Bogoliubov Theory of Dipolar Bose Gas in Weak Random Potential

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Berlin, January 5, 2015
New Year Seminar

M. Ghabour and A. Pelster, *PRA* **90**, 063636 (2014)

Outline

Introduction

- **Superfluid Helium in Porous Media:**
  - Experiment: Reppy et al., *PRL* **61**, 1950 (1988)

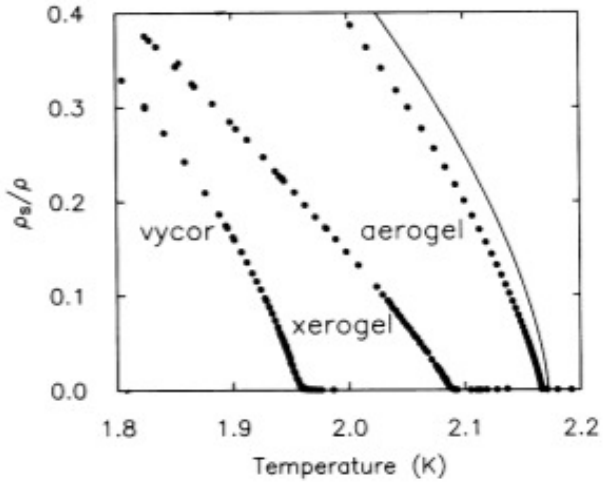
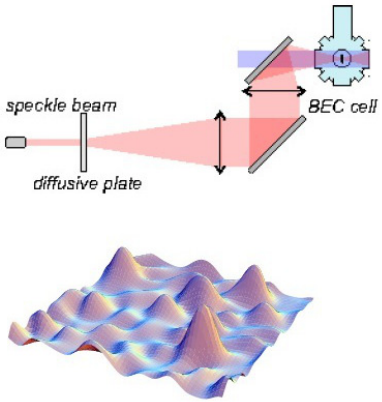


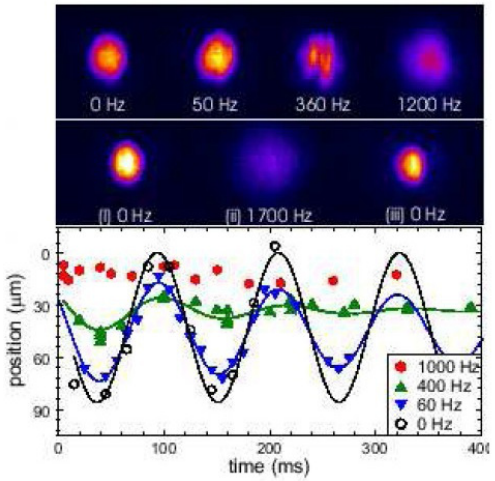
FIG. 1. The normalized superfluid-density data for helium contained in the three porous media are shown as a function of temperature. The solid curve gives the temperature dependence for the superfluid density of bulk helium.

  - Theory: K. Huang and H. F. Meng, *PRL* **69**, 644 (1992)

Introduction

- **Laser Speckles:**
  - Experiment: Inguscio et al., *PRL* **95**, 070401 (2005)

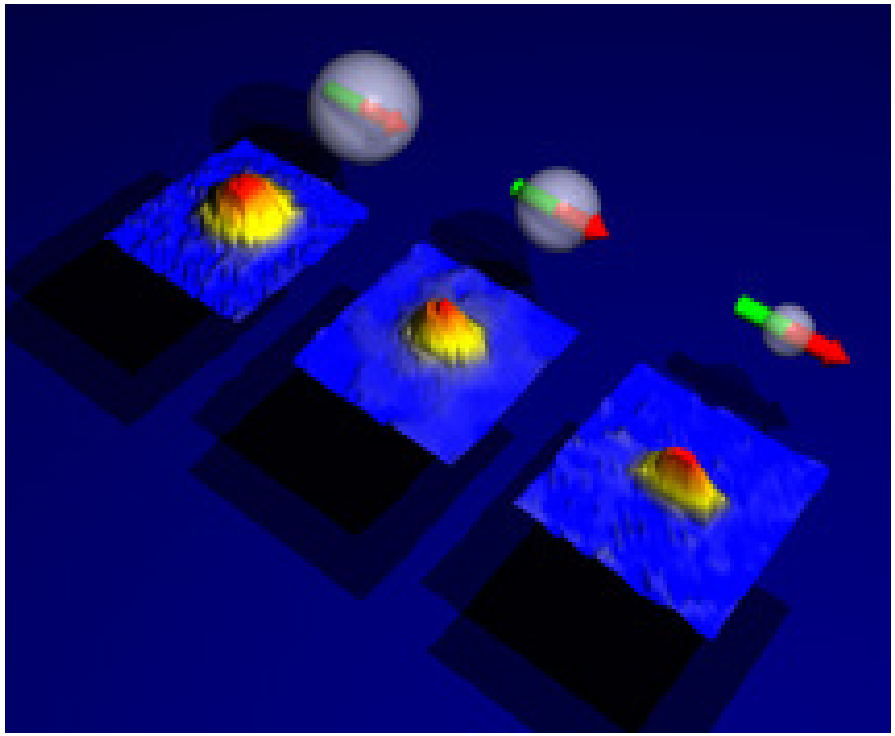




  - Theory: J. W. Goodman. *Speckle Phenomena in Optics: Theory and Applications*. Roberts & Co Publ (2010)

Introduction

- **Dipolar BEC:**
  - Experiment: Griesmaier et al., *PRL* **94**, 160401 (2005)



  - Theory: K. Góral et al., *PRA* **61**, 051601(R) (2000)

Grand-Canonical Hamiltonian

- Grand-Canonical Hamiltonian of dipolar Bose gas in weak random potential

$$\hat{\mathcal{K}} = \int d^3\mathbf{x}\, \hat{\psi}^\dagger(\mathbf{x}) \left[\frac{-\hbar^2 \nabla^2}{2m} - \mu + U(\mathbf{x}) \right] \hat{\psi}(\mathbf{x}) + \frac{1}{2} \int d^3\mathbf{x} \int d^3\mathbf{x}' \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}^\dagger(\mathbf{x}') V(\mathbf{x}, \mathbf{x}') \hat{\psi}(\mathbf{x}') \hat{\psi}(\mathbf{x})$$

- Properties
  - Disorder potential $U(\mathbf{x})$
  - Chemical potential μ
  - Two-Body Interaction $V(\mathbf{x}, \mathbf{x}') = V_\delta(\mathbf{x} - \mathbf{x}') + V_{\mathrm{dd}}(\mathbf{x} - \mathbf{x}')$
  - Bose quantized fields $\left[\hat{\psi}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{x}')\right] = \delta(\mathbf{x} - \mathbf{x}'), \left[\hat{\psi}(\mathbf{x}), \hat{\psi}(\mathbf{x}')\right] = \left[\hat{\psi}^\dagger(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{x}')\right] = 0$

Random Potential

- Disorder Ensemble Average

$$\langle \bullet \rangle_{\text{dis}} = \int DU\, P[U](\bullet), \qquad \int \mathcal{D}U P[U] = 1$$

$P[U]$ denotes disorder probability distribution

- Assumption

$$\langle U(\mathbf{x}) \rangle_{\text{dis}} = 0, \qquad \langle U(\mathbf{x}) U(\mathbf{x}') \rangle_{\text{dis}} = R(\mathbf{x} - \mathbf{x}')$$

