

# Critical Temperature of Dipolar Interacting Bose Gases



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1. Bose-Einstein Condensation
2. Description for Ideal Bose Gases
3. Case of Dipolar Interacting Bosons

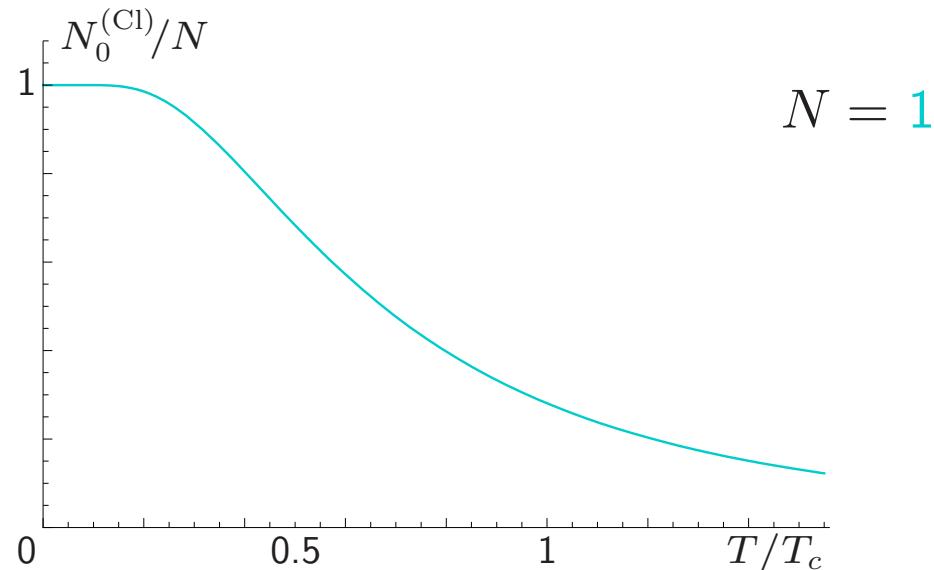
K. Glaum, A. Pelster, H. Kleinert, and T. Pfau, PRL **98**, 080407 (2007)

K. Glaum and A. Pelster, PRA **76**, 023604 (2007)

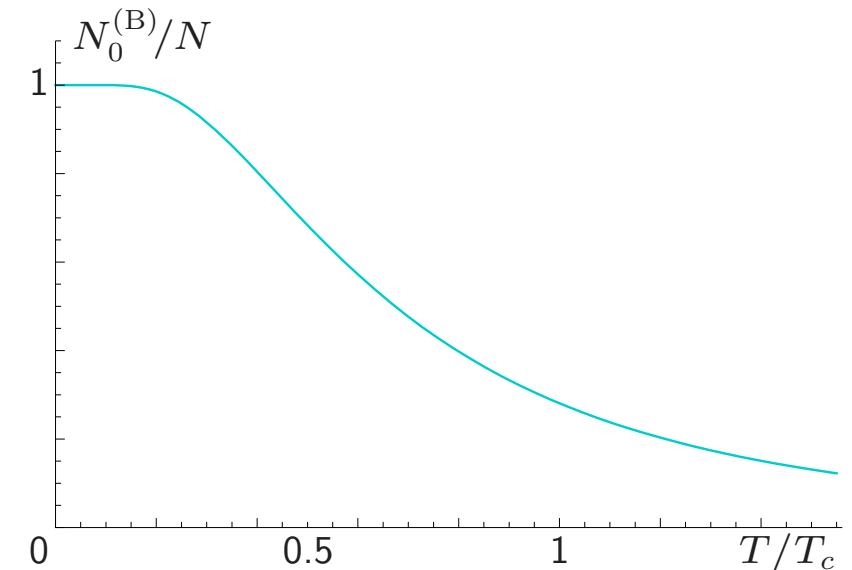
# 1. What is BEC ?

Ground-State Occupancy for Constant Density in Harmonic Trap:

for Classical Particles



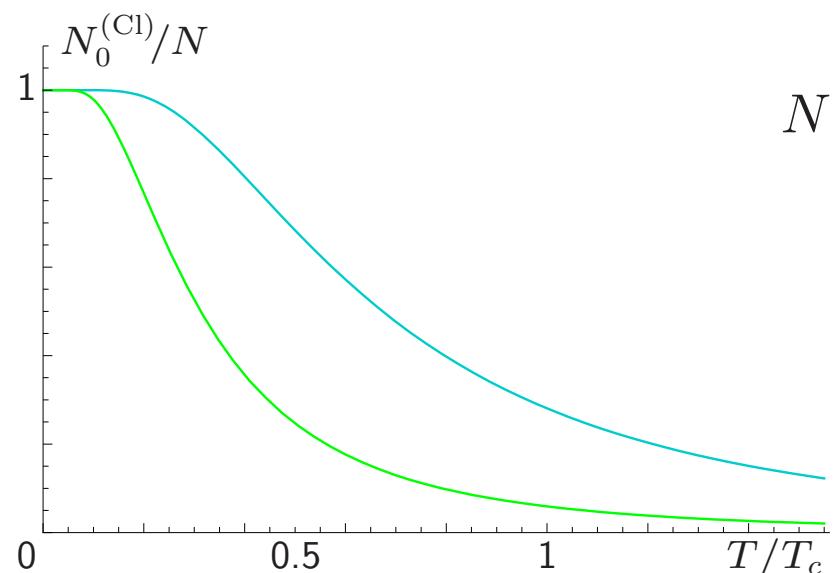
for Bosons



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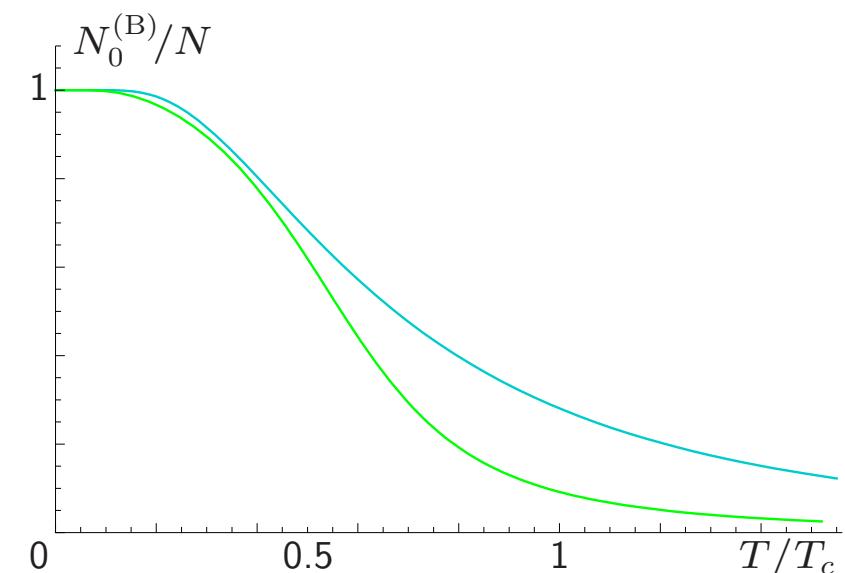
Ground-State Occupancy for Constant Density in Harmonic Trap:

for Classical Particles



$N = 1$   
10

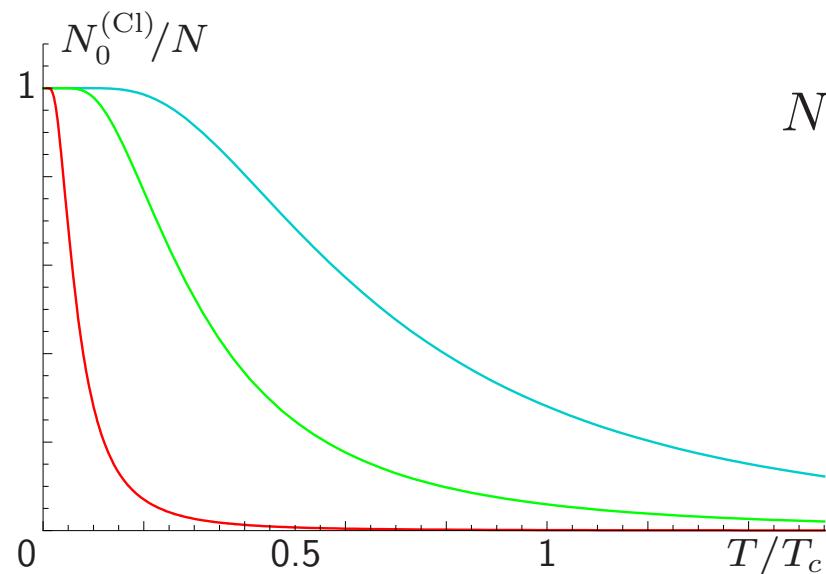
for Bosons



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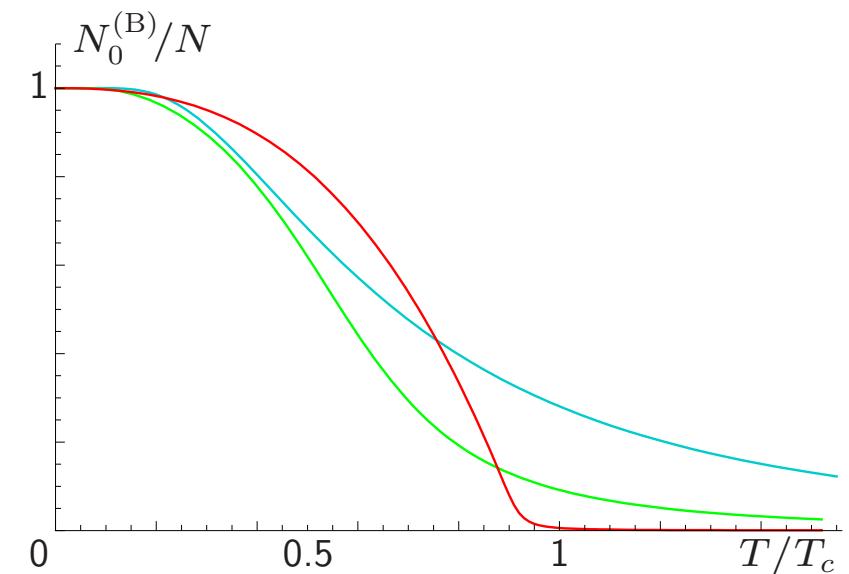
Ground-State Occupancy for Constant Density in Harmonic Trap:

for Classical Particles



$$N = \begin{array}{l} 1 \\ 10 \\ 1000 \end{array}$$

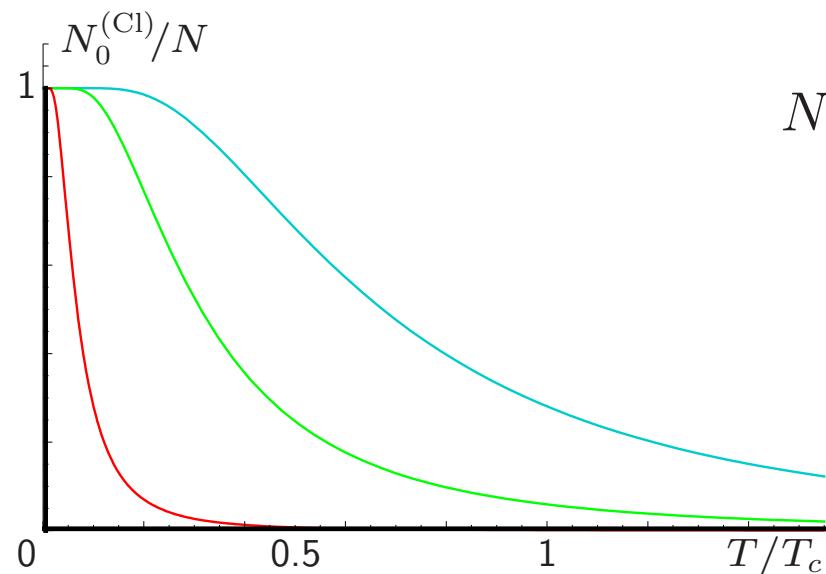
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# 1. What is BEC ?

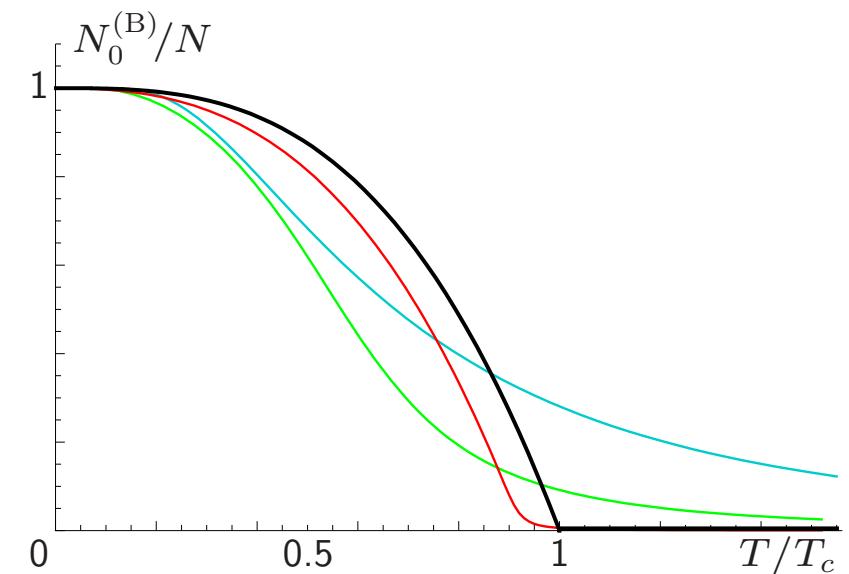
Ground-State Occupancy for Constant Density in Harmonic Trap:

for Classical Particles



$$N = \begin{matrix} 1 \\ 10 \\ 1000 \\ \infty \end{matrix}$$

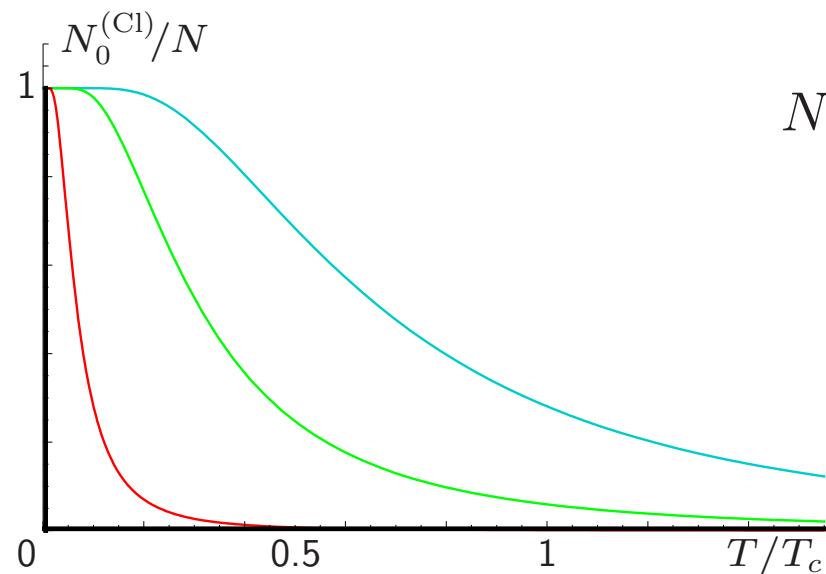
for Bosons



BEC is the Macroscopic Ground-State Occupancy Below a Certain Finite Critical Temperature  $T_c$

