

Ginzburg-Landau Theory for Bosonic Lattices

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Landau theory:

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Ginzburg-Landau theory:

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1 - Generating functional

Hamiltonian with inhomogeneous sources:

$$\hat{H}(\tau) = \hat{H}_0 - \sum_{ij} t_{ij} \hat{a}_i^\dagger \hat{a}_j + \sum_i \left[j_i(\tau) \hat{a}_i^\dagger + j_i^*(\tau) \hat{a}_i \right]$$

here: $\hat{H}_0 = \sum_i f_i(\hat{a}_i^\dagger \hat{a}_i)$

Generalized grand-canonical partition function:

$$\mathcal{Z} = \text{tr} \left\{ \hat{T} e^{- \int_0^\beta d\tau \hat{H}(\tau)} \right\} = \text{tr} \left\{ e^{-\beta \hat{H}_0} \hat{U}_{\text{D}}(\beta, 0) \right\}$$

Generalized free energy: $\mathcal{F} = -\frac{1}{\beta} \log \mathcal{Z}$

Order parameter field:

$$\psi_i(\tau) = \beta \frac{\delta \mathcal{F}}{\delta j_i^*(\tau)}; \quad \psi_i^*(\tau) = \beta \frac{\delta \mathcal{F}}{\delta j_i(\tau)}$$

2 - Calculation tools

Double expansion in currents $j_i(\tau), j_i^*(\tau)$
and hopping parameters t_{ij}

$$\mathcal{F} = F_0$$

$$\begin{aligned} & - \frac{1}{\beta} \sum_i \left\{ \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \left[a_2^{(0)}(i, \tau_1 | i, \tau_2) j_i(\tau_1) j_i^*(\tau_2) + \sum_j a_2^{(1)}(i, \tau_1 | j, \tau_2) t_{ij} j_i(\tau_1) j_j^*(\tau_2) \right] \right. \\ & + \frac{1}{4} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int_0^\beta d\tau_3 \int_0^\beta d\tau_4 a_4^{(0)}(i, \tau_1; i, \tau_2 | i, \tau_3; i, \tau_4) j_i(\tau_1) j_i(\tau_2) j_i^*(\tau_3) j_i^*(\tau_4) \\ & + \frac{1}{2} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int_0^\beta d\tau_3 \int_0^\beta d\tau_4 \sum_j t_{ij} \left[a_4^{(1)}(i, \tau_1; i, \tau_2 | j, \tau_3; i, \tau_4) j_i(\tau_1) j_i(\tau_2) j_j^*(\tau_3) j_i^*(\tau_4) \right. \\ & \left. \left. + a_4^{(1)}(i, \tau_1; j, \tau_2 | i, \tau_3; i, \tau_4) j_i(\tau_1) j_j(\tau_2) j_i^*(\tau_3) j_i^*(\tau_4) \right] \right\} \end{aligned}$$

2.1 - Hopping expansion

Coefficients in diagrammatic notation:

$$a_2^{(0)}(i, \tau_1 | i, \tau_2) = \tau_1 \xrightarrow{i} \tau_2 = {}_i C_2^{(0)}(\tau_1 | \tau_2)$$

$$a_2^{(1)}(i, \tau_1 | j, \tau_2) = \tau_1 \xrightarrow{i} \tau_2 = \int_0^\beta d\tau_i C_2^{(0)}(\tau_1 | \tau)_j C_2^{(0)}(\tau | \tau_2)$$

$$a_4^{(0)}(i, \tau_1; i, \tau_2 | i, \tau_3; i, \tau_4) = \begin{array}{c} \tau_2 \\ \tau_1 \end{array} \xrightarrow{i} \begin{array}{c} \tau_3 \\ \tau_4 \end{array} = {}_i C_4^{(0)}(\tau_1, \tau_2 | \tau_3, \tau_4)$$

$$a_4^{(1)}(i, \tau_1; i, \tau_2 | j, \tau_3; i, \tau_4) = \begin{array}{c} \tau_2 \\ \tau_1 \end{array} \xrightarrow{i} \begin{array}{c} j \\ \tau_3 \\ \tau_4 \end{array} = \int_0^\beta d\tau_i C_4^{(0)}(\tau_1, \tau_2 | \tau, \tau_4)_j C_2^{(0)}(\tau | \tau_3)$$

3 - Effective action

Free energy in Matsubara space:

$$\begin{aligned} \mathcal{F} = F_0 - \frac{1}{\beta} & \left[\sum_{ij} \sum_{\omega_{m1}, \omega_{m2}} M_{ij}(\omega_{m1}, \omega_{m2}) j_i(\omega_{m1}) j_j^*(\omega_{m2}) \right. \\ & + \left. \sum_{ijkl} \sum_{\substack{\omega_{m1}, \omega_{m2} \\ \omega_{m3}, \omega_{m4}}} N_{ijkl}(\omega_{m1}, \omega_{m2}, \omega_{m3}, \omega_{m4}) j_i(\omega_{m1}) j_j(\omega_{m2}) j_k^*(\omega_{m3}) j_l^*(\omega_{m4}) \right] \end{aligned}$$

