \bar{R}^3} \left[f \left(\frac{R_x}{R_z}, \frac{R_y}{R_z} \right) \right.$$

$$\left. - f \left(\frac{K_z}{K_x}, \frac{K_z}{K_y} \right) - \frac{R_y}{R_z} f_2 \left(\frac{R_x}{R_z}, \frac{R_y}{R_z} \right) \right] = 0$$

$$\omega_z^2 R_z^2 - \frac{1}{3} \sum_j \frac{\hbar^2 K_j^2}{m^2} + \frac{48 N c_0}{m \bar{R}^3} \left[f \left(\frac{R_x}{R_z}, \frac{R_y}{R_z} \right) \right.$$

$$\left. - f \left(\frac{K_z}{K_x}, \frac{K_z}{K_y} \right) + \frac{R_x}{R_z} f_1 \left(\frac{R_x}{R_z}, \frac{R_y}{R_z} \right) + \frac{R_y}{R_z} f_2 \left(\frac{R_x}{R_z}, \frac{R_y}{R_z} \right) \right] = 0$$

Properties of Static Solution



- Lima and Pelster, PRA **81**, 021606(R) and 063629 (2010)

Treatment of Collision Integral

- Relaxation time approximation:

$$I_{\text{coll}}[f] = -\frac{f - f^{\text{le}}}{\tau}$$

- Scaling ansatz:

$$\dot{\Theta}_i + 2 \frac{\dot{b}_i}{b_i} \Theta_i = -\frac{1}{\tau} (\Theta_i - \Theta_i^{\text{le}})$$

$$\ddot{b}_i + \omega_i^2 b_i - \frac{\hbar^2 K_i^2 \Theta_i}{m^2 b_i R_i^2} + \frac{48 N c_0}{m b_i R_i^2 \prod_j b_j R_j} g_1(b_x R_x, b_y R_y, b_z R_z)$$

$$- \frac{48 N c_0}{m b_i R_i^2 \prod_j b_j R_j} g_2 \left(\Theta_x^{\frac{1}{2}} K_x, \Theta_y^{\frac{1}{2}} K_y, \Theta_z^{\frac{1}{2}} K_z \right) = 0$$

Local Equilibrium

- Relaxation does not influence the spatial dependence of distribution function
- Energy minimization with respect to momentum coordinates

$$\frac{\hbar^2 \Theta_x^{\text{le}} K_z^2}{2m} - \frac{\hbar^2 \Theta_z^{\text{le}} K_x^2}{2m} = -\frac{72 N c_0}{\prod_j b_j R_j} g \left[(\Theta_x^{\text{le}})^{\frac{1}{2}} K_x, (\Theta_z^{\text{le}})^{\frac{1}{2}} K_z \right]$$

- Coupling through particle conservation

$$1 = \bar{b}^3 \left(\overline{\Theta}^{\text{le}} \right)^{\frac{3}{2}}$$

Limiting Cases

$$\dot{\Theta}_i + 2 \frac{\dot{b}_i}{b_i} \Theta_i = -\frac{1}{\tau} (\Theta_i - \Theta_i^{\text{le}})$$

- Collisionless limit: $\tau \longrightarrow \infty$
 - $\Theta_i^{\text{cl}} = (b_i^{\text{cl}})^{-2}$
 - Sogo *et al.*, NJP **11**, 055017 (2009)
- Hydrodynamic limit: $\tau \longrightarrow 0$
 - equivalent to local equilibrium
 - Lima and Pelster, PRA **81**, 021606(R) and 063629 (2010)

Summary

- Derivation of BVE via perturbation theory
- Determination of limiting function of relaxation
- Scaling ansatz and taking moments of BVE to find ordinary differential equations for respective scaling parameters
- Determining static solution via energy minimization
- Using relaxation time approximation for collision integral
- Determining symmetry of local equilibrium via energy minimization

Outlook

- Calculation of low-lying excitation modes and time-of-flight dynamics from collisionless to hydrodynamic regime
- Clear derivation of relaxation time approximation and corresponding relaxation time
- Include temperature dependence with Sommerfeld expansion
- Apply derivation and scaling solution to dipolar Bose gas: extension of ZNG theory