$R(\mathbf{x} - \mathbf{x}')$ denotes correlation function

- Delta-Correlated Potential

$$R(\mathbf{x} - \mathbf{x}') = R_0 \delta(\mathbf{x} - \mathbf{x}')$$

R_0 denotes disorder strength

Two-Body Interaction

- Contact Interaction

$$V_\delta(\mathbf{x}-\mathbf{x}') = g\delta(\mathbf{x}-\mathbf{x}')$$

$g = 4\pi a\hbar^2/m$ with s-wave scattering length a

- Dipolar Interaction

$$V_{\mathrm{dd}}(\mathbf{x}) = \frac{C_{\mathrm{dd}}}{4\pi}\,\frac{(x^2+y^2+z^2)-3z^2}{(x^2+y^2+z^2)^{5/2}}$$

for dipoles aligned along z-axis direction

- Dipolar Interaction Strength due to Magnetic or Electric dipole moments

$$C_{\mathrm{dd}} = \mu_0 d_{\mathrm{m}}^2, \quad C_{\mathrm{dd}} = d_{\mathrm{e}}^2/\epsilon_0$$

μ_0 denotes vacuum magnetic permeability and ϵ_0 denotes vacuum dielectric constant

Momentum Space Representation

- Hamiltonian

$$\hat{\mathcal{K}} = \sum_{\mathbf{k}} \left(\frac{\hbar^2 \mathbf{k}^2}{2m} - \mu \right) \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \frac{1}{v} \sum_{\mathbf{p},\mathbf{k}} U_{\mathbf{p}-\mathbf{k}} \, \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{k}}$$
$$+ \frac{1}{2v} \sum_{\mathbf{p},\mathbf{k},\mathbf{q}} V_{\mathbf{q}} \, \hat{a}^\dagger_{\mathbf{k}+\mathbf{q}} \hat{a}^\dagger_{\mathbf{p}-\mathbf{q}} \hat{a}_{\mathbf{p}} \hat{a}_{\mathbf{k}}$$

- Creation/Annihilation Operators

$$\left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'} \right] = \delta_{\mathbf{k}\mathbf{k}'}, \; \left[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'} \right] = \left[\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'} \right] = 0$$

- Two-Body Interaction

$$V_{\mathbf{q}} = g \left[1 + \epsilon_{\mathrm{dd}} (3 \cos^2 \theta - 1) \right]$$

$\epsilon_{\mathrm{dd}} = C_{\mathrm{dd}}/3g$ denotes relative dipolar interaction strength

Bogoliubov Prescription

- Creation/annihilation operators

$$\hat{a}_0|N_0\rangle_0 = \sqrt{N_0}|N_0-1\rangle_0,\ \hat{a}_0^\dagger|N_0\rangle_0 = \sqrt{N_0+1}|N_0+1\rangle_0$$

- $N_0 \gg 1$, we replace operators by c-number

$$\hat{a}_0 \approx \hat{a}_0^\dagger \approx \sqrt{N_0}$$

- Simplified Hamiltonian

$$\begin{aligned}\hat{\mathcal{K}}' &= \left(-\mu + \frac{1}{v}U_0\right)N_0 + \frac{1}{2v}V_0 N_0^2 \\ &+ \frac{1}{2}\sum_{\mathbf{k}}{}' \left(\frac{\hbar^2\mathbf{k}^2}{2m} - \mu\right)(\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}} + \hat{a}_{-\mathbf{k}}^\dagger\hat{a}_{-\mathbf{k}}) + \frac{1}{v}\sqrt{N_0}\sum_{\mathbf{k}}{}' U_{\mathbf{k},0}\,(\hat{a}_{\mathbf{k}}^\dagger + \hat{a}_{-\mathbf{k}}) \\ &+ \frac{1}{2v}N_0\sum_{\mathbf{k}}{}'(V_0 + V_{\mathbf{k}})\,(\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}} + \hat{a}_{-\mathbf{k}}^\dagger\hat{a}_{-\mathbf{k}}) + \frac{1}{2v}N_0\sum_{\mathbf{k}}{}' V_{\mathbf{k}}\,(\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{-\mathbf{k}}^\dagger + \hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}})\end{aligned}$$

- This approximation is justified in weakly interacting systems
- for weak disorder, disorder fluctuations decouple in lowest order

Bogoliubov Transformation

- Diagonalizing the simplified Hamiltonian via inhomogeneous Bogoliubov transformation

$$\begin{aligned}\hat{a}_{\mathbf{k}} &= u_{\mathbf{k}}\hat{\alpha}_{\mathbf{k}} - v_{\mathbf{k}}\hat{\alpha}^{\dagger}_{-\mathbf{k}} - z_{\mathbf{k}},\\ \hat{a}^{\dagger}_{\mathbf{k}} &= u_{\mathbf{k}}\hat{\alpha}^{\dagger}_{\mathbf{k}} - v_{\mathbf{k}}\hat{\alpha}_{-\mathbf{k}} - z^{*}_{\mathbf{k}}\end{aligned}$$

- New operators $\hat{\alpha}_{\mathbf{k}}$, $\hat{\alpha}^{\dagger}_{\mathbf{k}}$ satisfy bosonic commutation relations

$$\left[\hat{\alpha}_{\mathbf{k}}, \hat{\alpha}^{\dagger}_{\mathbf{k}'}\right] = \delta_{\mathbf{k}\mathbf{k}'}, \left[\hat{\alpha}_{\mathbf{k}}, \hat{\alpha}_{\mathbf{k}'}\right] = \left[\hat{\alpha}^{\dagger}_{\mathbf{k}}, \hat{\alpha}^{\dagger}_{\mathbf{k}'}\right] = 0$$

K. Huang and H. F. Meng, *PRL* **69**, 644 (1992)

Bogoliubov Amplitudes And Translation

- Bogoliubov amplitudes $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ read ($n_0 = N_0/v$)

$$u_{\mathbf{k}}^2 = \frac{1}{2}\left[\frac{\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + n_0(V_0 + V_{\mathbf{k}})}{E_{\mathbf{k}}} + 1\right], \; v_{\mathbf{k}}^2 = \frac{1}{2}\left[\frac{\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + n_0(V_0 + V_{\mathbf{k}})}{E_{\mathbf{k}}} - 1\right]$$

- Translation $z_{\mathbf{k}}$ reads

$$z_{\mathbf{k}} = \frac{\frac{1}{v}\sqrt{N_0}\,U_{\mathbf{k}}\left(\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + n_0 V_0\right)}{E_{\mathbf{k}}^2}$$

- Bogoliubov quasi-particle dispersion

$$E_{\mathbf{k}} = \sqrt{\left[\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + n_0\left(V_0 + V_{\mathbf{k}}\right)\right]^2 - \left(n_0 V_{\mathbf{k}}\right)^2}$$

Diagonalized Hamiltonian

- Hamiltonian after disorder ensemble average

$$
\begin{aligned}
\left\langle \hat{\mathcal{K}}' \right\rangle_{\text{dis}} = & v\left(-\mu n_{\mathbf{0}} + \frac{1}{2} V_{\mathbf{0}} n_{\mathbf{0}}^2\right) \\
& + \frac{1}{2}\sum_{\mathbf{k}}{}' \left\{ E_{\mathbf{k}} - \left[\frac{\hbar^2 \mathbf{k}^2}{2m} - \mu + n_{\mathbf{0}}(V_{\mathbf{0}} + V_{\mathbf{k}})\right]\right\} \\
& + \frac{1}{2}\sum_{\mathbf{k}}{}' E_{\mathbf{k}}(\hat{\alpha}_{\mathbf{k}}^{\dagger}\hat{\alpha}_{\mathbf{k}} + \hat{\alpha}_{-\mathbf{k}}^{\dagger}\hat{\alpha}_{-\mathbf{k}}) \\
& - \sum_{\mathbf{k}}{}' \frac{n_{\mathbf{0}} R_{\mathbf{k}}}{E_{\mathbf{k}}^2}\left(\frac{\hbar^2 \mathbf{k}^2}{2m} - \mu + n_{\mathbf{0}} V_{\mathbf{0}}\right)
\end{aligned}
$$