Effective action:

$$\Gamma[\psi_i(\omega_m), \psi_i^*(\omega_m)] = \mathcal{F} - \frac{1}{\beta} \sum_i \sum_{\omega_m} [\psi_i(\omega_m) j_i^*(\omega_m) + \psi_i^*(\omega_m) j_i(\omega_m)]$$

Equations of motion:

$$\frac{\partial \Gamma}{\partial \psi_i(\omega_m)} = 0; \quad \frac{\partial \Gamma}{\partial \psi_i^*(\omega_m)} = 0$$

3.1 - Ginzburg-Landau theory

Hopping expansion gives:

$$\begin{aligned}\Gamma = F_0 + \frac{1}{\beta} \sum_i \left\{ \sum_{\omega_m} \left[\frac{|\psi_i(\omega_m)|^2}{a_2^{(0)}(i, \omega_m)} - \sum_j t_{ij} \psi_i(\omega_m) \psi_j^*(\omega_m) \right] \right. \\ \left. - \sum_{\substack{\omega_{m1}, \omega_{m2} \\ \omega_{m3}, \omega_{m4}}} \frac{a_4^{(0)}(i, \omega_{m1}; i, \omega_{m2} | i, \omega_{m3}; i, \omega_{m4})}{4a_2^{(0)}(i, \omega_{m1}) a_2^{(0)}(i, \omega_{m2}) a_2^{(0)}(i, \omega_{m3}) a_2^{(0)}(i, \omega_{m4})} \right. \\ \left. \times \psi_i(\omega_{m1}) \psi_i(\omega_{m2}) \psi_i^*(\omega_{m3}) \psi_i^*(\omega_{m4}) \right\}\end{aligned}$$

Static case:

$$\psi_i(\omega_m) = \psi \sqrt{\beta} \delta_{\omega_m, 0}$$

4 - Landau theory

Effective potential:

$$\Gamma = F_0 + N_s \left[\frac{|\psi|^2}{a_2^{(0)}(0)} - \frac{\beta a_4^{(0)}(0, 0|0, 0)}{4 \left[a_2^{(0)}(0) \right]^4} |\psi|^4 \right] - |\psi|^2 \sum_{ij} t_{ij}$$

Condensate density:

$$|\psi|_{\text{eq}}^2 = \frac{2(a_2^{(0)}(0))^3 \left[N_s - a_2^{(0)}(0) \sum_{ij} t_{ij} \right]}{\beta N_s a_4^{(0)}(0, 0|0, 0)}$$

Particle density and compressibility:

$$\langle n \rangle = - \frac{1}{N_s} \frac{\partial \Gamma}{\partial \mu} \Bigg|_{\psi=\psi_{\text{eq}}} ; \quad \kappa = - \frac{1}{N_s} \frac{\partial^2 \Gamma}{\partial \mu^2} \Bigg|_{\psi=\psi_{\text{eq}}}$$

5 - Superfluid density

Peierls phase factor: $\hat{a}_i \rightarrow \hat{a}_i e^{i \frac{\vec{x}_i}{L} \cdot \vec{\phi}}$

Superfluid density:

$$\rho = \lim_{|\vec{\phi}| \rightarrow 0} \frac{2m^* L^2}{N_s |\vec{\phi}|^2} \left[\Gamma(\vec{\phi}) \Big|_{\psi=\psi_{\text{eq}}(\vec{\phi})} - \Gamma(\vec{0}) \Big|_{\psi=\psi_{\text{eq}}(\vec{0})} \right]$$

First hopping order:

$$\begin{aligned} \rho = & \lim_{|\vec{\phi}| \rightarrow 0} \frac{2m^* L^2}{N_s |\vec{\phi}|^2} \left\{ \sum_{ij} \left[t_{ij} \left(\left| \psi_{\text{eq}}(\vec{\phi}) \right|^2 e^{i \frac{\vec{x}_j - \vec{x}_i}{L} \cdot \vec{\phi}} - \left| \psi_{\text{eq}}(\vec{0}) \right|^2 \right) \right] \right. \\ & \left. + N_s \left[\frac{1}{a_2^{(0)}(0)} \left(\left| \psi_{\text{eq}}(\vec{\phi}) \right|^2 - \left| \psi_{\text{eq}}(\vec{0}) \right|^2 \right) - \frac{\beta a_4^{(0)}(0, 0|0, 0)}{4(a_2^{(0)}(0))^4} \left(\left| \psi_{\text{eq}}(\vec{\phi}) \right|^4 - \left| \psi_{\text{eq}}(\vec{0}) \right|^4 \right) \right] \right\}. \end{aligned}$$