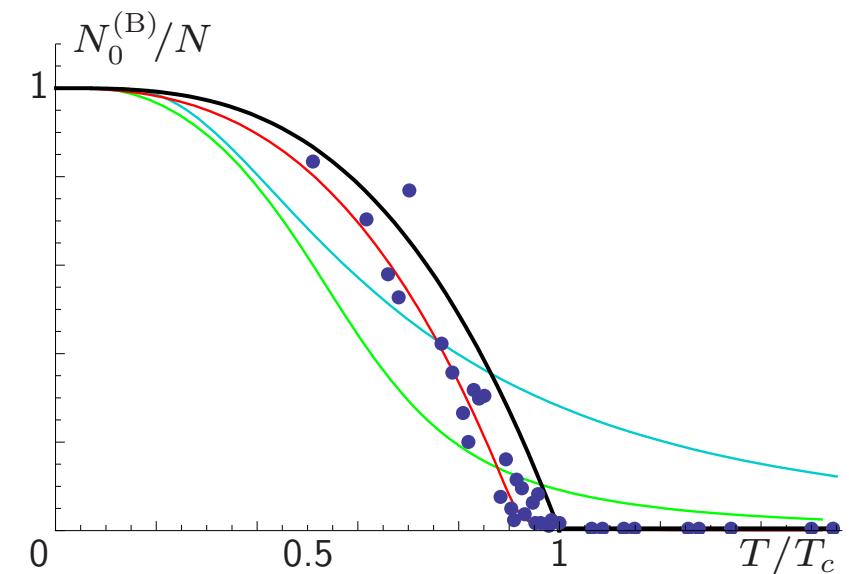
# 1. What is BEC ?

Ground-State Occupancy for Constant Density in Harmonic Trap:  
for Classical Particles



$$N = \begin{matrix} 1 \\ 10 \\ 1000 \\ \infty \end{matrix}$$

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BEC is the Macroscopic Ground-State Occupancy Below a Certain Finite Critical Temperature  $T_c$

● ← J.R. Ensher, D.S. Jin, M.R. Matthews, C.E. Wieman, and E.A. Cornell, PRL 77, 4984 (1996)

## 2.1. Ideal Bose Gas

Partition Function:

$$\mathcal{Z}^{(0)}(\beta) = \oint \mathcal{D}\psi^* \mathcal{D}\psi e^{-\mathcal{A}^{(0)}[\psi^*, \psi]/\hbar}$$

Free Action:

$$\mathcal{A}^{(0)} = \int_0^{\hbar\beta} d\tau d\tau' \int d^3x d^3x' \psi^*(\mathbf{x}, \tau) G^{(0)-1}(\mathbf{x}, \tau; \mathbf{x}', \tau') \psi(\mathbf{x}', \tau')$$

Integral Kernel:

$$G^{(0)-1}(\mathbf{x}, \tau; \mathbf{x}', \tau') \equiv \delta(\tau' - \tau) \delta(\mathbf{x}' - \mathbf{x}) \left\{ \hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2 \Delta}{2M} + V(\mathbf{x}) - \mu \right\}$$

Free Propagator:

$$G^{(0)}(\mathbf{x}, \tau; \mathbf{x}', \tau') = \frac{1}{\mathcal{Z}^{(0)}} \oint \mathcal{D}\psi^* \mathcal{D}\psi \psi(\mathbf{x}, \tau) \psi^*(\mathbf{x}', \tau') e^{-\mathcal{A}^{(0)}[\psi^*, \psi]/\hbar}$$

Grand-Canonical Free Energy:

$$\mathcal{F}^{(0)} \equiv -\frac{1}{\beta} \ln \mathcal{Z}^{(0)}$$

## 2.2. Calculations for Ideal Bose Gas

Field Decomposition:

$$\psi(\mathbf{x}, \tau) = \Psi_0(\mathbf{x}) + \sum_{\mathbf{n} \neq \mathbf{0}} \sum_{m=-\infty}^{\infty} a_{\mathbf{n}m} \psi_{\mathbf{n}}(\mathbf{x}) e^{-i 2\pi m \tau / \hbar \beta}$$

Free Energy:

$$\mathcal{F}^{(0)} \equiv (E_0 - \mu) \int d^3x |\Psi_0(\mathbf{x})|^2 + \frac{1}{\beta} \sum_{\mathbf{n} \neq \mathbf{0}} \ln \left\{ 1 - e^{-\beta(E_{\mathbf{n}} - \mu)} \right\}$$

Particle Number:

$$N \equiv -\frac{\partial \mathcal{F}^{(0)}}{\partial \mu} = N_0 + \sum_{\mathbf{n} \neq \mathbf{0}} \frac{1}{e^{\beta(E_{\mathbf{n}} - \mu)} - 1}$$

Order Parameter:

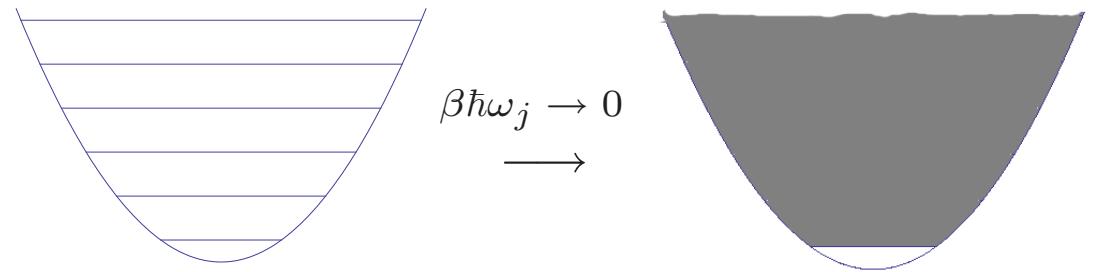
$$N_0 \equiv \int d^3x |\Psi_0(\mathbf{x})|^2$$

