

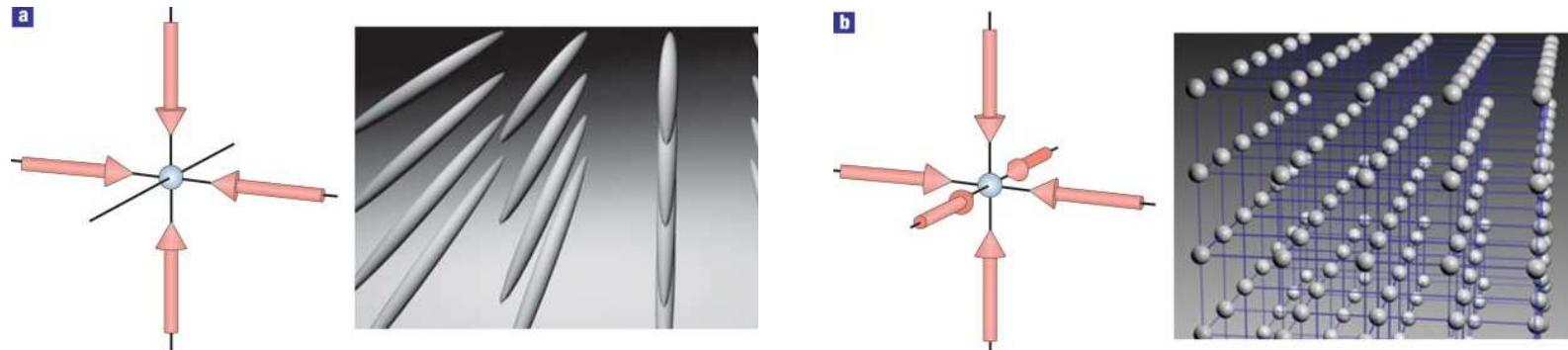
Visibility of an Atomic Cloud Released from Optical Lattice

Alexander Hoffmann



- 1. Bose-Hubbard Model**
- 2. Mean-Field Theory**
- 3. Corrections to Mean-Field Theory**
- 4. Harmonic Trap**
- 5. Visibility**

1.1 Experimental Realization



$$V_{\text{ext}} = V_0 \sum_{i=1}^3 \sin^2\left(\frac{\pi}{a}x_i\right)$$

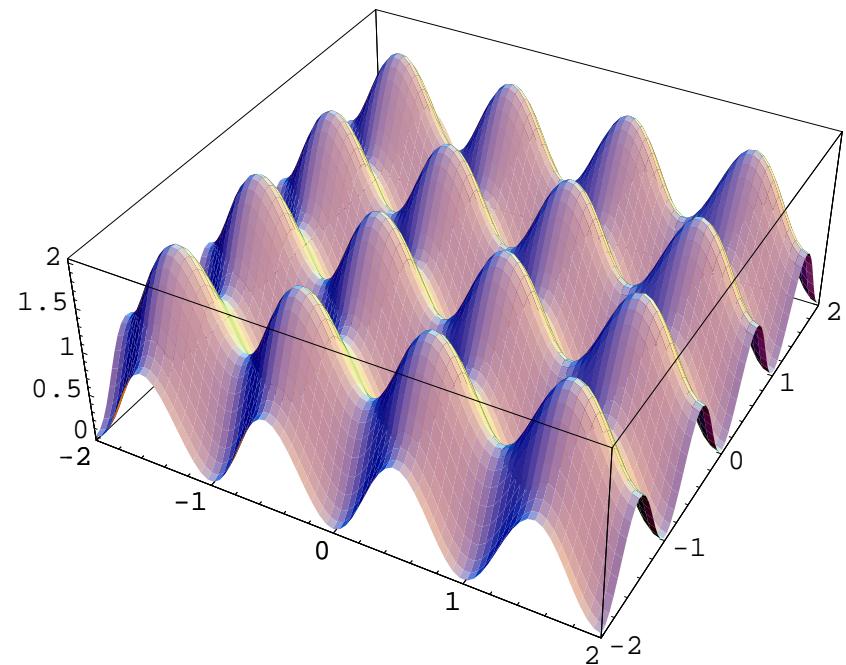
$$a = \frac{\lambda}{2}, \quad \lambda = 1030 \text{ nm}$$

$$\text{Recoil energy: } E_R = \frac{\hbar^2 \pi^2}{2ma^2}$$

$$\text{Dimensionless energies: } \tilde{E} = E/E_R$$

$$\text{Typical range: } 0 \leq \tilde{V}_0 \leq 30$$

Bosons used in experiment: ^{87}Rb



1.2 Derivation of Bose-Hubbard Model

Second-quantized Hamiltonian for bosons:

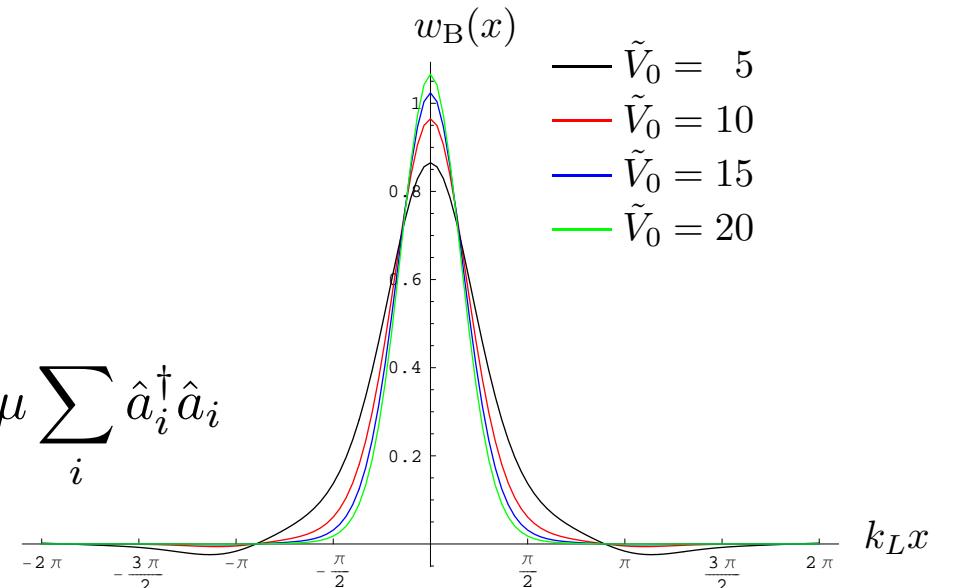
$$\hat{H} = \int d^3x \left\{ \hat{\psi}^\dagger(\mathbf{x}) \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}} - \mu \right] \hat{\psi}(\mathbf{x}) + \frac{2\pi a_{\text{BB}} \hbar^2}{m} \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \hat{\psi}(\mathbf{x}) \right\}$$

Expansion in Wannier states:

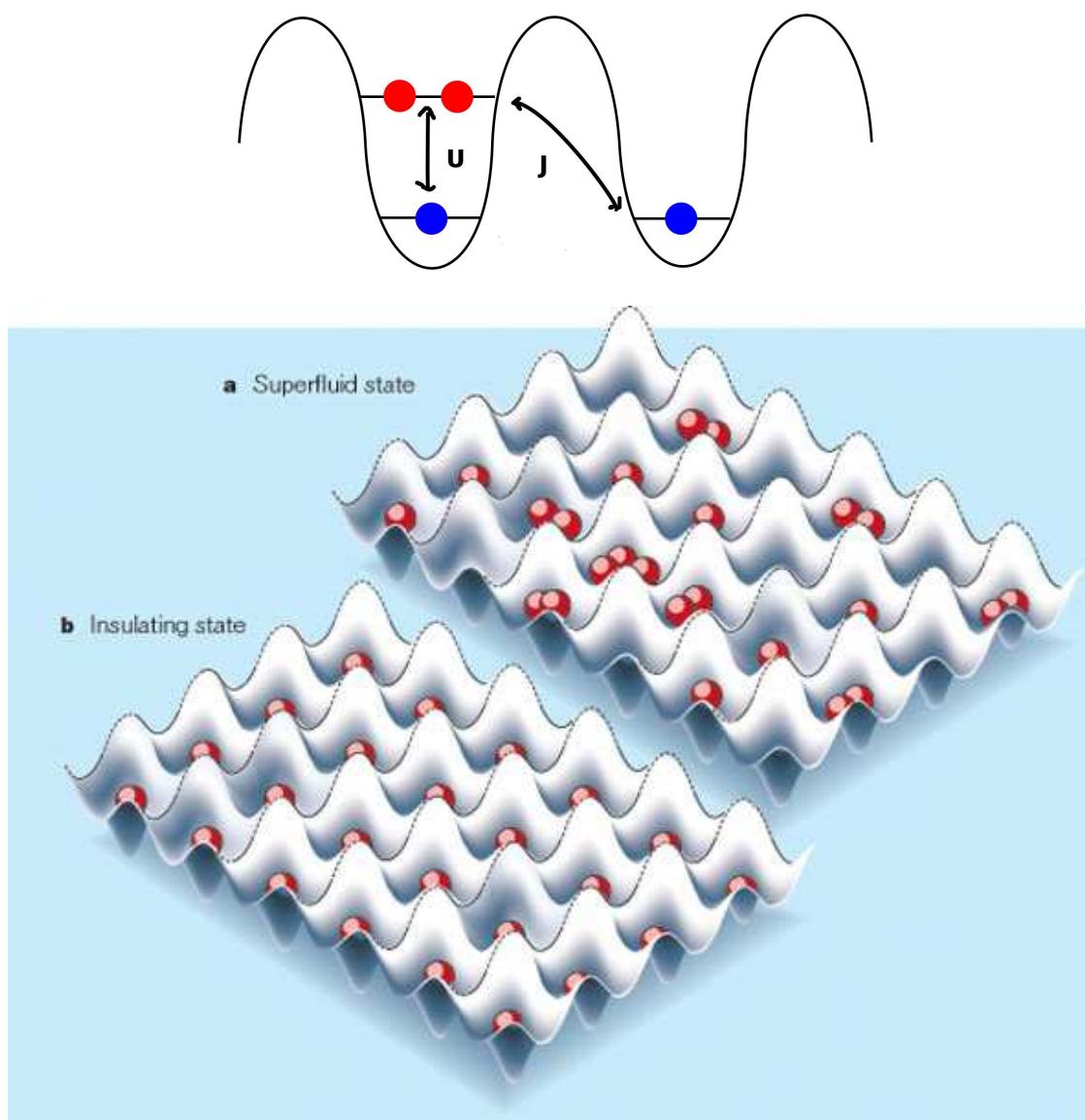
$$\hat{\psi}(\mathbf{x}) = \sum_i \hat{a}_i w_B(\mathbf{x} - \mathbf{x}_i)$$

Bose-Hubbard Hamiltonian:

$$\hat{H}_{\text{BHM}} = -J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + \frac{U}{2} \sum_i \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i - \mu \sum_i \hat{a}_i^\dagger \hat{a}_i$$



1.3 Parameters of the Bose-Hubbard Model I



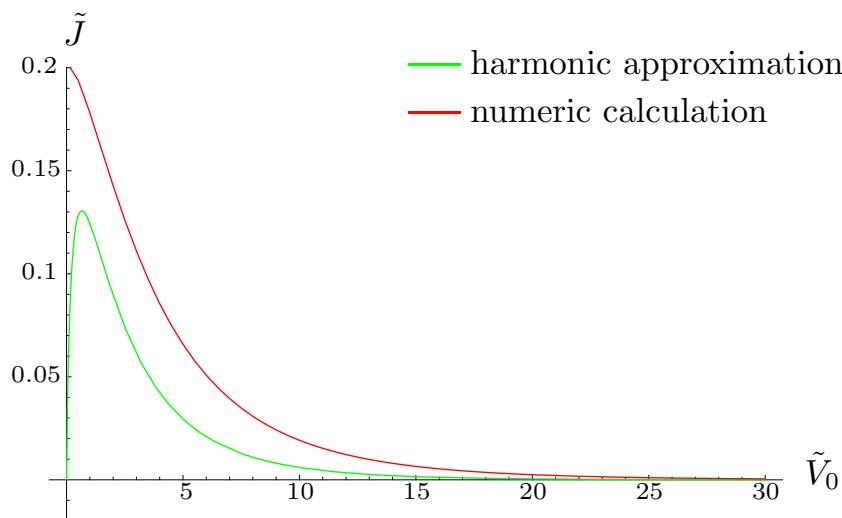
1.4 Parameters of the Bose-Hubbard Model II

Hopping-energy:

$$J = - \int d^3x w_B^*(\mathbf{x} - \mathbf{x}_i) \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}} \right] w_B(\mathbf{x} - \mathbf{x}_j) \quad U = \frac{4\pi a_{\text{BB}} \hbar^2}{m} \int d^3x |w_B(\mathbf{x} - \mathbf{x}_i)|^4$$

Harmonic approximation:

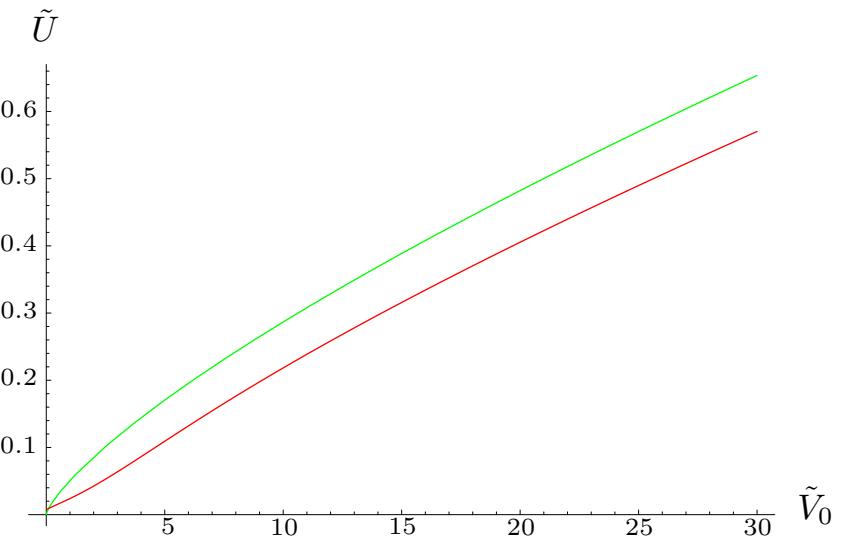
$$= \left(\frac{\pi^2}{4} - 1 \right) V_0 \exp \left[-\frac{\pi^2}{4} \sqrt{\frac{V_0}{E_R}} \right]$$