Grand-Canonical Potential

- Grand-canonical potential $\Omega_{\text{eff}} = -\beta^{-1} \ln \mathcal{Z}_G$, where $\mathcal{Z}_G = \mathrm{Tr}\, e^{-\beta \langle \hat{\mathcal{K}}' \rangle_{\text{dis}}}$, reduces to

$$\begin{aligned}\Omega_{\text{eff}} =& v \left(-\mu n_0 + \frac{1}{2} V_0 n_0^2 \right) \\ &+ \frac{1}{2} \sum_{\mathbf{k}}' \left\{ E_{\mathbf{k}} - \left[\frac{\hbar^2 \mathbf{k}^2}{2m} - \mu + n_0 (V_0 + V_{\mathbf{k}}) \right] \right\} \\ &+ \sum_{\mathbf{k}}' \frac{1}{\beta} \ln \left(1 - e^{-\beta E_{\mathbf{k}}} \right) \\ &- \sum_{\mathbf{k}}' \frac{n_0 R_{\mathbf{k}}}{E_{\mathbf{k}}^2} \left(\frac{\hbar^2 \mathbf{k}^2}{2m} - \mu + n_0 V_0 \right)\end{aligned}$$

Grand-Canonical Free Energy

- Extremizing with respect to n_0 for fixed chemical potential μ, we find that the grand-canonical potential Ω_{eff} reduces up to first order in all fluctuations to the grand-canonical free energy

$$\begin{aligned}\mathcal{F} = &-\frac{v\mu^2}{2V_0} \\ &+\frac{1}{2}\sum_{\mathbf{k}}{}' \left[E_{\mathbf{k}} - \left(\frac{\hbar^2\mathbf{k}^2}{2m} + \mu\frac{V_{\mathbf{k}}}{V_0}\right)\right] \\ &+\sum_{\mathbf{k}}{}' \frac{1}{\beta}\ln\left(1 - e^{-\beta E_{\mathbf{k}}}\right) \\ &-\sum_{\mathbf{k}}{}' \frac{R_{\mathbf{k}}}{E_{\mathbf{k}}^2}\frac{\hbar^2\mathbf{k}^2}{2m}\frac{\mu}{V_0}\end{aligned}$$

Condensate Depletion, $n - n_0 = n' + n_{\mathrm{th}} + n_{\mathrm{R}}$

- Due to quantum fluctuations

$$n' = \frac{1}{2v} \sum_{\mathbf{k}}' \left(\frac{\frac{\hbar^2 \mathbf{k}^2}{2m} + nV_{\mathbf{k}}}{E_{\mathbf{k}}} - 1 \right)$$

- Due to thermal fluctuations

$$n_{\mathrm{th}} = \frac{1}{v} \sum_{\mathbf{k}}' \frac{\frac{\hbar^2 \mathbf{k}^2}{2m} + nV_{\mathbf{k}}}{E_{\mathbf{k}}} \frac{1}{e^{\beta E_{\mathbf{k}}} - 1}$$

- Due to external random potential

$$n_{\mathrm{R}} = \frac{1}{v} \sum_{\mathbf{k}}' \frac{nR_{\mathbf{k}}}{E_{\mathbf{k}}^4} \left(\frac{\hbar^2 \mathbf{k}^2}{2m} \right)^2$$

$E_{\mathbf{k}} = \sqrt{\left(\frac{\hbar^2 \mathbf{k}^2}{2m}\right)^2 + nV_{\mathbf{k}} \frac{\hbar^2 \mathbf{k}^2}{m}}$ denotes Bogoliubov dispersion relation

Condensate Depletion ($T = 0$)

- Due to quantum fluctuations

$$n' = \frac{8}{3\sqrt{\pi}} (na)^{\frac{3}{2}} Q_{\frac{3}{2}}(\epsilon_{\mathrm{dd}})$$

A. R. P. Lima and A. Pelster, *PRA* **84**, 041604(R) (2011)

- Due to external random potential

$$n_{\mathrm{R}} = \frac{m^2 R_0}{8\hbar^4 \pi^{\frac{3}{2}}} \sqrt{\frac{n}{a}} Q_{-\frac{1}{2}}(\epsilon_{\mathrm{dd}})$$

C. Krumnow et al., *PRA* **84**, 021608(R) (2011); B. Nikolic et al., *PRA* **88**, 013624 (2013)

- Functions describe dipolar effect expressed analytically

$$Q_\alpha(\epsilon_{\mathrm{dd}}) = (1 - \epsilon_{\mathrm{dd}})^\alpha \, {}_2F_1\left(-\alpha, \frac{1}{2}; \frac{3}{2}; \frac{-3\epsilon_{\mathrm{dd}}}{1 - \epsilon_{\mathrm{dd}}}\right)$$

${}_2F_1$ denotes hypergeometric function

Function Plot

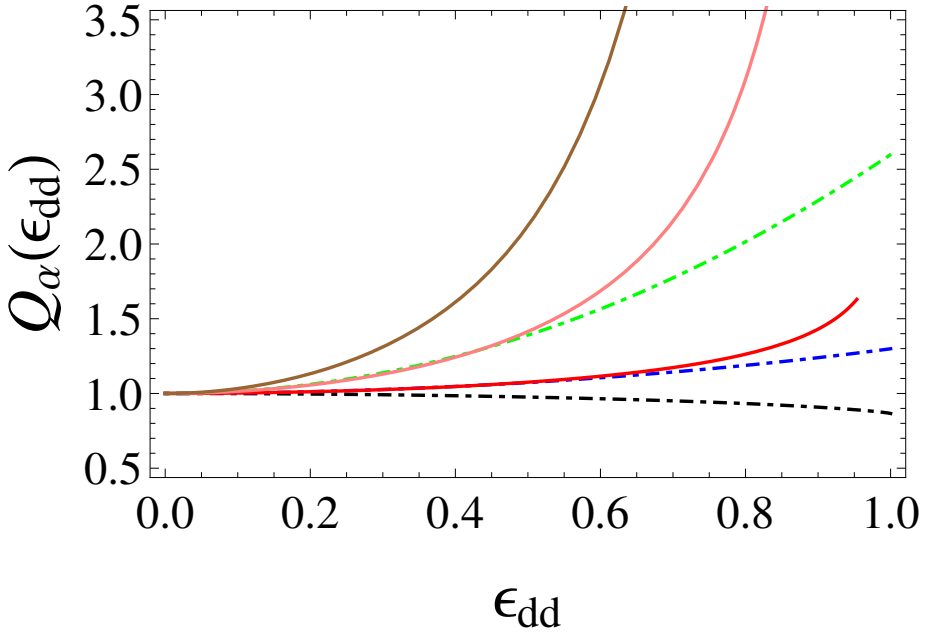


Figure: (1) Dipolar enhancement function $Q_\alpha(\epsilon_{dd})$ versus relative dipolar interaction strength ϵ_{dd} for different values of α: -5/2 (brown), -3/2 (pink), -1/2 (red), 1/2 (black), 3/2 (blue), 5/2 (green).

