Bose-Einstein condensation in compact astrophysical objects

Christine Gruber

Institut für Theoretische Physik, Freie Universität Berlin, Germany
In collaboration with PD Dr. Axel Pelster

Supported by the Erasmus Mundus Joint Doctorate Program by Grant Number 2010-1816
from the EACEA of the European Commission.

January 8th, 2013
New Year's seminar, FU Berlin, Germany

Outline

1. Motivation
   - BECs and astrophysics
2. Choice of treatment
   - What is the exact setting?
3. Hartree-Fock theory of interacting Bosons
   - Governing equations
   - Desired results
4. Summary & Outlook

Motivation: why BEC in stars?

Existence of a BEC:

$$T_{\text{crit}} = \left(\frac{n}{\zeta(3/2)} \right)^{2/3} \frac{2\pi\hbar^2}{m k_B}$$

Laboratory experiments $\leftrightarrow$ Astrophysics:
very different densities!

scenario	n $[\mathrm{m}^{-3}]$	T_{crit} $[\mathrm{K}]$	T_{typ} $[\mathrm{K}]$
neutron star (core)	10^{44}	10^{11}	$10^{10} \to 10^6$
neutron star (crust)	10^{36}	10^6	10^6
^{4}He white dwarf	10^{35}	10^5	$10^4 - 10^7$

Choice of treatment: 2 questions

- Energy of particles?

$$v = \sqrt{\frac{2k_B T}{m}} \lesssim c\,?$$

- Size scales of system?

$$r_S = \frac{2GM}{c^2} \lesssim r_{\text{typ}}\,?$$

Choice:

Gross-Pitaevskii equation $\Leftrightarrow$ Klein-Gordon equation
Newtonian gravity $\Leftrightarrow$ General relativity

Choice of treatment

Conditions in astrophysical objects:

scenario	v/c	r_{typ}/r_S
neutron star (core)	10^{-1}	2.7
neutron star (crust)	10^{-3}	2.7
^{4}He white dwarf	10^{-3}	10^3

Description of non-relativistic particles in Newtonian gravity:

Hartree-Fock theory of bosons with contact and gravitational interaction

$$\Downarrow$$

describes white dwarfs and cool neutron stars.

Hartree-Fock theory I

Hamilton operator of bosons with interactions:

$$\hat{\mathcal{H}} = \int d^3x\, \hat{\Psi}^+(\mathbf{x}) \left\{ -\frac{\hbar^2}{2m}\Delta - \mu + \int d^3x' \left[g\delta(\mathbf{x}-\mathbf{x}') - \frac{G\,m}{|\mathbf{x}-\mathbf{x}'|} \right] |\hat{\Psi}(\mathbf{x}')|^2 \right\} \hat{\Psi}(\mathbf{x})$$

($g = 4\pi\hbar^2 a/m$... contact interaction strength, G ... gravitational constant, m... particle mass)

- Expand field operator into unknown one-particle basis $\Psi_{\mathbf{n}}(\mathbf{x})$:

$$\hat{\Psi}(\mathbf{x}) = \sum_{\mathbf{n}} \hat{a}_{\mathbf{n}}\, \Psi_{\mathbf{n}}(\mathbf{x})$$

- Creation and annihilation operators obey commutation relations:

$$[\hat{a}_{\mathbf{n}}, \hat{a}_{\mathbf{m}}] = [\hat{a}^+_{\mathbf{n}}, \hat{a}^+_{\mathbf{m}}] = 0\,, \quad [\hat{a}^+_{\mathbf{n}}, \hat{a}_{\mathbf{m}}] = \delta_{\mathbf{n},\mathbf{m}}$$

Hartree-Fock theory II

Use grand-canonical description for the system:

$$Z = e^{-\beta F} = Tr\left(e^{-\beta\hat{\mathcal{H}}}\right)$$

- Introduce effective Hamiltonian with effective one-particle energies: $\hat{\mathcal{H}}_{eff} = \sum_{\mathbf{n}} (\epsilon_{\mathbf{n}} - \mu)\, \hat{a}^{+}_{\mathbf{n}} \hat{a}_{\mathbf{n}}$
- Approach inspired from Variational Perturbation Theory[1]: $\hat{\mathcal{H}} = \hat{\mathcal{H}}_{eff} + \eta\left(\hat{\mathcal{H}} - \hat{\mathcal{H}}_{eff}\right)$
- Expansion wrt. η of free energy: $F = F\left\{\eta, \Psi^{*}_{\mathbf{n}}(\mathbf{x}), \Psi_{\mathbf{n}}(\mathbf{x}), \epsilon_{\mathbf{n}}\right\}$

[1] Kleinert, *Path Integrals*, 2009.

Hartree-Fock theory III

Cut the series at first order and let $\eta = 1$:

$$F(1) = F\big|_{\eta=1} = F_{\mathit{eff}} + \langle \hat{\mathcal{H}} - \hat{\mathcal{H}}_{\mathit{eff}} \rangle_{\mathit{eff}}$$

How do we obtain equations?

Principle of minimal sensitivity:

$$\frac{\delta F(1)}{\delta \Psi^*_{\mathbf{n}}(\mathbf{x})} = \frac{\delta F(1)}{\delta \Psi_{\mathbf{n}}(\mathbf{x})} = \frac{\partial F(1)}{\partial \epsilon_{\mathbf{n}}} = 0$$

... to minimize dependence of F on $\{\Psi^*_{\mathbf{n}}, \Psi_{\mathbf{n}}, \epsilon_{\mathbf{n}}\}$ for finite order in η.

Hartree-Fock theory IV

- Macroscopic occupation of ground state:

$$\hat{a}_{\mathbf{0}}^{+},\, \hat{a}_{\mathbf{0}} \approx \sqrt{n_{\mathbf{0}}} \;\Rightarrow\; \Psi(\mathbf{x}) = \sqrt{n_{\mathbf{0}}}\,\Psi_{\mathbf{0}}(\mathbf{x})$$

- Define

$$n_0(\mathbf{x}) = |\Psi(\mathbf{x})|^2\,, \quad n_{\mathrm{th}}(\mathbf{x},\mathbf{x}') = \sum_{\mathbf{n}\neq\mathbf{0}} \frac{\Psi_{\mathbf{n}}^*(\mathbf{x})\,\Psi_{\mathbf{n}}(\mathbf{x}')}{e^{\beta(\epsilon_{\mathbf{n}}-\mu)}-1}$$

- Split all sums into condensate and thermal terms, use two equations:

$$\frac{\delta F(1)}{\delta \Psi_{\mathbf{0}}^*(\mathbf{x})} = 0\,, \quad \frac{\delta F(1)}{\delta \Psi_{\mathbf{n}}^*(\mathbf{x})} = 0$$

Condensate and thermal equation

$$
\begin{aligned}
0 &= \left[-\frac{\hbar^2}{2m}\Delta - \epsilon_0\right]\Psi(\mathbf{x}) + g\,\left[n_0(\mathbf{x}) + 2n_{\text{th}}(\mathbf{x},\mathbf{x})\right]\Psi(\mathbf{x}) \\
&\quad - \int d^3\mathbf{x}'\,\frac{G\,m}{|\mathbf{x}-\mathbf{x}'|}\left\{\left[n_0(\mathbf{x}') + n_{\text{th}}(\mathbf{x}',\mathbf{x}')\right]\Psi(\mathbf{x}) + n_{\text{th}}(\mathbf{x}',\mathbf{x})\Psi(\mathbf{x}')\right\} \\
0 &= \left[-\frac{\hbar^2}{2m}\Delta - \epsilon_{\mathbf{n}}\right]\Psi_{\mathbf{n}}(\mathbf{x}) + g\,\left[2n_0(\mathbf{x}) + 2n_{\text{th}}(\mathbf{x},\mathbf{x})\right]\Psi_{\mathbf{n}}(\mathbf{x}) \\
&\quad - \int d^3\mathbf{x}'\,\frac{G\,m}{|\mathbf{x}-\mathbf{x}'|}\Big\{\left[n_0(\mathbf{x}') + n_{\text{th}}(\mathbf{x}',\mathbf{x}')\right]\Psi_{\mathbf{n}}(\mathbf{x}) \\
&\quad + \left[\Psi^*(\mathbf{x}')\Psi(\mathbf{x}) + n_{\text{th}}(\mathbf{x}',\mathbf{x})\right]\Psi_{\mathbf{n}}(\mathbf{x}')\Big\}
\end{aligned}
$$

with Hartree- and Fock-terms of the gravitational interaction.

Status so far

Difficulties:

- Integrals in gravitational terms remain
- Fock-terms of gravitational parts bilocal

How to proceed further?

- Discard Fock-terms of gravitational interaction for the time being: $n_{\text{th}}(\mathbf{x},\mathbf{x}) \to n_{\text{th}}(\mathbf{x})$.
- Take into account spherical symmetry for the condensate, and introduce spherical coordinates.
- Carry out multipole expansion.

Final form: condensate equation

Condensate equation: $\frac{\partial^2}{\partial r^2}\left(r \cdot \ldots\right)$

$$\frac{\partial^2}{\partial r^2}\left\{ r\,\left[n_0(r) + 2n_{\mathrm{th}}(r)\right] \right\} = -\frac{4\pi\, Gm}{g}\, r\left[n_0(r) + n_{\mathrm{th}}(r)\right]$$

$$\Downarrow$$

Solution: still dependent on $n_{\mathrm{th}}(r)$

$$n_0(r) = C\,\frac{\sin(\sigma r)}{\sigma r} - 2n_{\mathrm{th}}(r) + \frac{\sigma}{r}\int_0^r dr'\;\sin\left[\sigma(r - r')\right]\, r'\, n_{\mathrm{th}}(r')$$

with $\sigma^2 = 4\pi Gm/g$.

Final form: thermal equation

Semi-classical approximation: $\Psi_{\mathbf{n}}(\mathbf{x}) = e^{i\,\mathbf{k}\cdot\mathbf{x}}$ with

$$\epsilon_{\mathbf{k}}(\mathbf{x}) := \frac{\hbar^2\mathbf{k}^2}{2m} + 2g\,\left[n_0(\mathbf{x}) + n_{\mathrm{th}}(\mathbf{x})\right] - \Phi(\mathbf{x})$$

$\Phi(\mathbf{x})$... gravitational potential.

Solution: integral over Bose-Einstein distribution

$$n_{\mathrm{th}}(r) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{e^{\beta[\epsilon_{\mathbf{k}}(r)-\mu]}-1} = \frac{1}{\lambda^3}\,\zeta_{3/2}\left(e^{-\beta\{\mu-2g\,[n_0(r)+n_{\mathrm{th}}(r)]+4\pi\Phi(r)\}}\right)$$

with $\zeta_\nu(z)$... Polylog-function, $\Phi(r)$... gravitational potential in spherical coordinates.

Final form: ground state energy

Ground state energy: from condensate equation

$$\begin{aligned}\epsilon_0 &= g\,n_0(r) + 2g\,n_{\rm th}(r,r)\\ &\quad -4\pi\,Gm\left\{\frac{1}{r}\int_0^r dr'\,r'^2\left[n_0(r')+n_{\rm th}(r')\right] + \int_r^\infty dr'\,r'\left[n_0(r')+n_{\rm th}(r')\right]\right\}\end{aligned}$$

– with $n_0(r)$ as a function of $n_{\rm th}(r)$, ϵ_0 must be independent of r.

Constant ground state energy:

$$\epsilon_0 = Cg\,\cos(\sigma R_0) + 4\pi Gm\int_0^\infty dr'\,\cos\left[\sigma(R_0-r')\right] r'\,n_{\rm th}(r')$$

Solution in two regimes:

$\boxed{0 \leq r \leq R_0}$

both condensate and thermal fluctuations.

Condition: $n_0(R_0) = 0$.

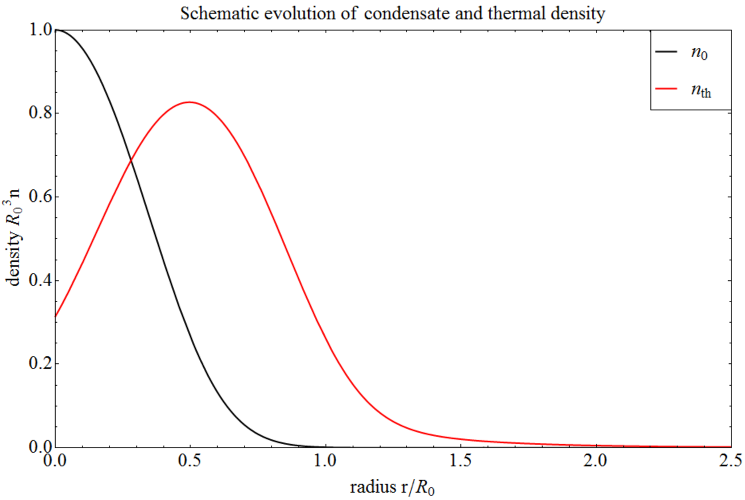
$\boxed{R_0 \leq r}$

outside of Thomas-Fermi radius R_0.

Conditions: $n_0(r) = 0$, $\mu(R_0) = \epsilon_0$.

μ fixed by total particle number N.

Expected results for condensate and thermal densities



Possible solutions

- Analytically: only by approximations
  - Robinson formula: expansion for small temperatures

$$\zeta_\nu\left(e^{\beta\mu_{eff}}\right) = \Gamma(1-\nu)\left(-\beta\mu_{eff}\right)^{\nu-1/2} + \sum_{k=0}^{\infty}\frac{1}{k!}\zeta(\nu-k)(\beta\mu_{eff})^k$$

with $\mu_{eff} = \mu - 2g\left(n_0 + n_{\text{th}}\right) + 4\pi\Phi(r)$.

  - Expansion around the critical temperature?
- Numerically: solve integro-differential equations
  - Iterative solution:

$$n_{\text{th}}^{(0)}(r) = \begin{cases} A + B\,\frac{r^2}{R_0^2} & 0 < r < R_0 \\ \frac{(A+B)\,R_0^3}{r^3} & r > R_0 \end{cases}$$

  - Exact solution??

Outline of iterative solution

From the ansatz for $n_{\mathrm{th}}^{(0)}$:

- Calculate other $k = 0-$quantities $n_0^{(0)}(r)$, $\epsilon_0^{(0)}$.
- Use conditions e.g. for chemical potential at $r = R_0$ and total particle number N to fix A, B, C, R_0.

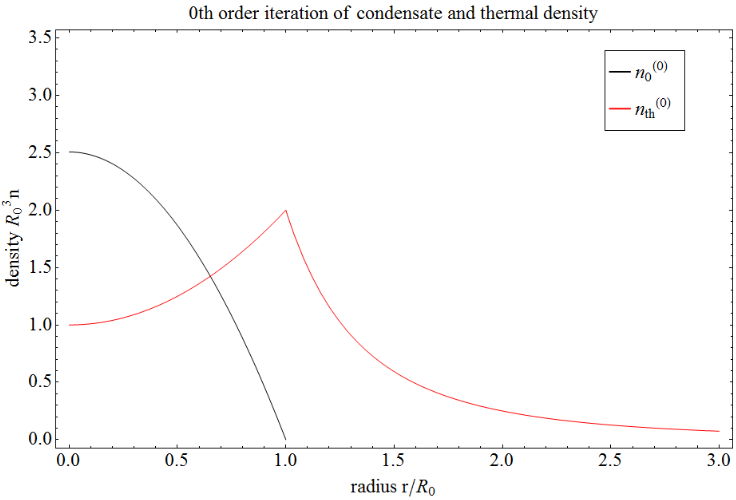
Then move on to order $k = 1$ and calculate

$$n_{\mathrm{th}}^{(1)}(r) = \frac{1}{\lambda^3}\,\zeta_{3/2}\left(e^{-\beta\left\{\mu-2g\left[n_0^{(0)}(r)+n_{\mathrm{th}}^{(0)}(r)\right]+4\pi\Phi^{(0)}(r)\right\}}\right)$$

and other quantities $n_0^{(1)}(r)$, $\epsilon_0^{(1)}$ in order $k = 1$.

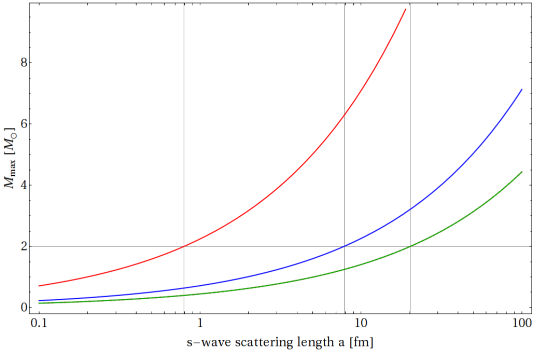
Continue and hope for convergence.

0th iteration for condensate and thermal densities



Maximum masses of compact objects

Maximum mass[2] of a neutron star for three different treatments: non-relativistic particles with Newton (blue), non-relativistic particles in GR (red), relativistic particles in GR (green), at $T = 0$.



[2] Chavanis & Harko 2011; Bilic & Nicolic 2000; Colpi, Shapiro & Wassermann 1986.

Summarizing...

- Conditions in compact objects admit formation of BEC.
- Hartree-Fock theory of interacting bosons
- $\Rightarrow$ governing equations for condensate and thermal fluctuations

To be done:

(Numerical) solution of equations.
Approximations to obtain qualitative confirmation?
Approximation of the bilocal Fock terms by local expressions?

That's it!

Thanks for your attention!

Any questions?

References

[1] H. Kleinert, *Path Integrals*, 5th ed., World Scientific, Singapore (2009).

[2] P. Chavanis, T. Harko, Phys. Rev. D **86**, 064011 (2012);
N. Bilic, H. Nicolic, Nucl. Phys. B **590**, 575 (2000);
M. Colpi *et al.*, Phys. Rev. Lett. **57**, 2485 (1986).