Onsite-energy:

$$= \sqrt{8\pi} E_R \frac{a_{\text{BB}}}{a} \left(\frac{V_0}{E_R} \right)^{3/4}$$

Harmonic approximation:



2.1 Mean-Field Theory T=0

Using the Mean-field ansatz:

$$\hat{a}_i^\dagger \hat{a}_j \approx \langle \hat{a}_i^\dagger \rangle \hat{a}_j + \langle \hat{a}_j \rangle \hat{a}_i^\dagger - \langle \hat{a}_i^\dagger \rangle \langle \hat{a}_j \rangle \quad \text{with} \quad \psi^* = \langle \hat{a}_i^\dagger \rangle, \psi = \langle \hat{a}_j \rangle$$

Mean-field hamiltonian:

$$\hat{H}_{\text{MF}} = \sum_i -Jz(\psi^* \hat{a}_i + \psi \hat{a}_i^\dagger - |\psi|^2) + \frac{U}{2} \hat{n}_i (\hat{n}_i - 1) - \mu \hat{n}_i$$

Perturbation theory:

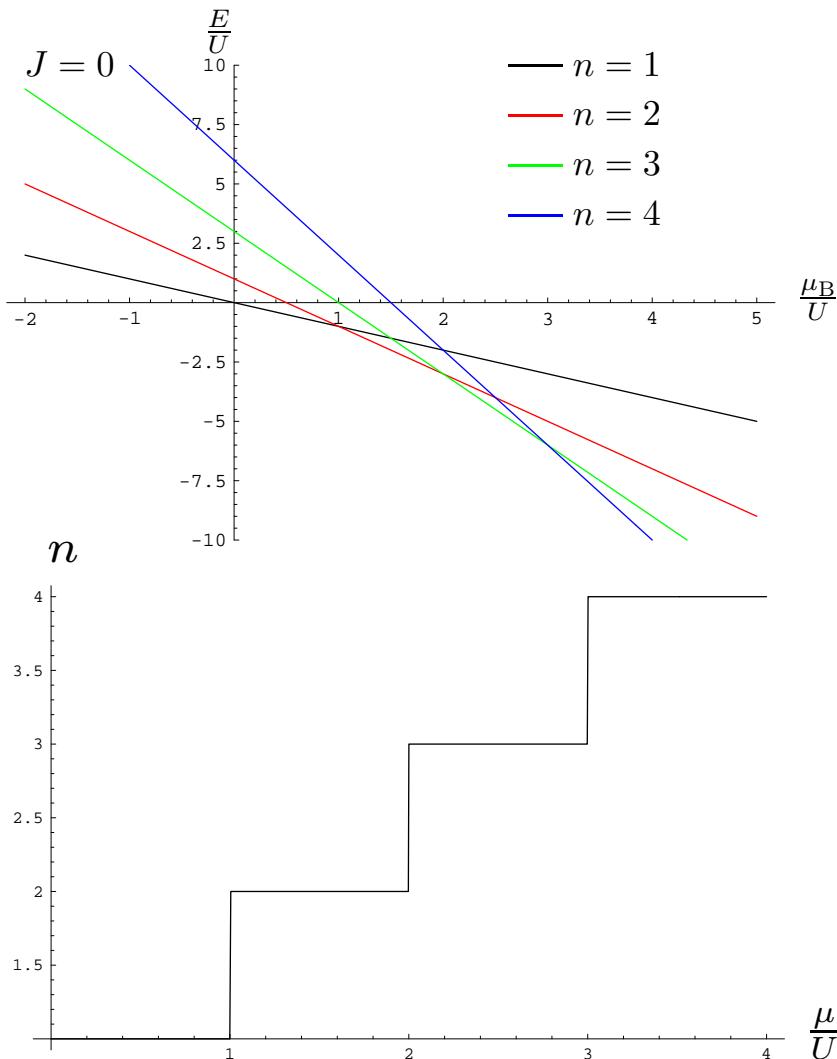
$$\bar{E} = a_0 + a_2 |\psi|^2 + a_4 |\psi|^4 + \dots$$

$$a_0 = E_n = \frac{U}{2} n(n-1) - \mu n$$

$$a_2 = Jz + J^2 z^2 \frac{U + \mu}{(\mu - Un)[U(n-1) - \mu]}$$

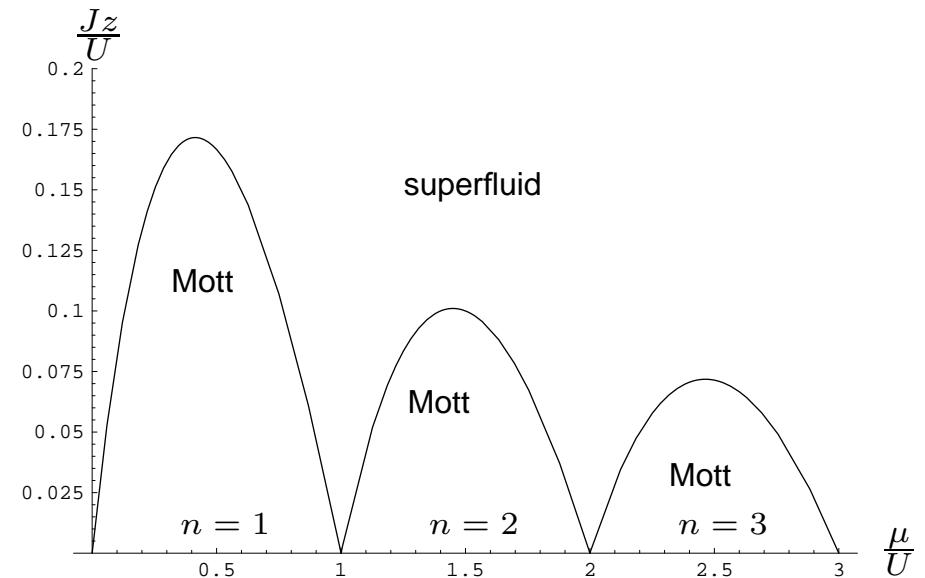
$$\begin{aligned} a_4 = & J^4 z^4 \left[\frac{n+1}{(E_{n+1} - E_n)^2} \left(\frac{n}{E_{n-1} - E_n} + \frac{n+1}{E_{n+1} - E_n} - \frac{n+2}{E_{n+2} - E_n} \right) \right. \\ & \left. + \frac{n}{(E_{n-1} - E_n)^2} \left(\frac{n+1}{E_{n+1} - E_n} + \frac{n}{E_{n-1} - E_n} - \frac{n-1}{E_{n-2} - E_n} \right) \right] \end{aligned}$$