Extremalization Condition:

$$(E_0 - \mu) \Psi_0(\mathbf{x}) = 0 \implies (E_0 - \mu) N_0 = 0$$

## 2.3. Semiclassical Approximation

$$V(x) = \frac{M}{2} \left( \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right)$$

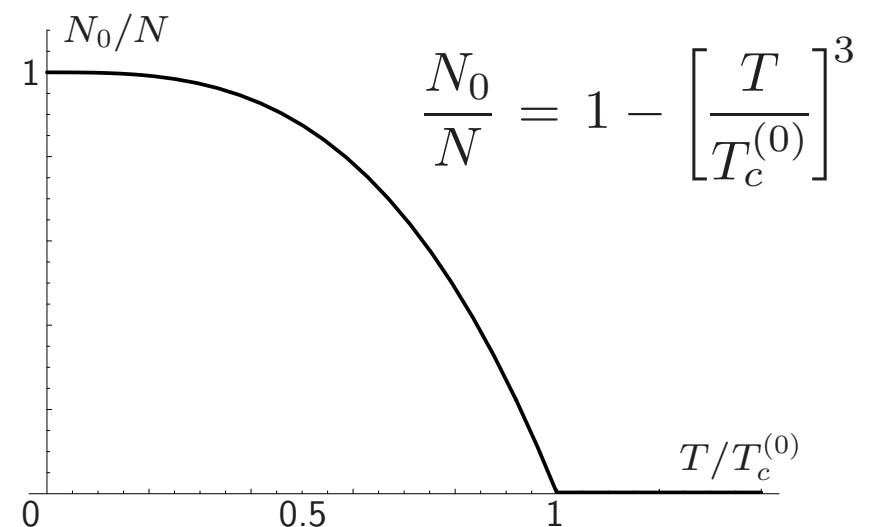


**Particle Number:**  $N = N_0 + \frac{1}{(\beta \hbar \tilde{\omega})^3} \zeta_3(e^{\beta \mu})$       with       $\zeta_\nu(z) \equiv \sum_{n=1}^{\infty} \frac{z^n}{n^\nu}$

**Two Phases:**

$$\begin{cases} \mu \leq 0, N_0/N = 0 & \text{for } T \geq T_c^{(0)}, \\ \mu = 0, N_0/N > 0 & \text{for } T < T_c^{(0)} \end{cases}$$

**Critical Value:**  $T_c^{(0)} = \frac{\hbar \tilde{\omega}}{k_B} \left[ \frac{N}{\zeta(3)} \right]^{1/3}$



## 2.4. Critical Point

**Propagator:**  $G^{(0)}(\mathbf{x}, \tau; \mathbf{x}', \tau') = \sum_{m=-\infty}^{\infty} \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{e^{\frac{i}{\hbar} [\mathbf{p}(\mathbf{x}-\mathbf{x}') - 2\pi m(\tau-\tau')/\beta]}}{\beta \left\{ -i\frac{2\pi m}{\beta} + \frac{\mathbf{p}^2}{2M} + V\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) - \mu \right\}}$

**Integral Kernel:**  $G^{(0)-1}(m, \mathbf{p}; \mathbf{X}) = \frac{1}{\hbar} \left\{ -i\frac{2\pi m}{\beta} + \frac{\mathbf{p}^2}{2M} + V(\mathbf{X}) - \mu \right\} = \frac{1}{G^{(0)}(m, \mathbf{p}; \mathbf{X})}$

**Condition for Ground State:**  $m = 0 \quad \text{and} \quad \mathbf{p} = \mathbf{0}$

**Phase Transition:**  $\mu \nearrow \mu_c = V(\mathbf{X}_{\min}) \implies G^{(0)-1}(m=0, \mathbf{p}=\mathbf{0}; \mathbf{X}_{\min}) \searrow 0$

$$\implies G^{(0)}(m=0, \mathbf{p}=\mathbf{0}; \mathbf{X}_{\min}) \rightarrow \infty$$

### 3.1. Dipolar Interacting Systems

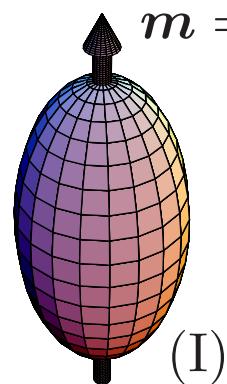
Harmonic Trap:

$$V(\mathbf{x}) = \frac{M}{2} \left( \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right)$$

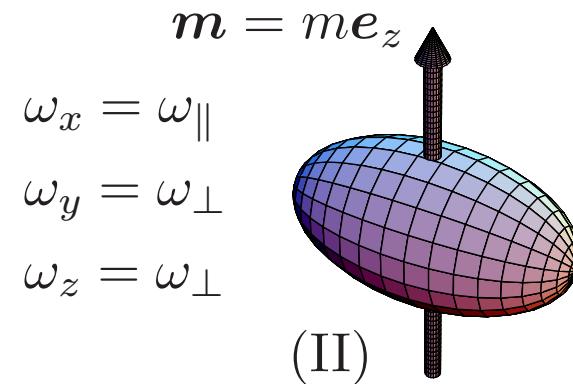
Interaction:

$$V^{(\text{int})}(\mathbf{x}) = \frac{4\pi\hbar^2 a_s}{M} \delta(\mathbf{x}) + \frac{\mu_0 m^2}{4\pi} \left( \frac{1}{|\mathbf{x}|^3} - \frac{3z^2}{|\mathbf{x}|^5} \right)$$