Model

- Inserting Galilean transformations $\mathbf{x}' = \mathbf{x} + \mathbf{u}t$, $t = t'$, and field operator in Heisenberg picture $\hat{\psi}'(\mathbf{x}', t') = \hat{\psi}(\mathbf{x}, t) e^{\frac{i}{\hbar} m \mathbf{v}_{\mathrm{s}} \mathbf{x}}$, Hamiltonian reads

$$\begin{aligned}\hat{\mathcal{K}} =& \frac{1}{2} \sum_{\mathbf{k}} \left[\frac{\hbar^2 \mathbf{k}^2}{2m} - \mu_{\mathrm{eff}} \right] (\hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{-\mathbf{k}} \hat{a}_{-\mathbf{k}}) \\ &+ \frac{1}{2} \sum_{\mathbf{k}} \hbar \mathbf{k} (\mathbf{u} - \mathbf{v}_{\mathrm{s}}) (\hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{-\mathbf{k}} \hat{a}_{-\mathbf{k}}) \\ &+ \frac{1}{2v} \sum_{\mathbf{p},\mathbf{k}} U_{\mathbf{p}-\mathbf{k}} (\hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{-\mathbf{k}} \hat{a}_{-\mathbf{p}}) \\ &+ \frac{1}{2v} \sum_{\mathbf{p},\mathbf{k},\mathbf{q}} V_{\mathbf{q}} \, \hat{a}^\dagger_{\mathbf{k}+\mathbf{q}} \hat{a}^\dagger_{\mathbf{p}-\mathbf{q}} \hat{a}_{\mathbf{p}} \hat{a}_{\mathbf{k}}\end{aligned}$$

$\mu_{\mathrm{eff}} = \mu - \frac{1}{2} m \mathbf{v}_{\mathrm{s}}^2 + m \mathbf{u} \mathbf{v}_{\mathrm{s}}$ denotes effective chemical potential

Diagonalized Hamiltonian

- Hamiltonian after disorder ensemble average

$$
\begin{aligned}
\left\langle \hat{\mathcal{K}}' \right\rangle_{\text{dis}} =& v\left(-\mu_{\text{eff}} n_{\mathbf{0}} + \frac{1}{2} V_{\mathbf{0}} n_{\mathbf{0}}^2\right) \\
&+ \frac{1}{2}\sum_{\mathbf{k}}{}' \left\{ E_{\mathbf{k}} - \left[\frac{\hbar^2 \mathbf{k}^2}{2m} - \mu_{\text{eff}} + n_{\mathbf{0}}(V_{\mathbf{0}} + V_{\mathbf{k}})\right]\right\} \\
&+ \sum_{\mathbf{k}}{}' \left[E_{\mathbf{k}} + \hbar\mathbf{k}(\mathbf{u} - \mathbf{v}_{\text{s}})\right] \hat{\alpha}_{\mathbf{k}}^{\dagger} \hat{\alpha}_{\mathbf{k}} \\
&- \sum_{\mathbf{k}}{}' \frac{n_{\mathbf{0}} R_{\mathbf{k}} \left(\frac{\hbar^2 \mathbf{k}^2}{2m} - \mu_{\text{eff}} + n_{\mathbf{0}} V_0\right)}{E_{\mathbf{k}}^2 - \left[\hbar\mathbf{k}(\mathbf{u} - \mathbf{v}_{\text{s}})\right]^2}
\end{aligned}
$$

Grand-Canonical Effective Potential

- Up to second order in $\hbar\mathbf{k}\,(\mathbf{u}-\mathbf{v}_{\mathrm{s}})$

$$
\begin{aligned}
\Omega_{\mathrm{eff}} =& v\left(-\mu_{\mathrm{eff}}\, n_0 + \frac{1}{2} V_0 n_0^2\right) \\
&+ \frac{1}{2}\sum_{\mathbf{k}}{}' \left\{ E_{\mathbf{k}} - \left[\frac{\hbar^2\mathbf{k}^2}{2m} - \mu_{\mathrm{eff}} + n_0(V_0 + V_{\mathbf{k}})\right]\right\} \\
&+ \sum_{\mathbf{k}}{}' \frac{1}{\beta}\ln\left(1 - e^{-\beta E_{\mathbf{k}}}\right) - \sum_{\mathbf{k}}{}' \frac{\beta e^{\beta E_{\mathbf{k}}}\left[\hbar\mathbf{k}(\mathbf{u}-\mathbf{v}_{\mathrm{s}})\right]^2}{2\left(e^{\beta E_{\mathbf{k}}} - 1\right)^2} \\
&- \sum_{\mathbf{k}}{}' \frac{n_0 R_{\mathbf{k}}}{E_{\mathbf{k}}^2}\left[\frac{\hbar^2\mathbf{k}^2}{2m} - \mu_{\mathrm{eff}} + n_0 V_0\right] \\
&- \sum_{\mathbf{k}}{}' \frac{n_0 R_{\mathbf{k}}}{E_{\mathbf{k}}^4}\left[\frac{\hbar^2\mathbf{k}^2}{2m} - \mu_{\mathrm{eff}} + n_0 V_0\right]\left[\hbar\mathbf{k}(\mathbf{u}-\mathbf{v}_{\mathrm{s}})\right]^2
\end{aligned}
$$

System Momentum

- Extremizing yields in zeroth order $\mu_{\rm eff} = n_0 V_0$
- From thermodynamic relation $\mathbf{P} = \left(-\frac{\partial \Omega_{\rm eff}}{\partial \mathbf{u}}\right)_{v,T,\mu}$ we find

$$\mathbf{P} = mv\left(n\mathbf{v}_s + n_{\rm n}\mathbf{v}_{\rm n}\right)$$

Normal fluid density decompose to $n_{{\rm n}ij} = n_{{\rm R}ij} + n_{{\rm th}ij}$, $\mathbf{v}_{\rm n} = \mathbf{u} - \mathbf{v}_{\rm s}$

- Contribution due to external random potential

$$n_{{\rm R}ij} = \frac{1}{v}\sum_{\mathbf{k}}{}' \frac{2nR_{\mathbf{k}}\hbar^2 k_i k_j}{m\left(\frac{\hbar^2\mathbf{k}^2}{2m}\right)\left(\frac{\hbar^2\mathbf{k}^2}{2m} + 2nV_{\mathbf{k}}\right)^2}$$

- Contribution due to thermal fluctuations

$$n_{{\rm th}ij} = \frac{1}{v}\sum_{\mathbf{k}}{}' \frac{\beta}{m}\hbar^2 k_i k_j \frac{e^{\beta E_{\mathbf{k}}}}{\left(e^{\beta E_{\mathbf{k}}} - 1\right)^2}$$

Superfluid Depletion ($T = 0$)

- In direction parallel to dipole polarization

$$n_{\mathrm{R}\parallel} = \frac{m^2 R_0}{2\hbar^4 \pi^{\frac{3}{2}}} \sqrt{\frac{n}{a}}\, J_{-\frac{1}{2}}(\epsilon_{\mathrm{dd}})$$

- In direction perpendicular to dipole polarization

$$n_{\mathrm{R}\perp} = \frac{m^2 R_0}{4\hbar^4 \pi^{\frac{3}{2}}} \sqrt{\frac{n}{a}} \left[Q_{-\frac{1}{2}}(\epsilon_{\mathrm{dd}}) - J_{-\frac{1}{2}}(\epsilon_{\mathrm{dd}}) \right]$$

C. Krumnow et al., *PRA* **84**, 021608(R) (2011); B. Nikolic et al., *PRA* **88**, 013624 (2013)

- Functions describe dipolar effect expressed analytically

$$J_\alpha(\epsilon_{\mathrm{dd}}) = \frac{1}{3}(1-\epsilon_{\mathrm{dd}})^\alpha \, {}_2F_1\left(-\alpha, \frac{3}{2}; \frac{5}{2}; \frac{-3\epsilon_{\mathrm{dd}}}{1-\epsilon_{\mathrm{dd}}}\right)$$