Taking the limit: $\rho = |\psi_{\text{eq}}|^2$

6 - Dynamical quantities

Real time effective action from: $\tau \rightarrow it$

Dispersion relations are defined through:

$$\frac{\delta^2 \Gamma_R}{\delta(\delta\psi_i(\omega_1))\delta(\delta\psi_j^*(\omega_2))} \Bigg|_{\delta\psi \equiv 0} = 0$$

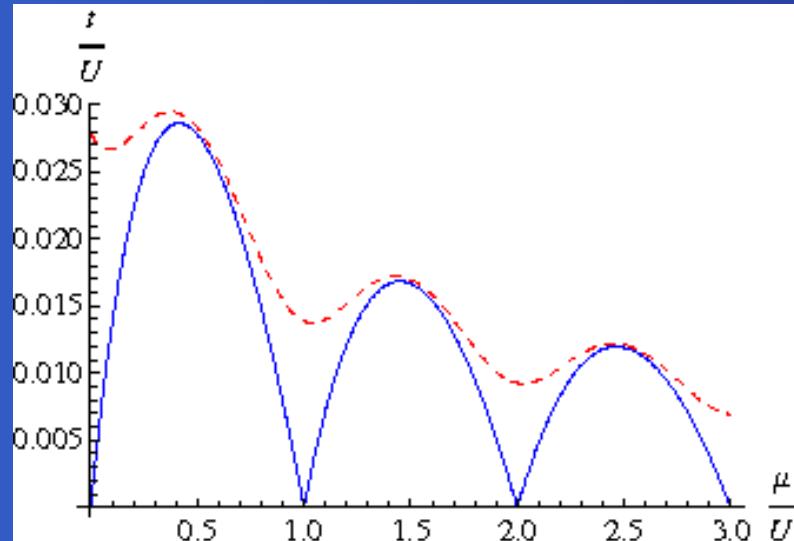
Amplitude $\omega_A(\vec{k})$ and phase $\omega_\theta(\vec{k})$ dispersions from:

$$\psi \rightarrow \psi_{\text{eq}} \sqrt{\beta} \delta_{\omega_m, 0} + \delta\psi_i(\omega_m); \quad \psi \rightarrow \psi_{\text{eq}} e^{i\theta_i(\tau)}$$

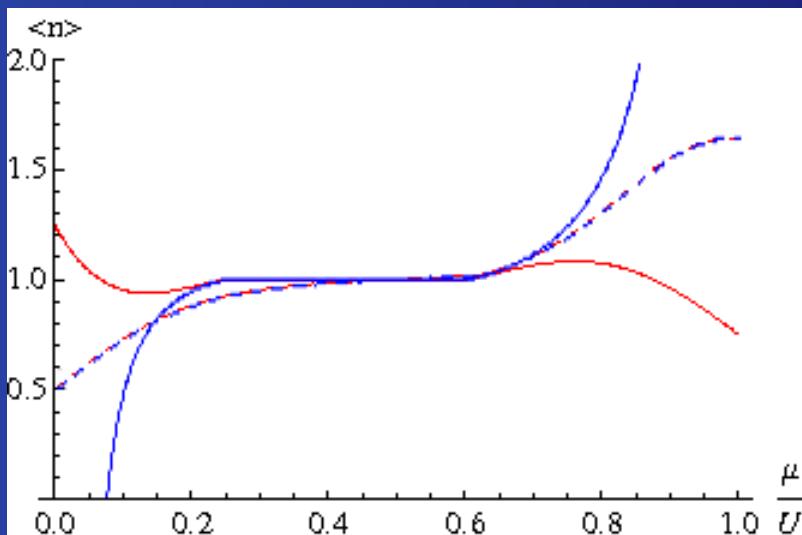
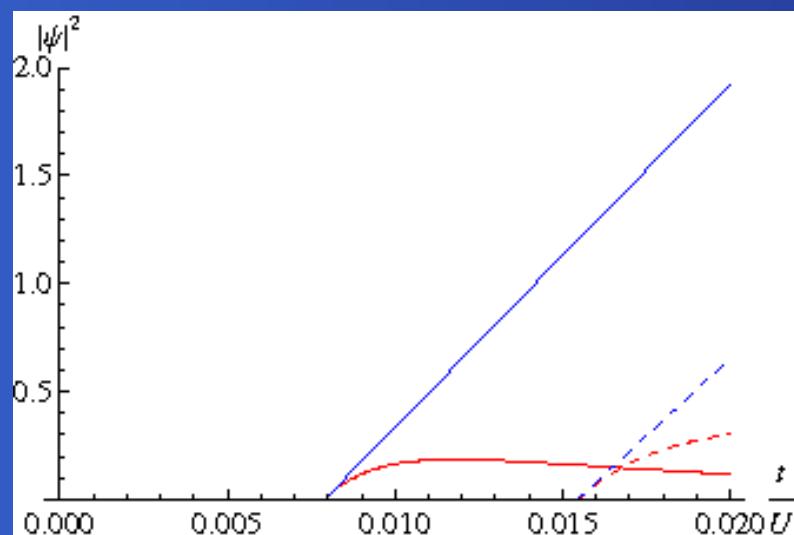
Phonon dispersion: $\omega_s(\vec{k}) = \sqrt{\omega_A(\vec{k})\omega_\theta(\vec{k})} \approx c|\vec{k}|$

here c is the sound velocity

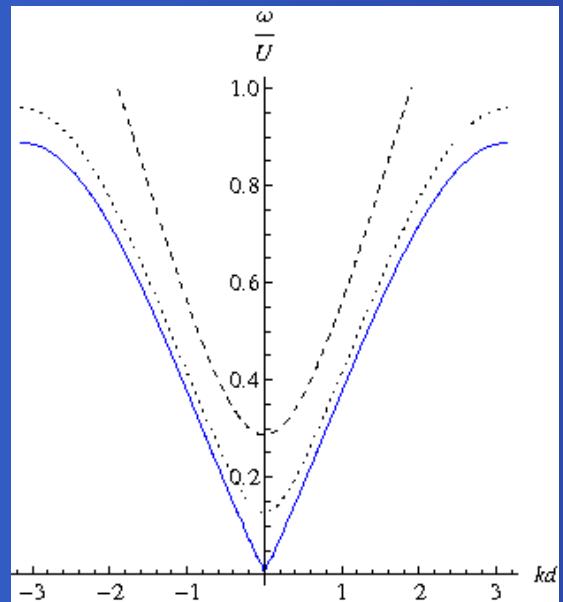
7 - Results



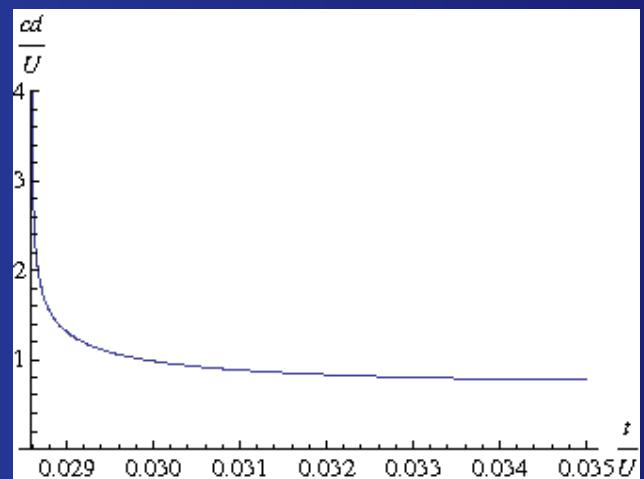
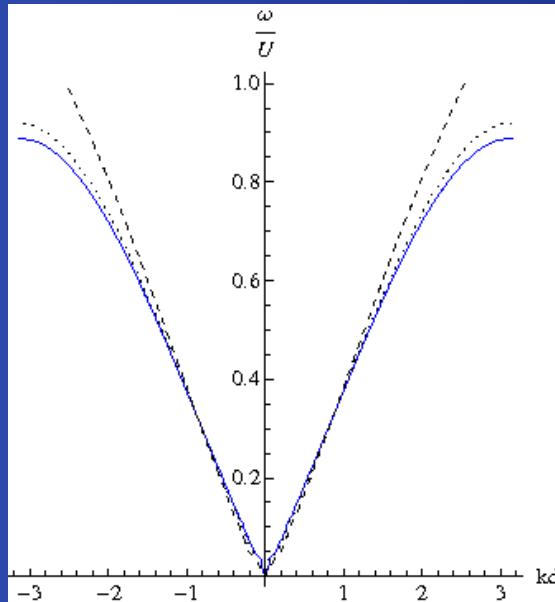
Red: Mean-field Blue: Landau theory
Solid: $T/U = 0$ Dashed: $T/U = 0.1/k_B$
Left: MI-SF phase boundary
reproduces the mean field result.
Left below: Condensate density
Below: Particles per site



7.1 - Results



Amplitude and phase dispersion relations



Second sound velocity

Solid: $t = t_c \approx 0.028 U$

Dotted: $t = 0.03 U$

Dashed: $t = 0.035 U$