Configurations in  $^{52}\text{Cr}$ -Experiment:



$$\begin{aligned} \mathbf{m} &= m\mathbf{e}_z \\ \omega_x &= \omega_{\perp} \\ \omega_y &= \omega_{\perp} \\ \omega_z &= \omega_{\parallel} \end{aligned}$$



$$\begin{aligned} \mathbf{m} &= m\mathbf{e}_z \\ \omega_x &= \omega_{\parallel} \\ \omega_y &= \omega_{\perp} \\ \omega_z &= \omega_{\perp} \end{aligned}$$

## 3.2. Theory

Partition Function:

$$\mathcal{Z} = \oint \mathcal{D}\psi^* \mathcal{D}\psi e^{-\mathcal{A}[\psi^*, \psi]/\hbar}$$

Action:

$$\mathcal{A}[\psi^*, \psi] = \int_0^{\hbar\beta} d\tau d\tau' \int d^3x d^3x' \psi^*(\mathbf{x}, \tau) G^{-1}(\mathbf{x}, \tau; \mathbf{x}', \tau') \psi(\mathbf{x}', \tau')$$

Full Integral Kernel:

$$G^{-1}(\mathbf{x}, \tau; \mathbf{x}', \tau') = G^{(0)-1}(\mathbf{x}, \tau; \mathbf{x}', \tau')$$

$$+ \frac{1}{2} \delta(\tau - \tau') \psi(\mathbf{x}, \tau) V^{(\text{int})}(\mathbf{x} - \mathbf{x}') \psi^*(\mathbf{x}', \tau')$$

Grand-Canonical Free Energy:

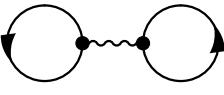
$$\mathcal{F} \equiv -\frac{1}{\beta} \ln \mathcal{Z}$$

### 3.3. Perturbative Approach

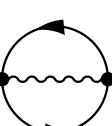
Approximation:

$$\mathcal{F} = \mathcal{F}^{(0)} + \mathcal{F}^{(D)} + \mathcal{F}^{(E)} + \dots$$

Direct Term:

$$\begin{aligned} \mathcal{F}^{(D)} &= \frac{1}{2\hbar\beta} \int_0^{\hbar\beta} d\tau \int d^3x d^3x' G^{(0)}(\mathbf{x}, \tau; \mathbf{x}', \tau) V^{(\text{int})}(\mathbf{x} - \mathbf{x}') \\ &\quad \times G^{(0)}(\mathbf{x}', \tau; \mathbf{x}', \tau) \equiv -\frac{1}{2\beta} \text{Diagram} \end{aligned}$$


Exchange Term:

$$\begin{aligned} \mathcal{F}^{(E)} &= \frac{1}{2\hbar\beta} \int_0^{\hbar\beta} d\tau \int d^3x d^3x' G^{(0)}(\mathbf{x}, \tau; \mathbf{x}', \tau) V^{(\text{int})}(\mathbf{x} - \mathbf{x}') \\ &\quad \times G^{(0)}(\mathbf{x}', \tau; \mathbf{x}, \tau) \equiv -\frac{1}{2\beta} \text{Diagram} \end{aligned}$$


Feynman Rules:

$$G^{(0)}(\mathbf{x}, \tau; \mathbf{x}', \tau') \equiv \mathbf{x} \tau \xleftarrow{} \mathbf{x}' \tau' , \quad V^{(\text{int})}(\mathbf{x} - \mathbf{x}') \equiv \mathbf{x} \nearrow \text{wavy line} \swarrow \mathbf{x}'$$

### 3.4. Perturbative Result

Particle Number for  $T > T_c$ :

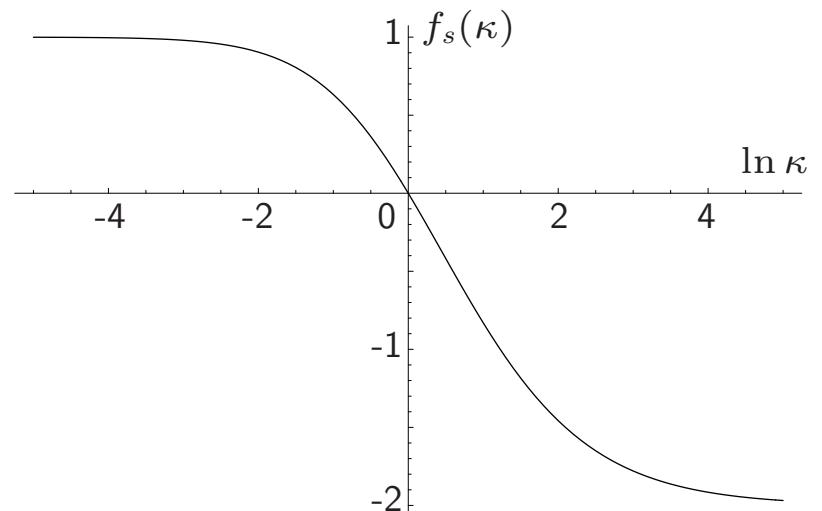
$$N^{(\text{I/II})} = -\frac{\partial \mathcal{F}}{\partial \mu}$$

$$= \frac{\zeta_3(e^{\beta\mu})}{(\beta\hbar\tilde{\omega})^3} - \frac{4a_s}{\lambda_T} \left[ 1 - \frac{\epsilon_{DD}}{2} f_s^{(\text{I/II})} \left( \sqrt{\frac{\omega_{||}}{\omega_{\perp}}} \right) \right] \frac{\zeta_{1/2,3/2,3/2}(e^{\beta\mu})}{(\beta\hbar\tilde{\omega})^3}$$