${}_2F_1$ denotes hypergeometric function

Ratios

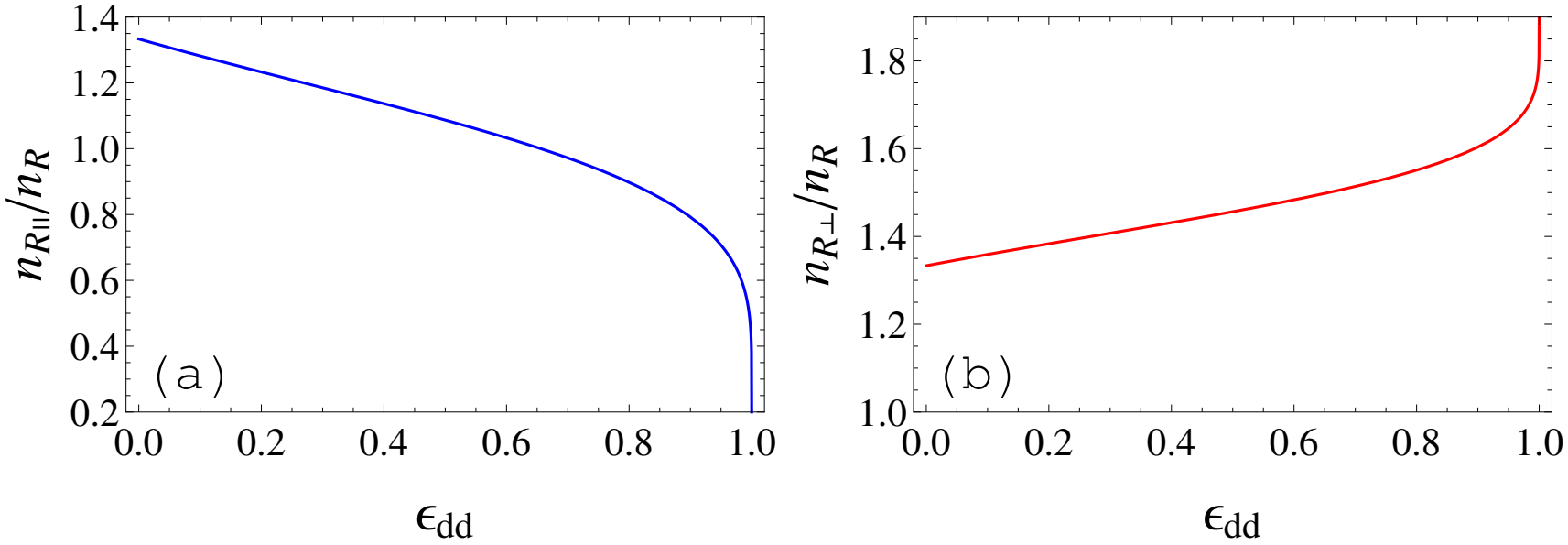


Figure: (2) Ratios of superfluid depletions $n_{\mathrm{R}\parallel}$ and $n_{\mathrm{R}\perp}$ and condensate depletion n_{R} versus relative dipolar interaction strength ϵ_{dd}.

==> finite localization time

R. Graham and A. Pelster, *Int. J. Bif. Chaos* **19**, 2745 (2009)

Ratio

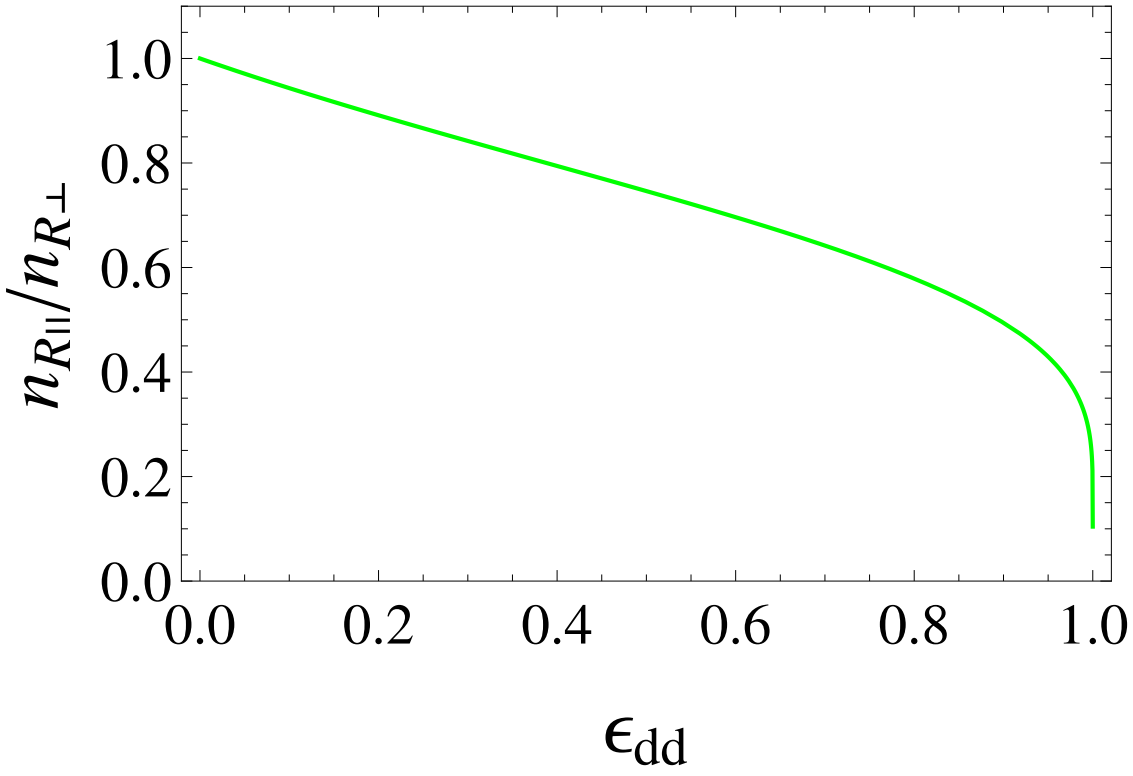


Figure: (3) Superfluid depletions ratio $n_{\mathrm{R}\parallel}/n_{\mathrm{R}\perp}$ versus relative dipolar interaction strength ϵ_{dd}.

Condensate Depletion ($T > 0$)

- Condensate depletion due to thermal excitations

$$\frac{n_{\mathrm{th}}}{n} = \frac{\gamma^{-\frac{1}{6}} t^2}{2\pi^{\frac{1}{2}} \left(\zeta(\frac{3}{2})\right)^{\frac{4}{3}}} I(\gamma, \epsilon_{\mathrm{dd}}, t)$$

Gas parameter $\gamma = na^3$, and relative temperature $t = T/T_c^0$,
$T_c^0 = 2\pi\hbar^2 n^{\frac{2}{3}} / \left(\zeta(\frac{3}{2})\right)^{\frac{2}{3}} m k_{\mathrm{B}}$ non-interacting Bose gas critical temperature

- The integral $I(\gamma, \epsilon_{\mathrm{dd}}, t)$ reads

$$I(\gamma, \epsilon_{\mathrm{dd}}, t) = \int_0^\infty dx \int_0^\pi d\theta \frac{x \sin\theta \left(1 + \frac{\alpha x^2}{8\Theta^2}\right)}{\sqrt{\Theta + \frac{\alpha x^2}{16\Theta}} \left(e^{\sqrt{x^2 + \frac{\alpha x^4}{16\Theta^2}}} - 1\right)}$$

with abbreviations $\alpha = \left[t/\gamma^{\frac{1}{3}} \left(\zeta(\frac{3}{2})\right)^{\frac{2}{3}}\right]^2$ and $\Theta = 1 + \epsilon_{\mathrm{dd}} \left(3\cos^2\theta - 1\right)$

Depletion Plot

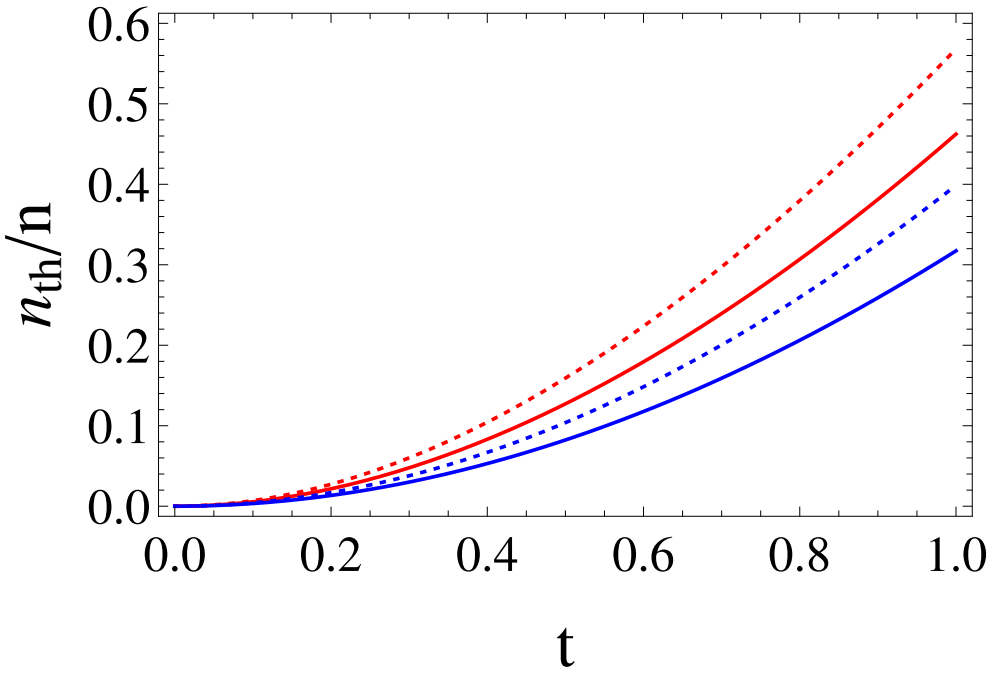


Figure: (4) Thermal fractional depletion $n_{\rm th}/n$ versus relative temperature t for different values of relative dipolar interaction strength $\epsilon_{\rm dd} = 0$ (solid), $\epsilon_{\rm dd} = 0.8$ (dotted), and gas parameter $\gamma = 0.01$ (red), $\gamma = 0.20$ (blue).

Zero-Temperature Vicinity ($T \approx 0$)

- Below the critical temperature we approximate the condensate depletion analytically

$$\frac{n_{\mathrm{th}}}{n} = \frac{\pi^{\frac{3}{2}}\gamma^{-\frac{1}{6}}t^2}{6\left(\zeta(\frac{3}{2})\right)^{\frac{4}{3}}} Q_{-\frac{1}{2}}(\epsilon_{\mathrm{dd}}) - \frac{\pi^{\frac{7}{2}}\gamma^{-\frac{5}{6}}t^4}{480\left(\zeta(\frac{3}{2})\right)^{\frac{8}{3}}} Q_{-\frac{5}{2}}(\epsilon_{\mathrm{dd}}) + \ldots$$

This reproduces the isotropic contact interaction case for vanishing dipolar interaction due to $Q_\alpha(0) = 1$

Validity Range

- For fractional condensate depletion values limited to $\frac{n-n_0}{n} \leq \frac{1}{2}$

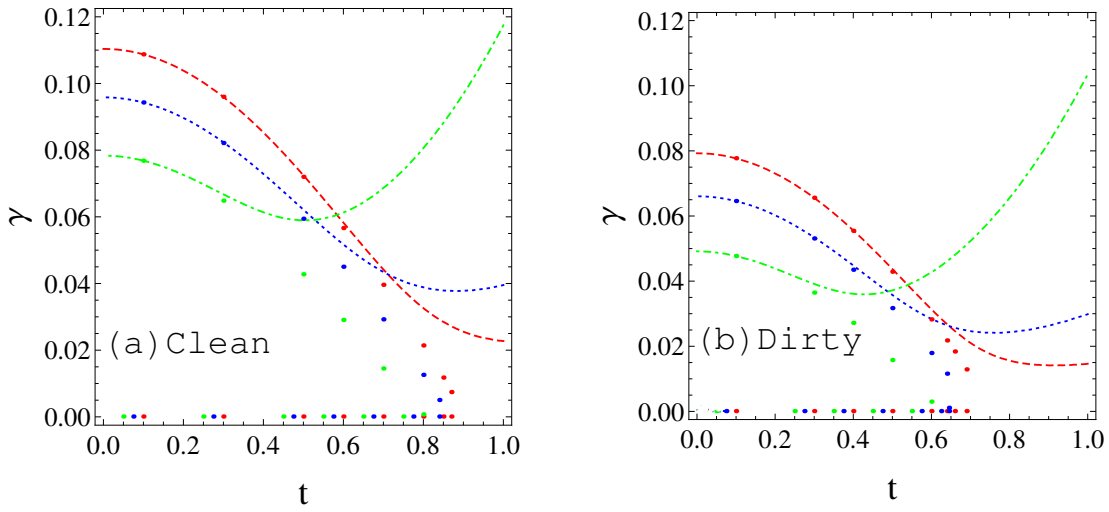


Figure: (5) Validity range of Bogoliubov theory in the $t-\gamma$ plane for (a) clean case, i.e. $R_0 = 0$, and (b) dirty case with $R_0 = \frac{2\hbar^4 \pi^{\frac{3}{2}} n^{\frac{1}{3}}}{5m^2}$ for different values of relative dipolar interaction strength $\epsilon_{\rm dd} = 0$ (red), $\epsilon_{\rm dd} = 0.5$ (blue), $\epsilon_{\rm dd} = 0.8$ (green).

Condensate Depletion Versus Gas Parameter

- The total fractional condensate depletion $\frac{\Delta n}{n}$ versus γ

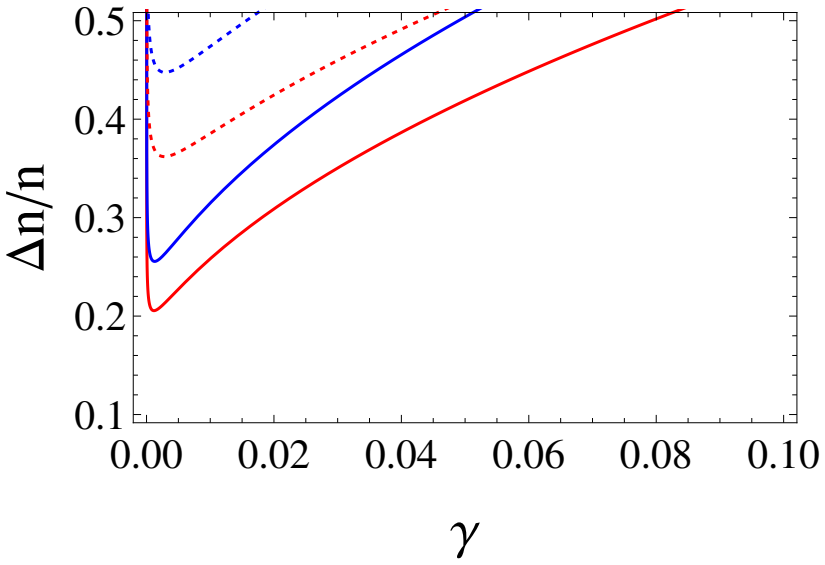


Figure: (6) Fractional depletion $\Delta n/n$ versus gas parameter γ for different values of relative dipolar interaction strength $\epsilon_{\rm dd} = 0$ (red), $\epsilon_{\rm dd} = 0.8$ (blue) and relative temperature $t = 0$ (solid), $t = 0.5$ (dotted) with the disorder strength $R_0 = \frac{2\hbar^4 \pi^{\frac{3}{2}} n^{\frac{1}{3}}}{5m^2}$.

Superfluid Depletion ($T > 0$)

- Parallel to the dipoles

$$\frac{n_{\mathrm{th}\parallel}}{n} = \frac{\gamma^{-\frac{5}{6}} t^4}{8\pi^{\frac{1}{2}} \left(\zeta(\frac{3}{2})\right)^{\frac{8}{3}}} \int_0^\infty dx \int_0^\pi d\theta \frac{x^4 \sin\theta \cos^2\theta e^{\sqrt{x^2+\frac{\alpha x^4}{16\Theta^2}}}}{\Theta^{\frac{5}{2}} \left(e^{\sqrt{x^2+\frac{\alpha x^4}{16\Theta^2}}} - 1\right)^2}$$

- Perpendicular to the dipoles

$$\frac{n_{\mathrm{th}\perp}}{n} = \frac{\gamma^{-\frac{5}{6}} t^4}{8\pi^{\frac{1}{2}} \left(\zeta(\frac{3}{2})\right)^{\frac{8}{3}}} \int_0^\infty dx \int_0^\pi d\theta \frac{x^4 \sin^3\theta e^{\sqrt{x^2+\frac{\alpha x^4}{16\Theta^2}}}}{2\Theta^{\frac{5}{2}} \left(e^{\sqrt{x^2+\frac{\alpha x^4}{16\Theta^2}}} - 1\right)^2}$$

with abbreviations $\alpha = \left[t/\gamma^{\frac{1}{3}} \left(\zeta(\frac{3}{2})\right)^{\frac{2}{3}}\right]^2$ and $\Theta = 1 + \epsilon_{\mathrm{dd}} \left(3\cos^2\theta - 1\right)$

Superfluid Depletion Plot I

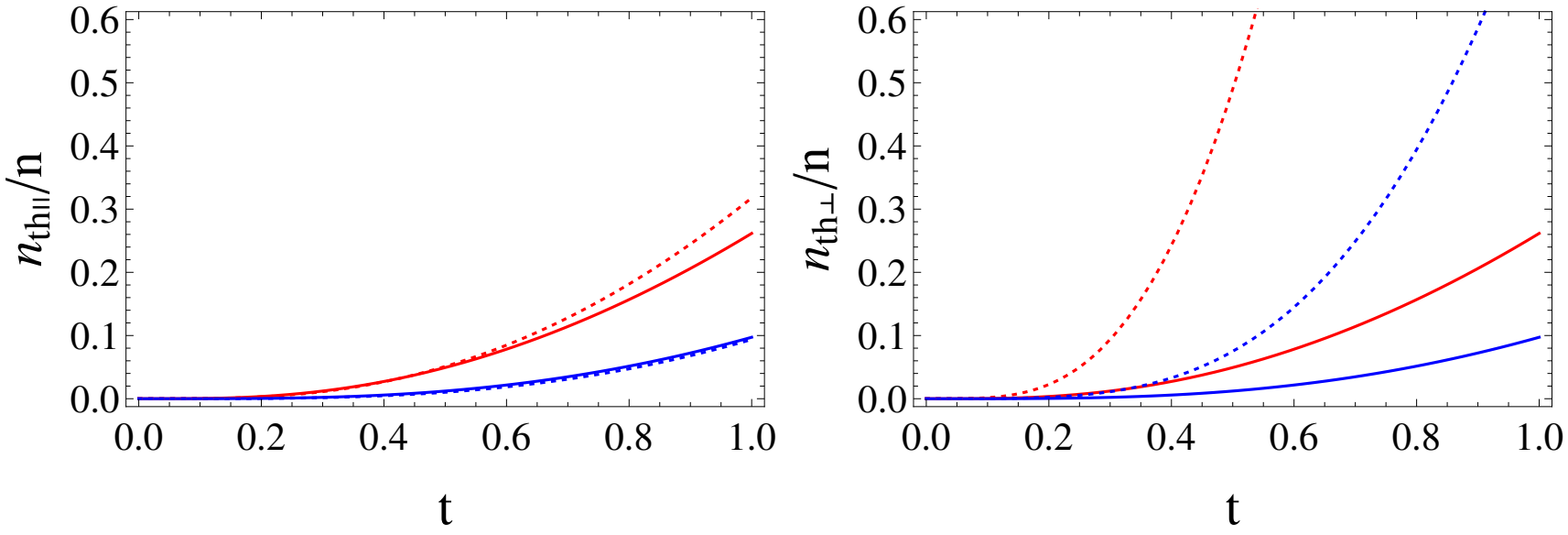


Figure: (7) Superfluid thermal fractional depletions $n_{\mathrm{th}\parallel}/n$ and $n_{\mathrm{th}\perp}/n$ versus relative temperature t for different values of relative dipolar interaction strength $\epsilon_{\mathrm{dd}} = 0$ (solid), $\epsilon_{\mathrm{dd}} = 0.6$ (dotted) and gas parameter $\gamma = 0.01$ (red), $\gamma = 0.20$ (blue).

Superfluid Depletion Plot II

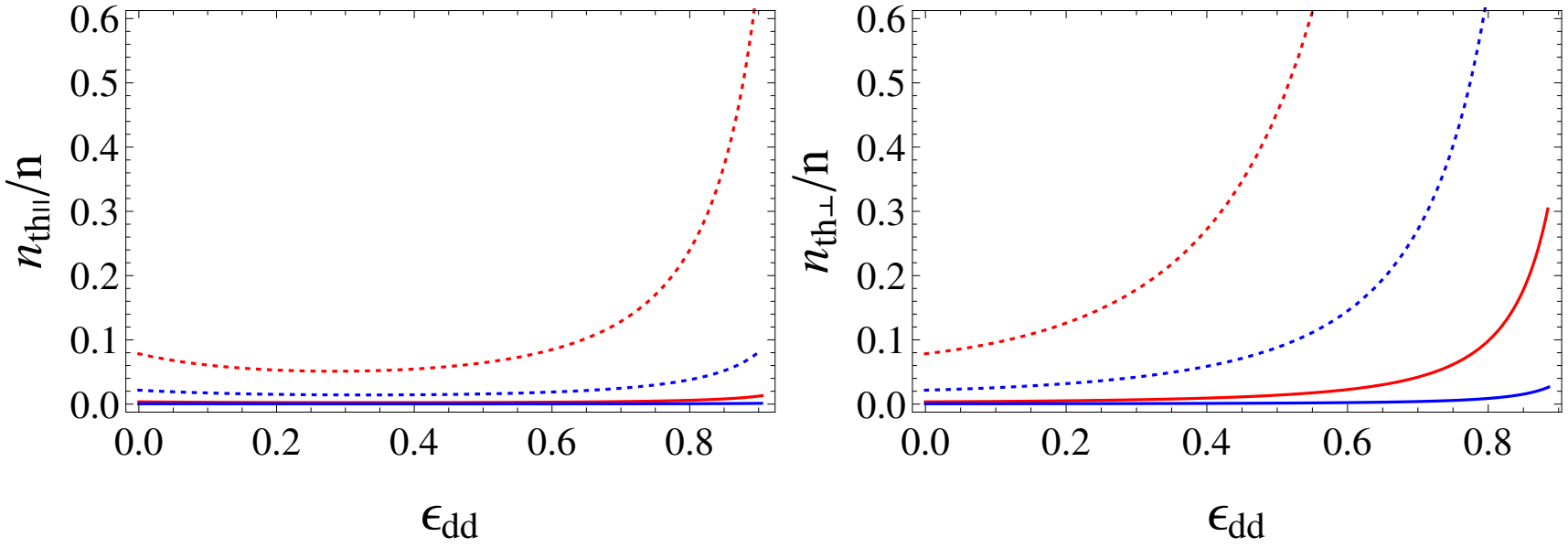


Figure: (8) Superfluid thermal fractional depletions $n_{\mathrm{th}\parallel}/n$ and $n_{\mathrm{th}\perp}/n$ versus relative dipolar interaction strength ϵ_{dd} for different values of relative temperature $t = 0.2$ (solid), $t = 0.6$ (dotted) and gas parameter $\gamma = 0.01$ (red), $\gamma = 0.20$ (blue).

Superfluid Depletion Plot III

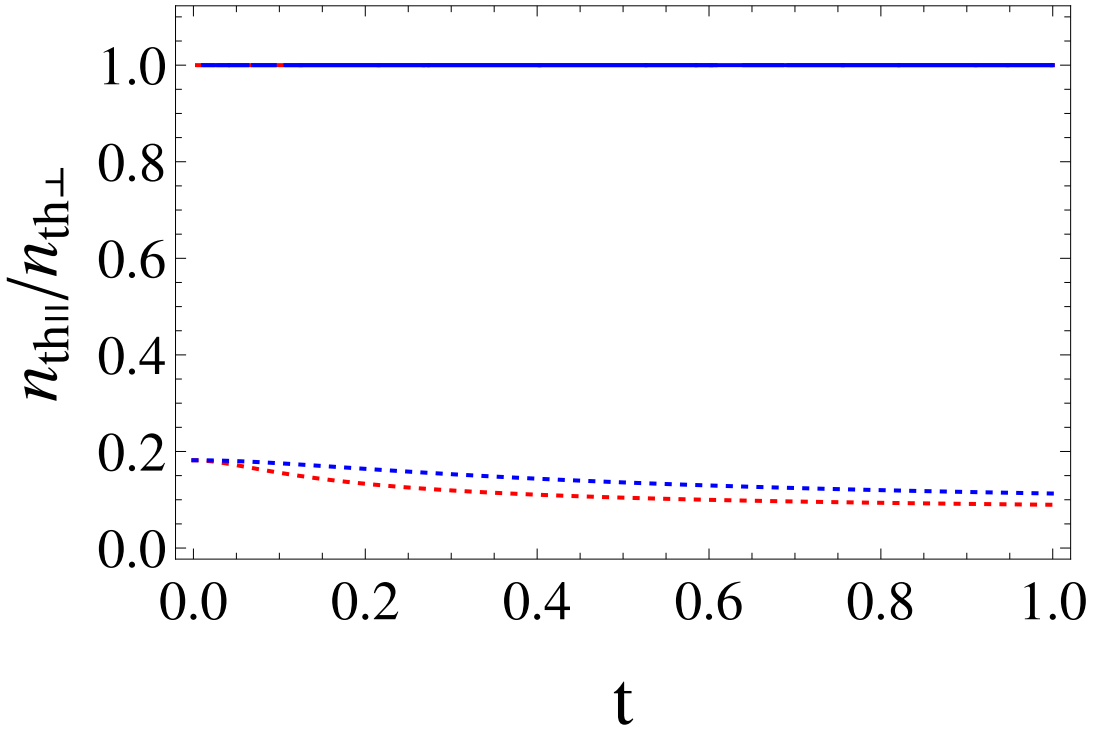


Figure: (9) Ratios of thermal superfluid depletions $n_{\mathrm{th}\parallel}/n_{\mathrm{th}\perp}$ versus relative temperature t for different values of relative dipolar strength $\epsilon_{\mathrm{dd}} = 0$ (solid), $\epsilon_{\mathrm{dd}} = 0.6$ (dotted) and gas parameter $\gamma = 0.01$ (red), $\gamma = 0.20$ (blue).

Zero-Temperature Vicinity ($T \approx 0$)

- Below the critical temperature we approximate the superfluid depletion analytically
- Parallel to the dipoles

$$\frac{n_{\mathrm{th}\parallel}}{n} = \frac{\pi^{\frac{7}{2}}\gamma^{-\frac{5}{6}}t^4}{15\left(\zeta(\frac{3}{2})\right)^{\frac{8}{3}}} J_{-\frac{5}{2}}(\epsilon_{\mathrm{dd}}) + ..$$

- Perpendicular to the dipoles

$$\frac{n_{\mathrm{th}\perp}}{n} = \frac{\pi^{\frac{7}{2}}\gamma^{-\frac{5}{6}}t^4}{30\left(\zeta(\frac{3}{2})\right)^{\frac{8}{3}}} \left[Q_{-\frac{5}{2}}(\epsilon_{\mathrm{dd}}) - J_{-\frac{5}{2}}(\epsilon_{\mathrm{dd}})\right] + ..$$

Superfluid Depletion Plot IV

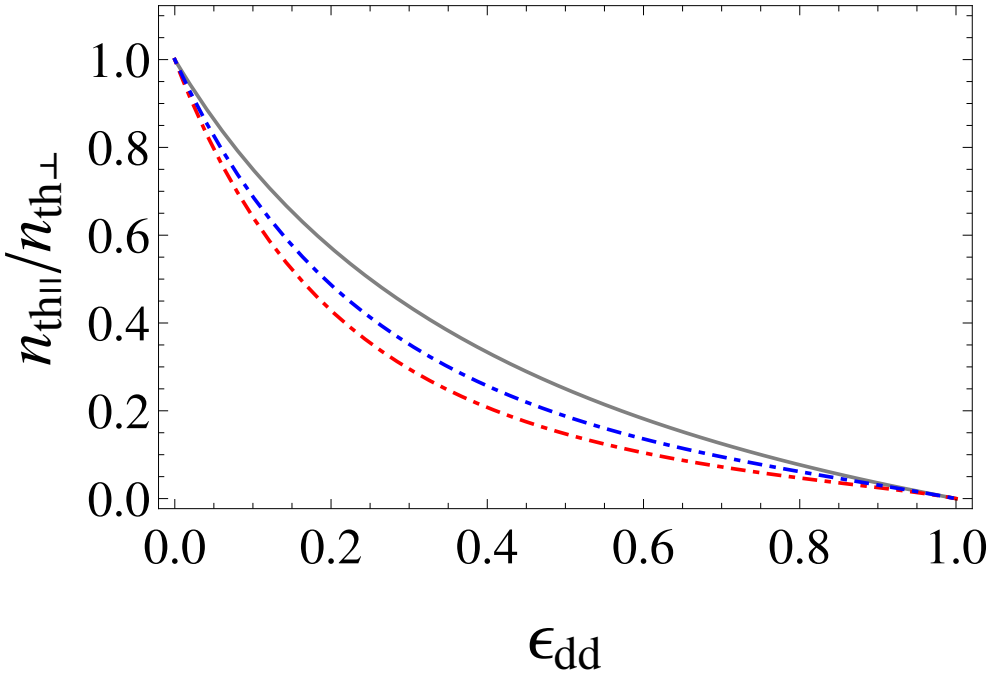


Figure: (10) Ratios of thermal superfluid depletions $n_{\mathrm{th}\parallel}/n_{\mathrm{th}\perp}$ versus relative dipolar interaction strength ϵ_{dd} for different values of the gas parameter $\gamma = 0.01$ (red), $\gamma = 0.20$ (blue) and relative temperature $t = 0$ (solid-gray), $t = 0.5$ (dotted-dashed).

Outlook

Further investigations

- Sound velocities in the anisotropic two-fluid model
- Anisotropic disorder potential
- Local density approximation
- Anisotropic trap potential

Thank You For Your Attention