2.2 Solutions T=0



Phase transition ($\psi = \psi^*$):

$$\left. \begin{array}{l} \frac{\partial E}{\partial \psi} = \psi(2a_2 + 4a_4\psi^2) = 0 \\ \frac{\partial^2 E}{\partial \psi^2} = 2a_2 + 12a_4\psi^2 > 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \psi = 0 \text{ if } a_2 > 0 \text{ Mott insulator} \\ \psi = \sqrt{-\frac{a_2}{2a_4}} \text{ if } a_2 < 0 \text{ superfluid} \end{array} \right.$$



2.3 Interaction Picture

For finite temperature we have to solve:

$$\mathcal{F} = -\frac{1}{\beta} \ln \mathcal{Z} \quad \text{with} \quad \mathcal{Z} = \text{Tr} \left\{ e^{-\beta \hat{H}} \right\}$$

Decomposition of the hamiltonian:

$$\hat{H} = \underbrace{-Jz(\psi^* \hat{a}_i + \psi \hat{a}_i^\dagger - |\psi|^2)}_{\hat{V}} + \underbrace{\frac{U}{2} \hat{n}_i (\hat{n}_i - 1) - \mu \hat{n}_i}_{\hat{H}_0}$$

Using the interaction picture in imaginary time:

$$\mathcal{Z} = \text{Tr} \left\{ e^{-\beta \hat{H}_0} \hat{U}_D(\hbar\beta, 0) \right\}$$

Dyson series:

$$\hat{U}_D(\hbar\beta, 0) = 1 + \frac{-1}{\hbar} \int_0^{\hbar\beta} d\tau \hat{V}_D(\tau) + \left(\frac{-1}{\hbar} \right)^2 \int_0^{\hbar\beta} d\tau \int_0^\tau d\tau_1 \hat{V}_D(\tau) \hat{V}_D(\tau_1) + \dots$$

$$\hat{V}_D(\tau) = e^{\hat{H}_0 \tau / \hbar} \hat{V} e^{-\hat{H}_0 \tau / \hbar}$$

2.4 Grand-Canonical Free Energy

All integrals are of the type:

$$\begin{aligned}
 & \gamma \int_0^{\hbar\beta} d\tau e^{a\tau} \int_0^\tau d\tau_1 e^{b\tau_1} \int_0^{\tau_1} d\tau_2 e^{c\tau_2} \int_0^{\tau_2} d\tau_3 e^{d\tau_3} \\
 = \gamma & \left[\frac{e^{(a+b+c+d)\hbar\beta} - 1}{(a+b+c+d)(b+c+d)(c+d)d} - \frac{e^{a\hbar\beta} - 1}{a(b+c+d)(c+d)d} - \frac{e^{(a+b)\hbar\beta} - 1}{(a+b)b(c+d)d} \right. \\
 & + \frac{e^{a\hbar\beta} - 1}{ab(c+d)d} - \frac{e^{(a+b+c)\hbar\beta} - 1}{(a+b+c)(b+c)cd} + \frac{e^{a\hbar\beta} - 1}{a(b+c)cd} + \frac{e^{(a+b)\hbar\beta} - 1}{(a+b)bcd} - \left. \frac{e^{a\hbar\beta} - 1}{abcd} \right]
 \end{aligned}$$

Result:

$$\mathcal{F}(\psi^*, \psi) = a_0 + a_2|\psi|^2 + a_4|\psi|^4 + \dots$$

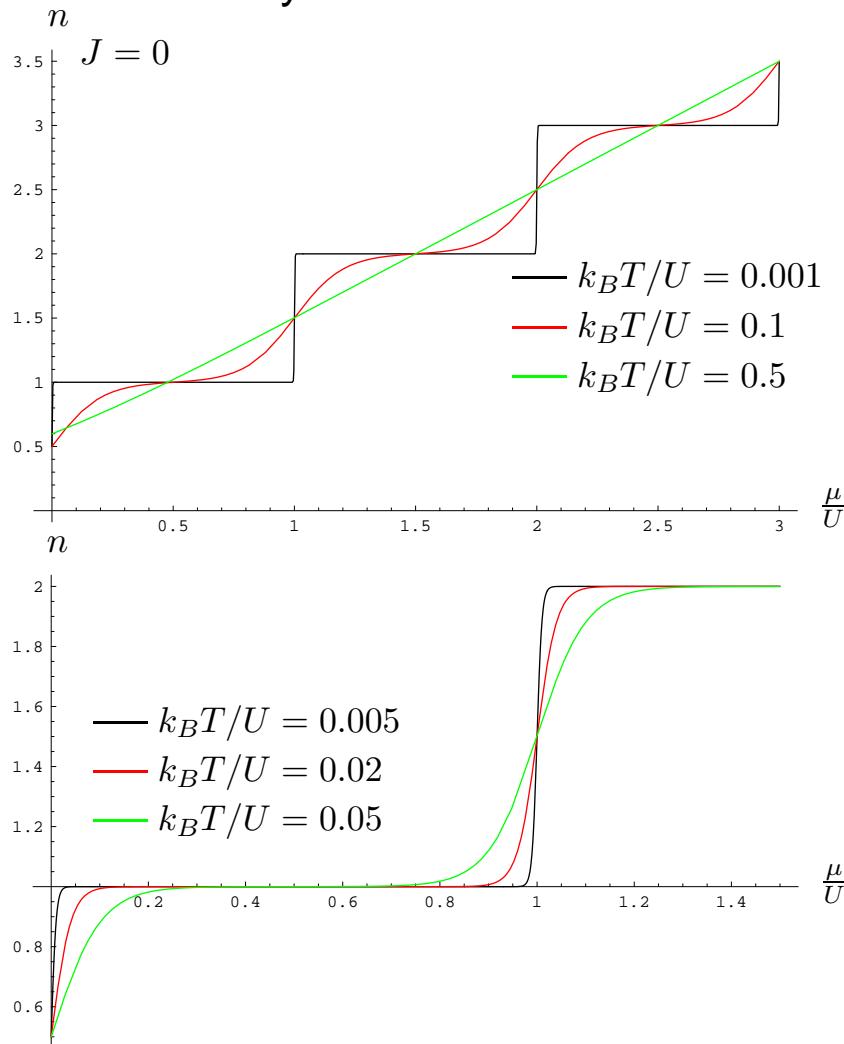
$$a_0 = -k_B T \ln \sum_{n=0}^{\infty} e^{-E_n/k_B T}$$

$$a_2 = \langle a_2(T=0) \rangle_T$$

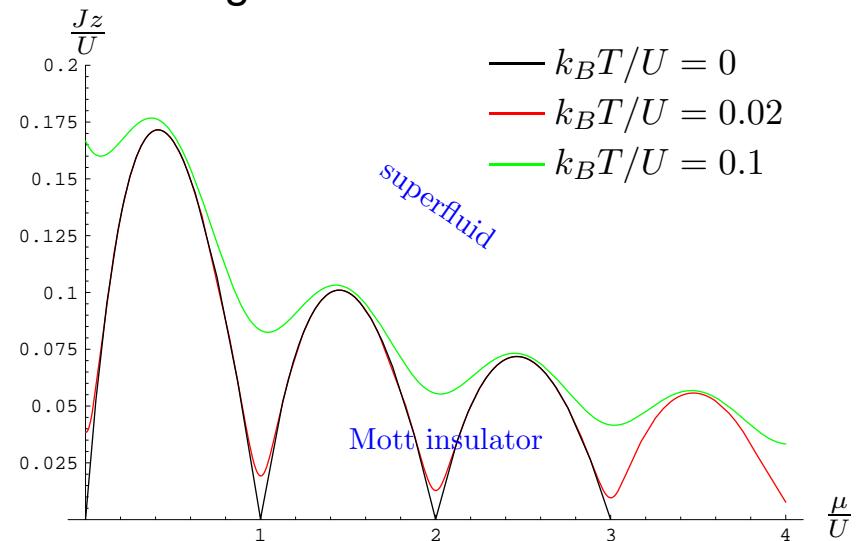
$$a_4 = \langle a_4(T=0) \rangle_T - \frac{\beta}{2} \left(\langle a_2(T=0)^2 \rangle_T - \langle a_2(T=0) \rangle_T^2 \right)$$

2.5 Solutions $T > 0$

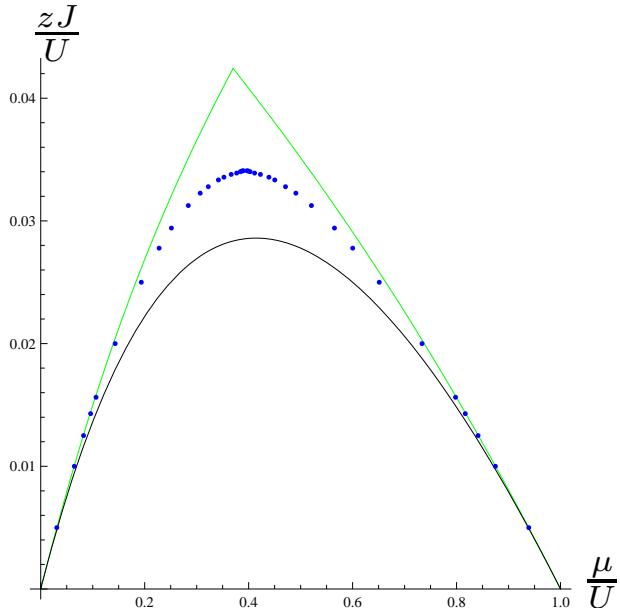
Boson density:



Phase diagram:



3.1 Variational Ansatz



Comparison of the phase border:
 Monte-Carlo simulation (blue dots)
 Mean-field result (black line)
 3rd order Strong-coupling approach
 (green line)

$$\hat{H} = \frac{U}{2} \sum_i \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i - \mu \sum_i \hat{a}_i^\dagger \hat{a}_i - \eta J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j - (1-\eta) J z \sum_i (\psi^* \hat{a}_i + \psi \hat{a}_i^\dagger - |\psi|^2)$$

- $\eta = 0$ gives mean-field hamiltonian
- $\eta = 1$ gives Bose-Hubbard hamiltonian

Self-consistency:

$$\frac{\partial \mathcal{F}}{\partial \psi^*} = 0 \iff \psi = \frac{\text{Tr} \left\{ \hat{a}_i e^{-\beta \hat{H}} \right\}}{\text{Tr} \left\{ e^{-\beta \hat{H}} \right\}} , \text{ proven up to first order in } \eta$$

3.2 Variational Ansatz

Dirac interaction picture $\hat{H} = \hat{H}_0 + \hat{V}$:

$$\begin{aligned}\hat{H}_0 &= \frac{U}{2} \sum_i \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i - \mu \sum_i \hat{a}_i^\dagger \hat{a}_i \\ \hat{V} &= -\eta J \sum_{} \hat{a}_i^\dagger \hat{a}_j - (1-\eta) J z \sum_i (\psi^* \hat{a}_i + \psi \hat{a}_i^\dagger - |\psi|^2)\end{aligned}$$

Result in zeroth order of η :

$$a_2 = JzN_S - J^2 z^2 N_S \left[\frac{\sum_{n=0}^{\infty} \frac{U+\mu}{(\mu-U n)[U(n-1)-\mu]} e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}} \right]$$

Result in first order of η :

$$a_2 = J^2 N_S \left[\frac{\sum_{n=0}^{\infty} \frac{U+\mu}{(\mu-U n)[U(n-1)-\mu]} e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}} \right] - J^3 z N_S \left[\frac{\sum_{n=0}^{\infty} \frac{U+\mu}{(\mu-U n)[U(n-1)-\mu]} e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}} \right]^2$$

\implies Same phase border for all temperatures.

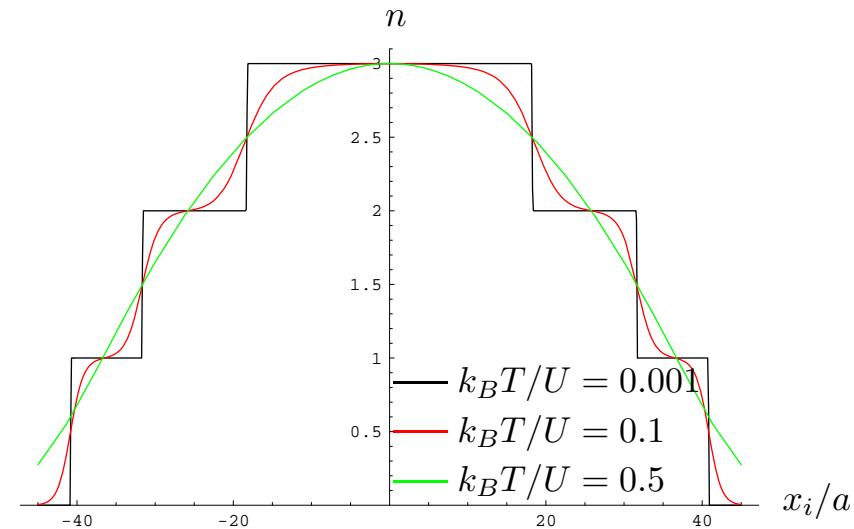
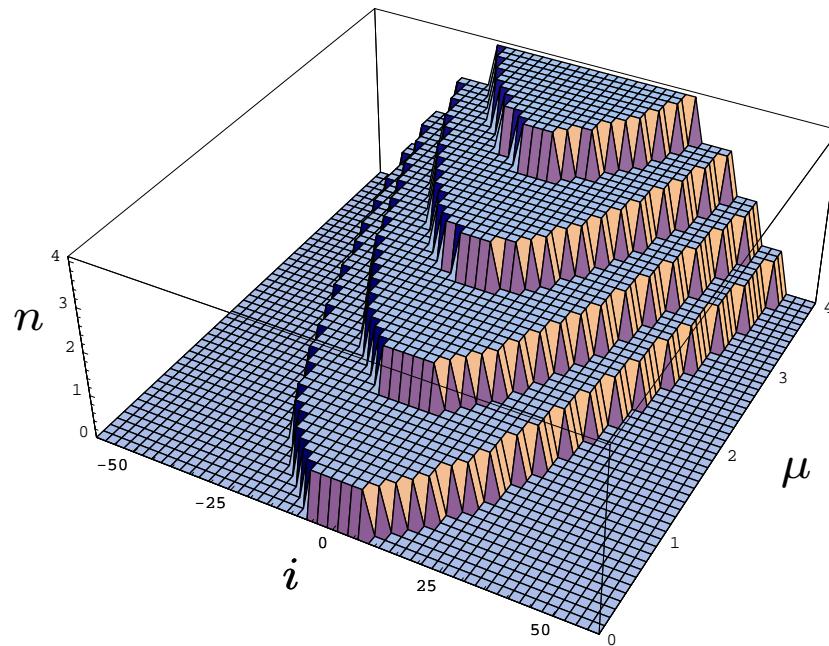
4.1 Influence of a Harmonic Trap

Hamiltonian with trap:

$$\hat{H}_{\text{BHM}} = -J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + \frac{U}{2} \sum_i \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i - \sum_i \mu_i \hat{a}_i^\dagger \hat{a}_i$$

with: $\mu_i = \mu - V_T \mathbf{r}_i^2$ and $V_T = \frac{1}{2} m \omega^2$

Site dependent density:



Effective frequency through inhomogeneous laser intensity

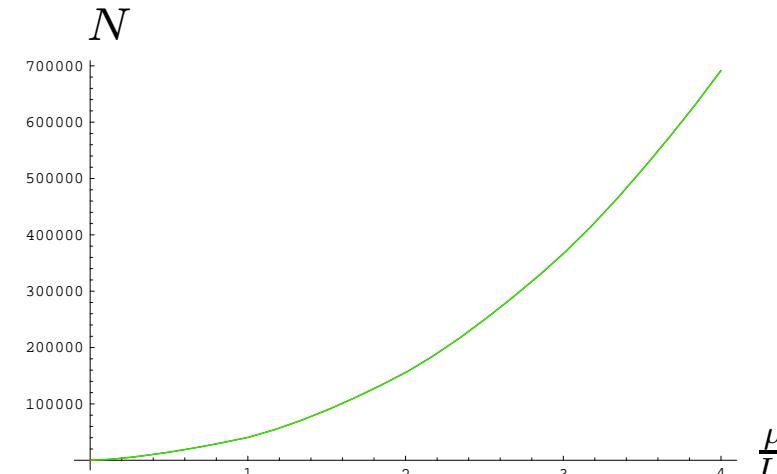
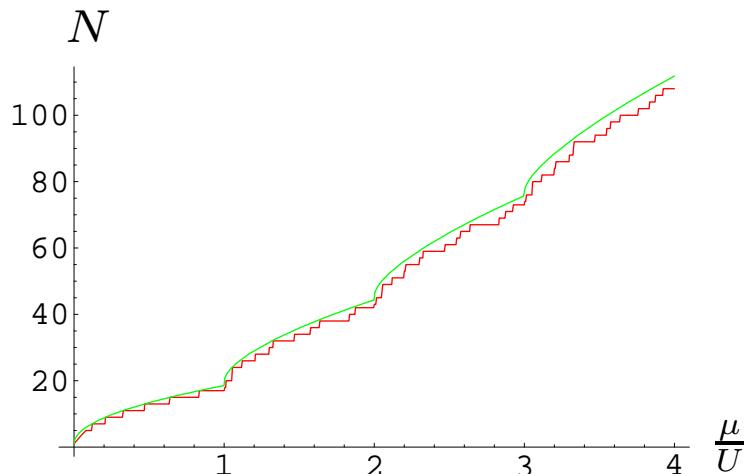
$$\omega \approx \sqrt{\omega_m^2 + \frac{8V_0 - 4E_R \sqrt{V_0/E_R}}{mw^2}}$$

Gerbier, Bloch et al. cond-mat/0701420

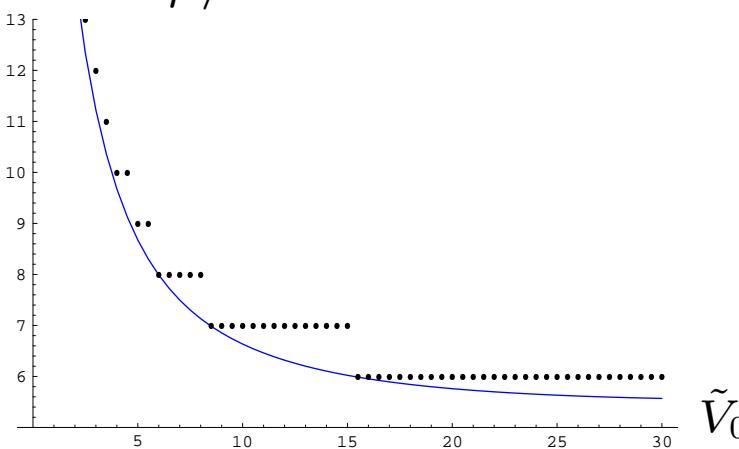
4.2 Particle Number ($T = 0$)

$$1D: N = \sum_{0 \leq c < \mu/U} 2 \sqrt{\frac{\mu - cU}{V_T a^2}} + 1$$

$$3D: N = \sum_{0 \leq c < \mu/U} \frac{4\pi}{3} \sqrt{\frac{\mu - cU}{V_T a^2}}^3$$



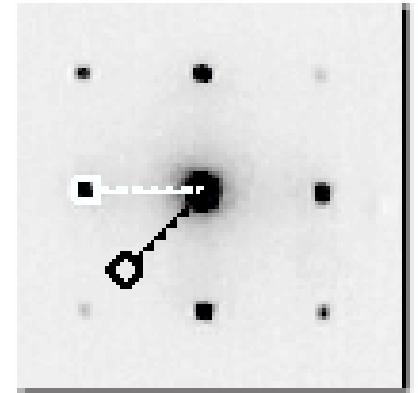
For the experimental setup in Hamburg:
 $n = 6$ or 7
 $(N = 200\,000)$



5.1 Basics

Density in momentum space:

$$\hat{n}_{\mathbf{k}} = \langle \hat{\psi}^\dagger(\mathbf{k}) \hat{\psi}(\mathbf{k}) \rangle = |w(\mathbf{k})|^2 \underbrace{\sum_{i,j} e^{i\mathbf{k}(\mathbf{r}_i - \mathbf{r}_j)} \langle \hat{a}_i^\dagger \hat{a}_j \rangle}_{S(\mathbf{k})}$$



Time of flight: $\mathbf{k} = \frac{m\mathbf{r}}{\hbar t}$

Definition of the visibility: $V = \frac{n_{\max} - n_{\min}}{n_{\max} + n_{\min}}$

Gerbier, Bloch et. al. PRL
95, 050404

Mean-field theory gives no result!

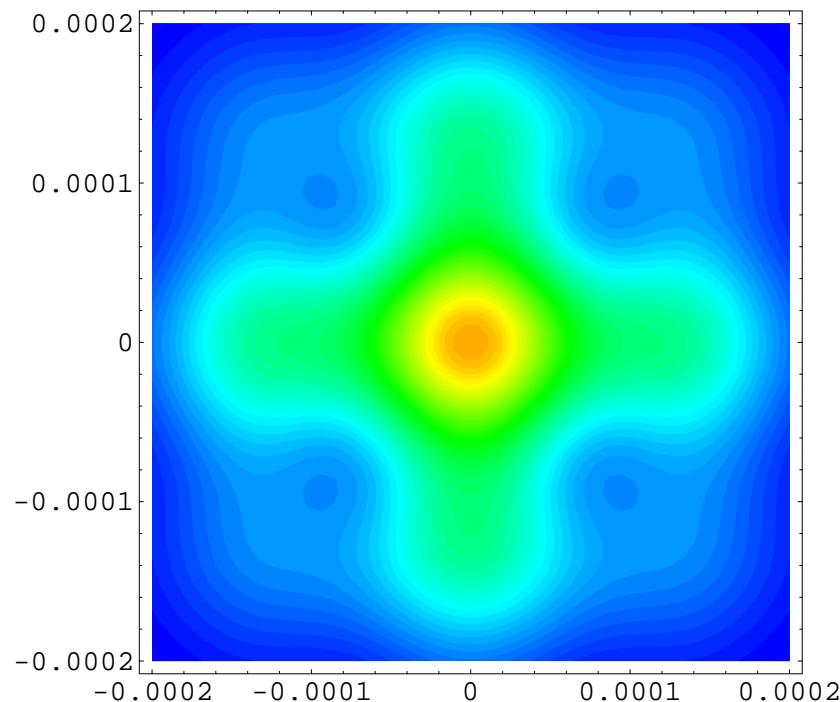
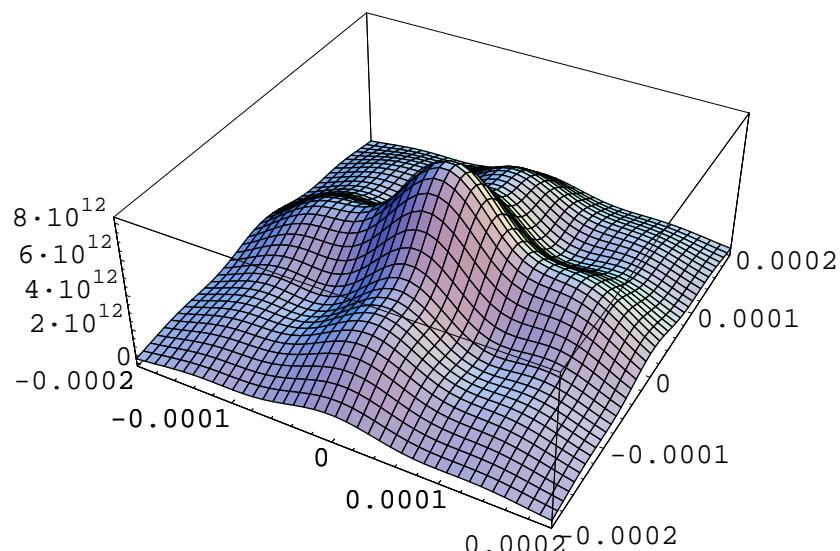
Monien ansatz:

$$\hat{H}_0 = \frac{U}{2} \sum_i \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i - \mu \sum_i \hat{a}_i^\dagger \hat{a}_i \quad \hat{V} = -J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j$$

Gives for $T = 0$:

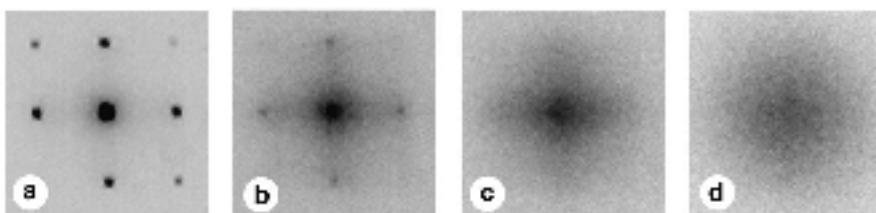
$$S(\mathbf{k}) = n + 4n(n+1) \frac{J}{U} \sum_{i=1}^3 \cos(k_i a)$$

5.2 Time of Flight



Experimental data:

$$n(x, y, t) = \int_{-\infty}^{\infty} dz n(\mathbf{k} = \frac{m\mathbf{r}}{\hbar t})$$



Gerbier, Bloch et. al. PRL **95**, 050404 (2005)

5.3 Finite Temperature

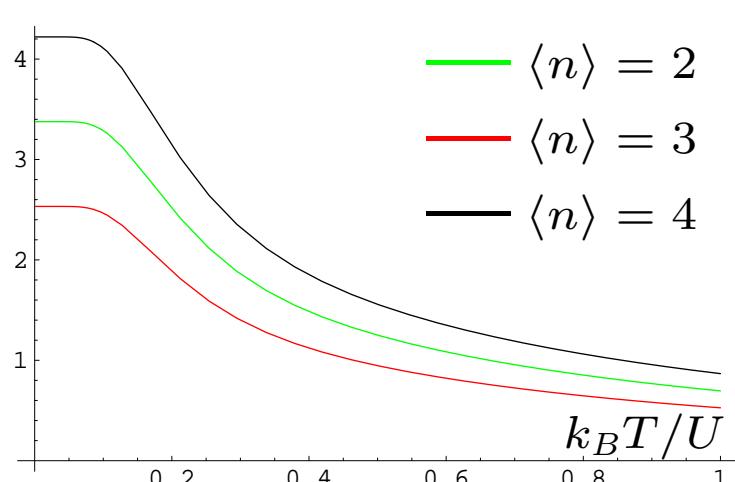
$$S(\mathbf{k}, T) = S_0(T) + 2 \frac{J}{U} S_1(T) \sum_{i=1}^3 \cos(k_i a)$$

$$S_0(T) = N_S \frac{\sum_{n=0}^{\infty} n e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}}$$

$$S_1(T) = N_S \frac{\sum_{n,m=0}^{\infty} \frac{(n+1)n+(m+1)m}{(n-m+1)(m-n+1)} e^{-\beta(E_n+E_m)}}{\sum_{n,m=0}^{\infty} e^{-\beta(E_n+E_m)}}$$

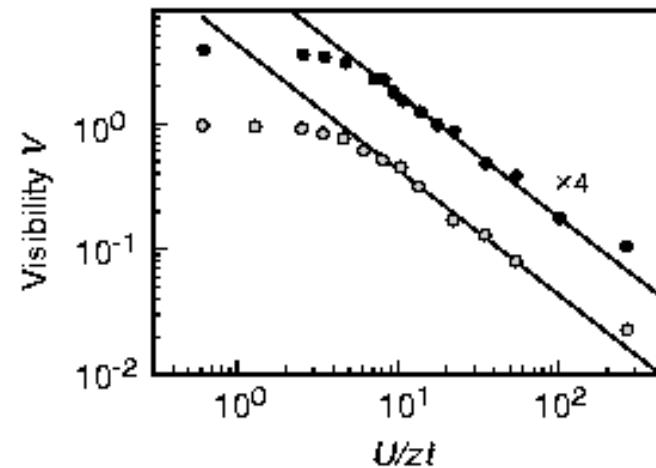
our theory:

$$V = \frac{zJ}{U} \underbrace{\frac{S_1(T)}{S_0(T)} \frac{1 - \cos(\sqrt{2}\pi)}{3}}_{\text{prefactor}}$$



Bloch result:

$$V = \frac{zJ}{U} \frac{S_1(T=0)}{S_0(T=0)} \frac{2}{3}$$



5.4 Superfluid Phase

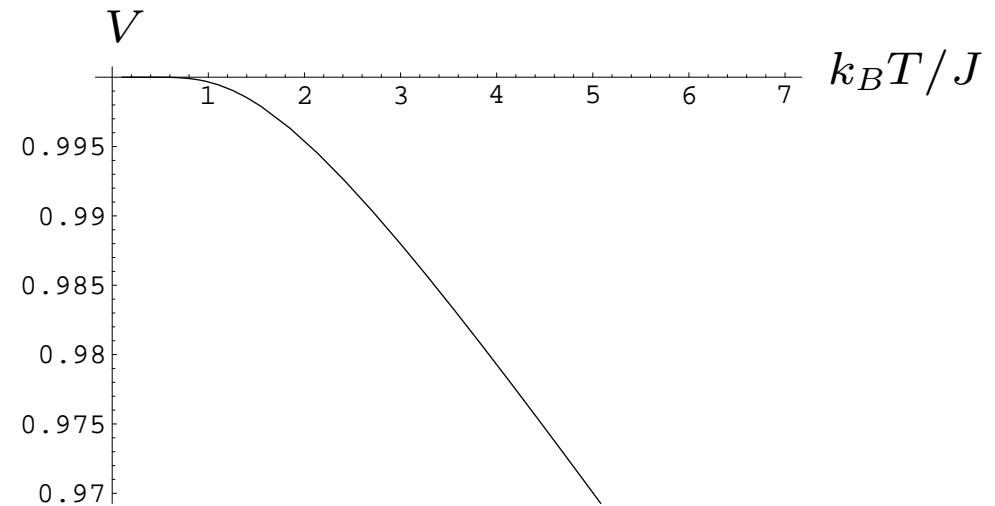
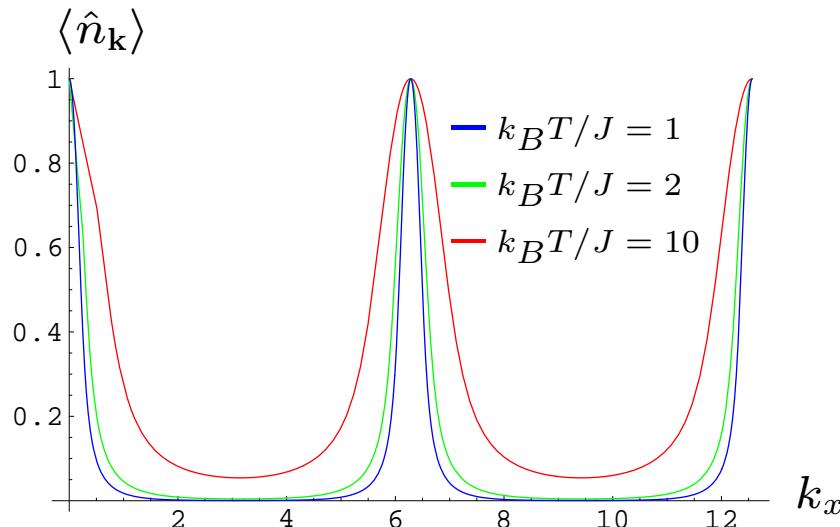
Valid for $V_0 \approx 0 \Rightarrow U = 0$.

Bose-Hubbard Hamiltonian:

$$\hat{H}_{\text{SF}} = \sum_{\mathbf{k}} [-2J \sum_{i=1}^3 \cos(k_i a) - \mu] \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}$$

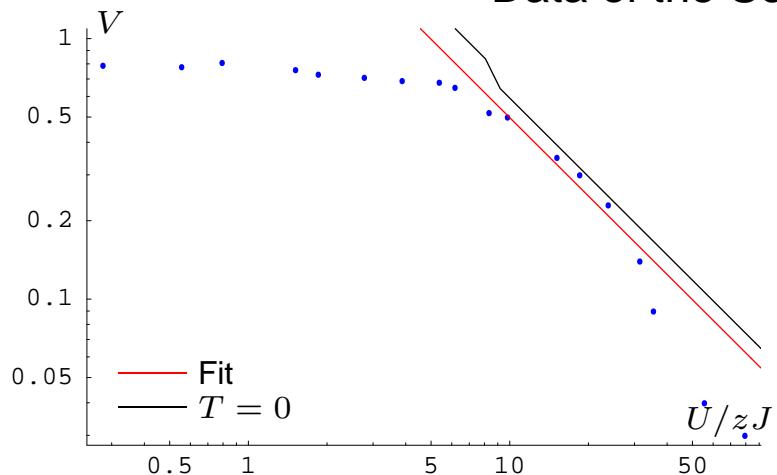
Visibility from:

$$\langle \hat{n}_{\mathbf{k}} \rangle = \frac{\sum_{n_{\mathbf{k}}} n_{\mathbf{k}} e^{-\beta E_{\text{SF}}}}{\sum_{n_{\mathbf{k}}} e^{-\beta E_{\text{SF}}}}$$



5.5 Quantitative Analysis

Temperature from Mott phase



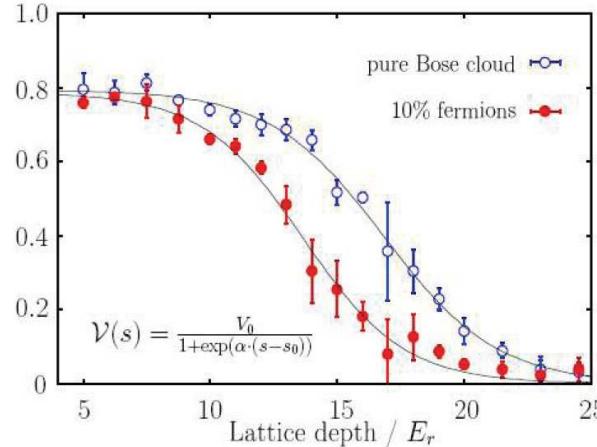
Prefactor gives: $k_B T / U = 0.15$

Corresponds to $T = 612$ nK for $\tilde{V}_0 = 20$
and to $T = 15.5$ nK for $\tilde{V}_0 = 0$.

Can be understood as an adiabatic
expansion.

Temperature from superfluid phase

Data of the Sengstock group in Hamburg:



Maximal visibility: 0.8

Corresponds to a temperature of 624 nK
Problem with critical temperature:

$$T_c^{(0)} \approx 0.94 \frac{\hbar\omega}{k_B} N^{1/3}$$

Gives $T_c^{(0)} = 132$ nK

Outlook

- Improvement in Mott phase:
second-order strong-coupling expansion
- Improvement in SF phase: finite U
- Influence of harmonic trap on visibility
- Visibility as a function of U/J :

quantitative comparison with experiment



thermometer for optical lattice