Anisotropy Factor:  $f_s^{(\text{I})}(\kappa) = \frac{2\kappa + 1}{1 - \kappa} - \frac{3\kappa \operatorname{Artanh}\sqrt{1 - \kappa}}{(1 - \kappa)^{3/2}} = -2f_s^{(\text{II})}(\kappa)$

$$\lambda_T \equiv \sqrt{\frac{2\pi\hbar^2\beta}{M}} , \quad \epsilon_{DD} \equiv \frac{\mu_0 m^2 M}{12\pi\hbar^2 a_s}$$

$$\zeta_{a,b,c}(z) \equiv \sum_{k,l=1}^{\infty} \frac{z^{k+l}}{k^a l^b (k+l)^c}$$



### 3.5. Renormalisation

Mean-Field Integral Kernel:  $G^{-1}(\mathbf{x}, \tau; \mathbf{x}', \tau) = G^{(0)-1}(\mathbf{x}, \tau; \mathbf{x}', \tau) - \Sigma(\mathbf{x}, \tau; \mathbf{x}', \tau)$

Self-Energy:

$$\Sigma(\mathbf{x}, \tau; \mathbf{x}', \tau') \equiv \text{Diagram: two external lines } \mathbf{x} \tau \text{ and } \mathbf{x}' \tau' \text{ connected by a wavy line with a loop above it.} + \text{Diagram: two external lines } \mathbf{x} \tau \text{ and } \mathbf{x}' \tau' \text{ connected by a wavy line with a loop below it.} + \dots$$

Fourier Representation:  $G^{-1}(m, \mathbf{p}; \mathbf{X}) = \frac{1}{\hbar} \left\{ -\frac{2\pi i m}{\beta} + \frac{\mathbf{p}^2}{2M} + V(\mathbf{X}) - \mu \right\} - \Sigma(m, \mathbf{p}; \mathbf{X})$

Renormalised Chemical Potential:  $\mu + \hbar \Sigma(m=0, \mathbf{p}=0; \mathbf{X}_{\min}) \equiv \mu_r$

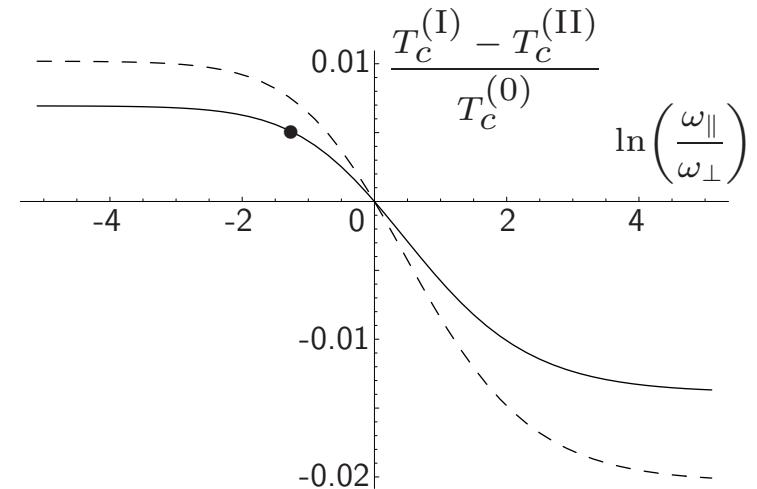
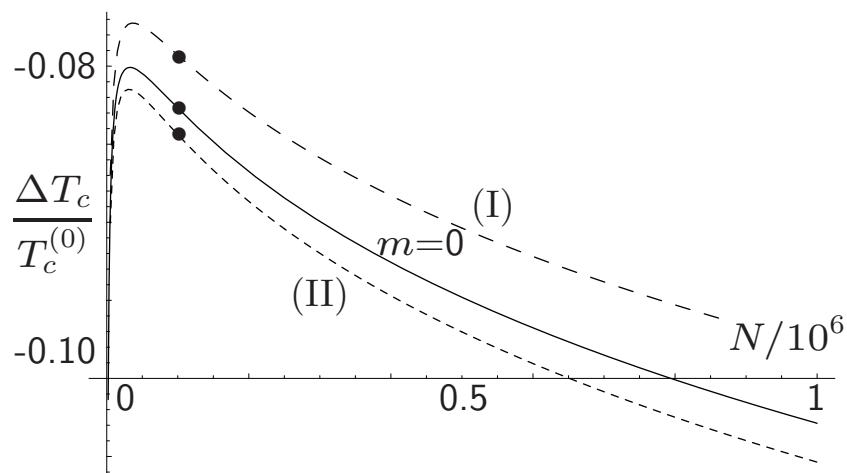
Result for  $T > T_c$ :  $\mu^{(\text{I/II})} = \mu_r + \frac{4a_s}{\beta \lambda_T} \left[ 1 - \frac{\epsilon_{DD}}{2} f_s^{(\text{I/II})} \left( \sqrt{\frac{\omega_{||}}{\omega_{\perp}}} \right) \right] \zeta_{3/2} (e^{\beta \mu_r})$

## 3.6. Results

Critical Point:  $G^{-1}(m=0, \mathbf{p}=\mathbf{0}; \mathbf{X}_{\min}) \rightarrow 0 \implies \mu_r \rightarrow 0$

Critical Temperature Shift:

$$\left(\frac{\Delta T_c}{T_c^{(0)}}\right)^{(I/II)} = -\frac{3.426 a_s}{\lambda_{T_c}^{(0)}(N)} \left[ 1 - \frac{\epsilon_{DD}}{2} f_s^{(I/II)} \left( \sqrt{\frac{\omega_{\parallel}}{\omega_{\perp}}} \right) \right]; \quad \left(\frac{\Delta T_c}{T_c^{(0)}}\right)_{\text{FS}} = -\frac{0.728}{N^{1/3}} \frac{2\omega_{\perp} + \omega_{\parallel}}{3\tilde{\omega}}$$

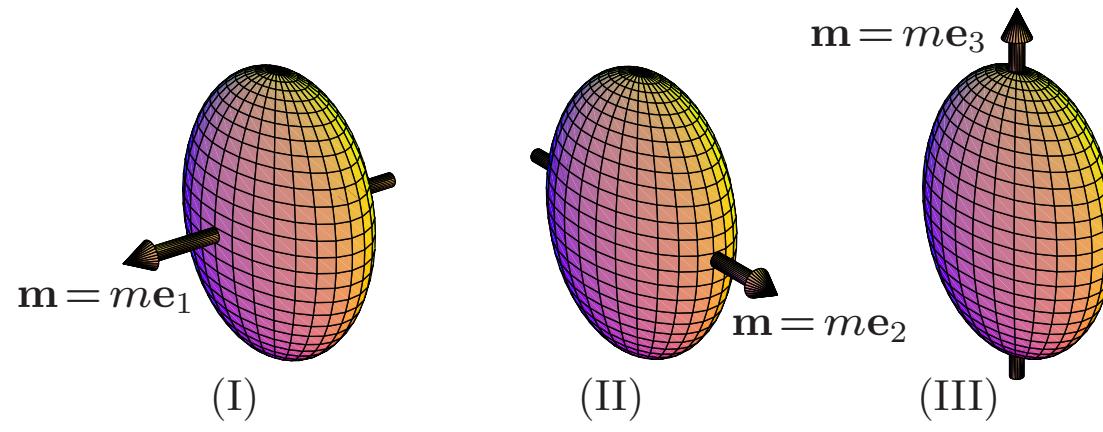


S. Giorgini, L. Pitaevskii, and S. Stringari, PRA **54**, R4633 (1996)

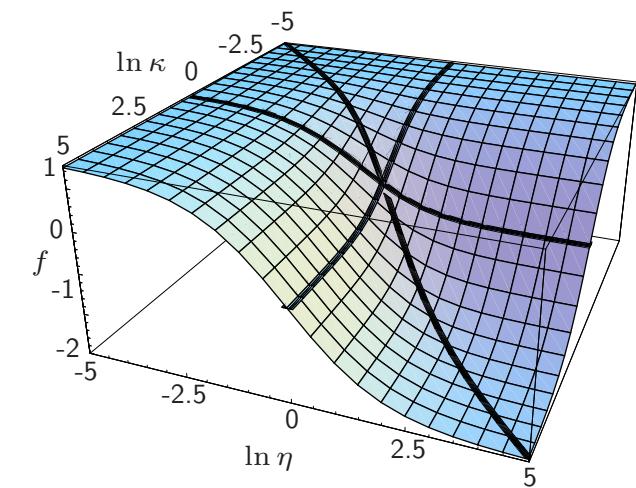
K. Glaum, A. Pelster, H. Kleinert, and T. Pfau, PRL **98**, 080407 (2007)

### 3.7. Totaly Anisotropic Case

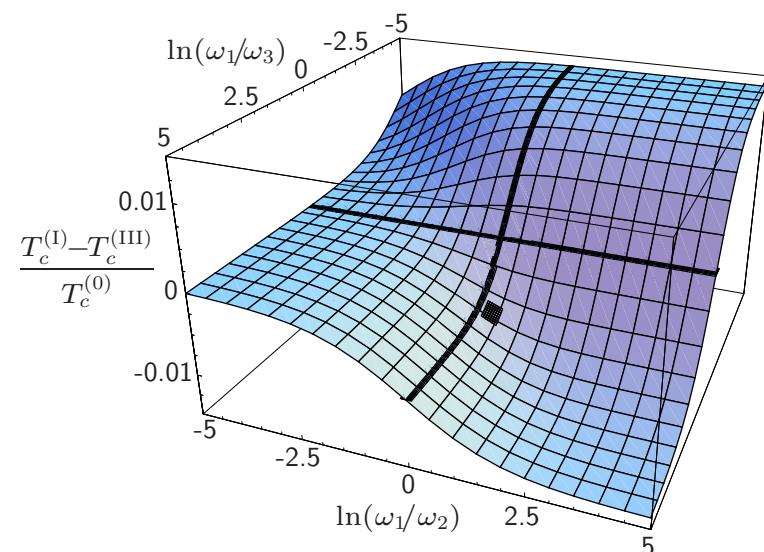
Trap Frequencies:  $\omega_1 > \omega_2 > \omega_3$



Anisotropy Function:  $f_s(\kappa) \mapsto f(\kappa, \eta)$



Dipolar Net Effect:



Magic Angle:

