Bose-Einstein Condensates with Strong Disorder: Replica Method

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New Year Seminar

Outline

Introduction

- **Superfluid Helium in Porous Media:**
  Reppy et al., PRL **51**, 666 (1983)
- **Laser Speckles:**
  Inguscio et al., PRL **95**, 070401 (2005)
  Aspect et al., PRL **95**, 170409 (2005)
- **Wire Traps:**
  Schmiedmayer et al., Phys. Rev. A **76**, 063621 (2007)
- **Localized Atomic Species:**
  Gavish and Castin, PRL **95**, 020401 (2005)
  Schneble et al., PRL **107**, 145306 (2011)
- **Incommensurate Lattices:**
  Lewenstein et al., PRL **91**, 080403 (2003)
  Ertmer et al., PRL **95**, 170411 (2005)

Model System

- Action of a Bose gas

$$\mathcal{A} = \int_0^{\hbar\beta} d\tau \int d\mathbf{r} \left\{ \psi^* (\mathbf{r}, \tau) \left[\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2M} \Delta + V(\mathbf{r}) + U(\mathbf{r}) - \mu \right] \psi (\mathbf{r}, \tau) + \frac{g}{2} \psi^* (\mathbf{r}, \tau)^2 \psi (\mathbf{r}, \tau)^2 \right\}$$

- Properties
  - trap potential $V(\mathbf{r})$
  - disorder potential $U(\mathbf{r})$
  - chemical potential μ
  - repulsive interaction $g = \frac{4\pi\hbar^2 a}{M}$
  - periodic Bose fields $\psi (\mathbf{r}, \tau + \hbar\beta) = \psi (\mathbf{r}, \tau)$

Random Potential

- Disorder Ensemble Average

$$\overline{\bullet} = \int \mathcal{D}U \bullet P[U], \qquad \int \mathcal{D}U P[U] = 1$$

- Assumption

$$\overline{U(\mathbf{r})} = 0 \qquad \overline{U(\mathbf{r_1})U(\mathbf{r_2})} = R^{(2)}(\mathbf{r_1} - \mathbf{r_2})$$

- Characteristic Functional

$$\overline{\exp\left\{i \int d\mathbf{r} J(\mathbf{r}) U(\mathbf{r})\right\}} = \exp\left\{\sum_{n=2}^{\infty} \frac{i^n}{n!} \int d\mathbf{r}_1 \cdots \int d\mathbf{r}_n R^{(n)}(\mathbf{r}_1 \cdots \mathbf{r}_n) J(\mathbf{r}_1) \cdots J(\mathbf{r}_n)\right\}$$

Grand-Canonical Potential

- Aim

$$F = -\frac{1}{\beta}\overline{\ln \mathcal{Z}}$$

$$\mathcal{Z} = \oint \mathcal{D}\psi^* \mathcal{D}\psi e^{-\mathcal{A}[\psi^*,\psi]/\hbar}$$

$\mathcal{Z}$: partition function

- Problem

$$\overline{\ln \mathcal{Z}} \neq \ln \overline{\mathcal{Z}}$$

- Solution: Replica Trick

$$F = -\frac{1}{\beta} \lim_{\mathcal{N}\to 0} \frac{\overline{\mathcal{Z}^{\mathcal{N}}} - 1}{\mathcal{N}}$$

G. Parisi, J. Phys. France **51**, 1595 (1990)
M. Mezard and G. Parisi, J. Phys. I France **1**, 809 (1991)

Replica Trick

- Disorder Averaged Partition Function

$$\overline{\mathcal{Z}^{\mathcal{N}}} = \overline{\left\{ \prod_{\alpha'=1}^{\mathcal{N}} \oint \mathcal{D}\psi^*_{\alpha'} \mathcal{D}\psi_{\alpha'} \right\} e^{-\sum_{\alpha=1}^{\mathcal{N}} \mathcal{A}[\psi^*_\alpha, \psi_\alpha]/\hbar}} = \oint \left\{ \prod_{\alpha=1}^{\mathcal{N}} \mathcal{D}\psi^*_\alpha \mathcal{D}\psi_\alpha \right\} e^{-\mathcal{A}^{(\mathcal{N})}/\hbar}$$

- Replicated Action

$$\mathcal{A}^{(\mathcal{N})} = \int_0^{\hbar\beta} d\tau \int d\mathbf{r} \sum_{\alpha=1}^{\mathcal{N}} \left\{ \psi^*_\alpha(\mathbf{r},\tau) \left[\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2M}\Delta + V(\mathbf{r}) - \mu \right] \psi_\alpha(\mathbf{r},\tau) \right.$$
$$\left. + \frac{g}{2} |\psi_\alpha(\mathbf{r},\tau)|^4 \right\} + \sum_{n=2}^{\infty} \frac{1}{n!} \left(\frac{-1}{\hbar} \right)^{n-1} \int_0^{\hbar\beta} d\tau_1 \cdots \int_0^{\hbar\beta} d\tau_n \int d\mathbf{r}_1 \cdots \int d\mathbf{r}_n$$
$$\times \sum_{\alpha_1=1}^{\mathcal{N}} \cdots \sum_{\alpha_n=1}^{\mathcal{N}} R^{(n)}(\mathbf{r}_1 \cdots \mathbf{r}_n) |\psi_{\alpha_1}(\mathbf{r}_1,\tau_1)|^2 \cdots |\psi_{\alpha_n}(\mathbf{r}_n,\tau_n)|^2$$

Remarks:

- In the replica limit $\mathcal{N} \to 0$ Higher-order disorder cumulants are negligible: only $R^{(2)}(\mathbf{r})$ contributes
- Disorder amounts to attractive interaction for $n = 2$

Assumptions

- Bogoliubov background method $\psi_\alpha(\mathbf{r},\tau) = \Psi_\alpha(\mathbf{r},\tau) + \delta\psi_\alpha(\mathbf{r},\tau)$
- Hartree-Fock theory
- Semiclassical approximations due to $V(\mathbf{r})$
- Replica symmetry

$$\Rightarrow \begin{cases} \Psi_\alpha(\mathbf{r},\tau) = \sqrt{n_0(\mathbf{r})} \\ \langle \delta\psi_\alpha(\mathbf{r},\tau)\,\delta\psi_{\alpha'}(\mathbf{r}',\tau')\rangle = \mathcal{Q}\left(\mathbf{r}-\mathbf{r}', \frac{\mathbf{r}+\mathbf{r}'}{2}, \tau-\tau'\right)\delta_{\alpha\alpha'} + q\left(\frac{\mathbf{r}+\mathbf{r}'}{2}, \tau-\tau'\right) \\ n(\mathbf{r}) = \Psi_\alpha(\mathbf{r},\tau)\Psi^*_\alpha(\mathbf{r},\tau) + \langle \delta\psi_\alpha(\mathbf{r},\tau)\,\delta\psi_\alpha(\mathbf{r},\tau)\rangle \end{cases}$$

Homogeneous case worked out in Ref.

R. Graham and A. Pelster, Int. J. Bif. Chaos **19**, 2745 (2009)

Self-Consistency equations for finite temperature T

$$n(\mathbf{r}) = n_0(\mathbf{r}) + q(\mathbf{r}) + \left(\frac{M}{2\pi\beta\hbar^2}\right)^{3/2} \varsigma_{3/2}\left(e^{\beta\ [\mu - d^2 - 2gn(\mathbf{r}) - V(\mathbf{r})]}\right)$$

$$q(\mathbf{r}) = \frac{d\left[n(\mathbf{r}) - \left(\frac{M}{2\pi\beta\hbar^2}\right)^{3/2} \varsigma_{3/2}\left(e^{\beta\ [\mu - d^2 - 2gn(\mathbf{r}) - V(\mathbf{r})]}\right)\right]}{d + \sqrt{-\mu + d^2 + 2gn(\mathbf{r}) + V(\mathbf{r})}}$$

$$\left\{-gn_0(\mathbf{r}) + \left[\sqrt{-\mu + d^2 + 2gn(\mathbf{r}) + V(\mathbf{r})} + d\right]^2 - \frac{\hbar^2}{2M}\Delta\right\}\sqrt{n_0(\mathbf{r})} = 0$$

$$N = \int n(\mathbf{r})d\mathbf{r}$$

where $R^{(2)}(\mathbf{r_1} - \mathbf{r_2}) = D\delta(\mathbf{r_1} - \mathbf{r_2})$, $d = \sqrt{\pi}D\left(\frac{M}{2\pi\hbar^2}\right)^{3/2}$

Assumptions

- $T = 0$
- $V(\mathbf{r}) = \frac{1}{2} M\Omega^2 r^2$
- Thomas-Fermi approximation
- length scale $l = \sqrt{\frac{\hbar}{M\Omega}}$
- Energy scale $\mu_0 = \frac{15^{2/5}}{2}\left(\frac{aN}{l}\right)^{2/5}\hbar\Omega$

- Dimensionless quantities

$$\tilde{r} = \sqrt{\frac{M\Omega^2}{2\mu_0}}\, r, \qquad \tilde{n}(\tilde{r}) = \frac{gn(\mathbf{r})}{\mu_0}, \qquad \tilde{\mu} = \frac{\mu - d^2}{\mu_0}, \qquad \tilde{d} = \frac{d}{\sqrt{\mu_0}}$$

$$\tilde{n}(\tilde{r}) = \tilde{n}_0(\tilde{r}) + \tilde{q}\,(\tilde{r})$$

$$\tilde{q}\,(\tilde{r}) = \frac{\tilde{d}\tilde{n}(\tilde{r})}{\sqrt{-\tilde{\mu} + 2\tilde{n}(\tilde{r}) + \tilde{r}^2} + \tilde{d}}$$

$$\left\{ -\tilde{n}_0(\tilde{r}) + \left[\sqrt{-\tilde{\mu} + 2\tilde{n}(\tilde{r}) + \tilde{r}^2} + \tilde{d} \right]^2 \right\} \sqrt{\tilde{n}_0(\tilde{r})} = 0$$

Bose-glass Phase $\tilde{n}_0(\tilde{r}) = 0$ and $\tilde{q}(\tilde{r}) = \tilde{n}(\tilde{r}) \neq 0$

$$-\tilde{\mu} + 2\tilde{n}(\tilde{r}) + \tilde{r}^2 = 0$$

Superfluid Phase $\tilde{n}_0(\tilde{r}) \neq 0$ and $\tilde{q}(\tilde{r}) \neq 0$

$$\tilde{n}_0(\tilde{r}) = \left[\sqrt{-\tilde{\mu} + 2\tilde{n}(\tilde{r}) + \tilde{r}^2} + \tilde{d}\right]^2$$

$$\tilde{q}(\tilde{r}) = \frac{\tilde{d}}{\sqrt{-\tilde{\mu} + 2\tilde{n}(\tilde{r}) + \tilde{r}^2}} \left[\sqrt{-\tilde{\mu} + 2\tilde{n}(\tilde{r}) + \tilde{r}^2} + \tilde{d}\right]^2$$

$$\tilde{n}(\tilde{r}) = \frac{1}{\sqrt{-\tilde{\mu} + 2\tilde{n}(\tilde{r}) + \tilde{r}^2}} \left[\sqrt{-\tilde{\mu} + 2\tilde{n}(\tilde{r}) + \tilde{r}^2} + \tilde{d}\right]^3$$

- Solving a cubic equation

$$Az^3 + Bz^2 + Cz + D = 0$$

Variable transformation

$$y = z - \frac{B}{3A}$$

gives normal form

$$y^3 + Py + Q = 0$$

where

$$\begin{cases} P = -\frac{B^2}{3A^2} + \frac{C}{A} \\ \\ Q = \frac{B}{27A}\left(\frac{2B^2}{A^2} - \frac{9C}{A}\right) + \frac{D}{A} \end{cases}$$

- Solution ansatz

$$y = u + v$$

$$\Rightarrow \begin{cases} u^3 + v^3 = -Q \\ u^3 v^3 = -\frac{P^3}{27} \end{cases}$$

$\Rightarrow$ u^3 and v^3 solve quadratic equation

$$X^2 + QX - \frac{P^3}{27} = 0$$

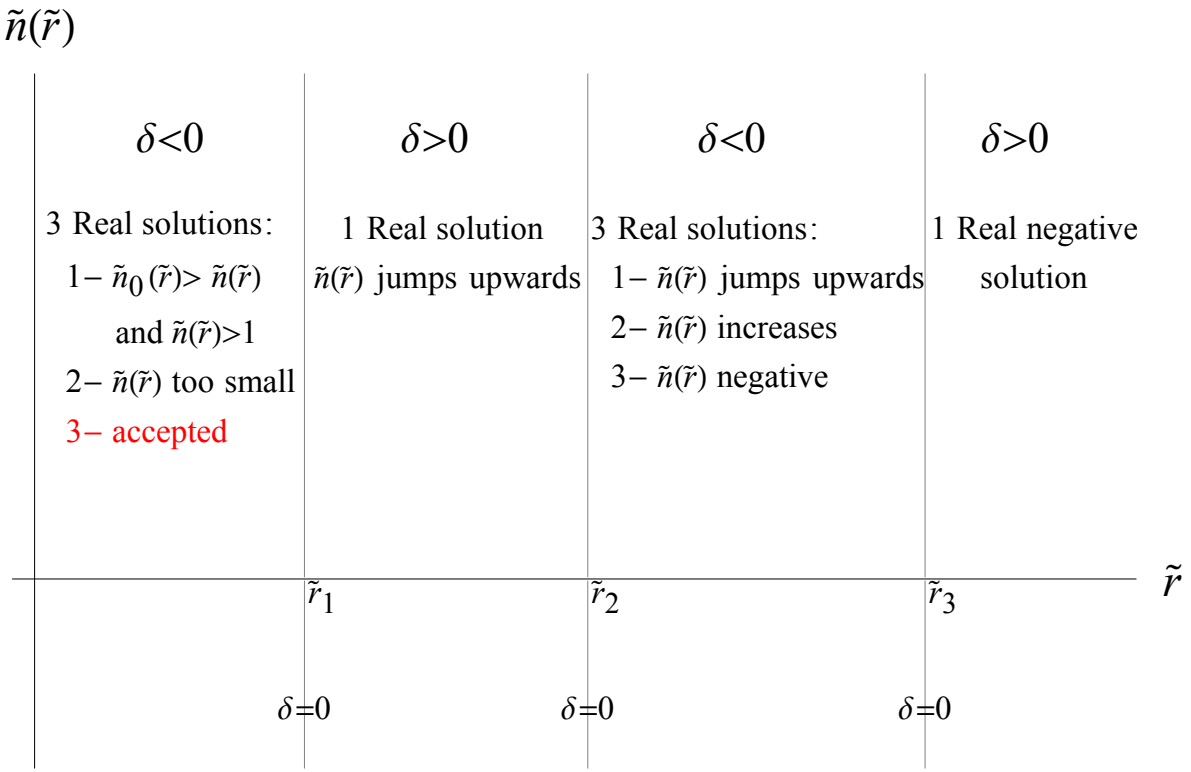
- Discriminant

$$\delta = Q^2 + \frac{4}{27} P^3$$

- $\delta > 0 \rightarrow$ 1 real solution + 2 complex solutions
  - $\sqrt[3]{\frac{-Q+\sqrt{\delta}}{2}} + \sqrt[3]{\frac{-Q-\sqrt{\delta}}{2}} - \frac{B}{3A}$
  - $e^{\frac{2i\pi}{3}} \sqrt[3]{\frac{-Q+\sqrt{\delta}}{2}} + e^{-\frac{2i\pi}{3}} \sqrt[3]{\frac{-Q-\sqrt{\delta}}{2}} - \frac{B}{3A}$
  - $e^{-\frac{2i\pi}{3}} \sqrt[3]{\frac{-Q+\sqrt{\delta}}{2}} + e^{\frac{2i\pi}{3}} \sqrt[3]{\frac{-Q-\sqrt{\delta}}{2}} - \frac{B}{3A}$
- $\delta < 0 \rightarrow$ 3 real solutions
  - $\sqrt[3]{\frac{-Q+i\sqrt{-\delta}}{2}} + \sqrt[3]{\frac{-Q-i\sqrt{-\delta}}{2}} - \frac{B}{3A}$
  - $e^{\frac{2i\pi}{3}} \sqrt[3]{\frac{-Q+i\sqrt{-\delta}}{2}} + e^{-\frac{2i\pi}{3}} \sqrt[3]{\frac{-Q-i\sqrt{-\delta}}{2}} - \frac{B}{3A}$
  - $e^{-\frac{2i\pi}{3}} \sqrt[3]{\frac{-Q+i\sqrt{-\delta}}{2}} + e^{\frac{2i\pi}{3}} \sqrt[3]{\frac{-Q-i\sqrt{-\delta}}{2}} - \frac{B}{3A}$
- $\delta = 0 \rightarrow$ 2 real solutions
  - $\frac{3Q}{P} - \frac{B}{3A}$
  - $-\frac{3Q}{2P} - \frac{B}{3A}$ (doubly degenerate)

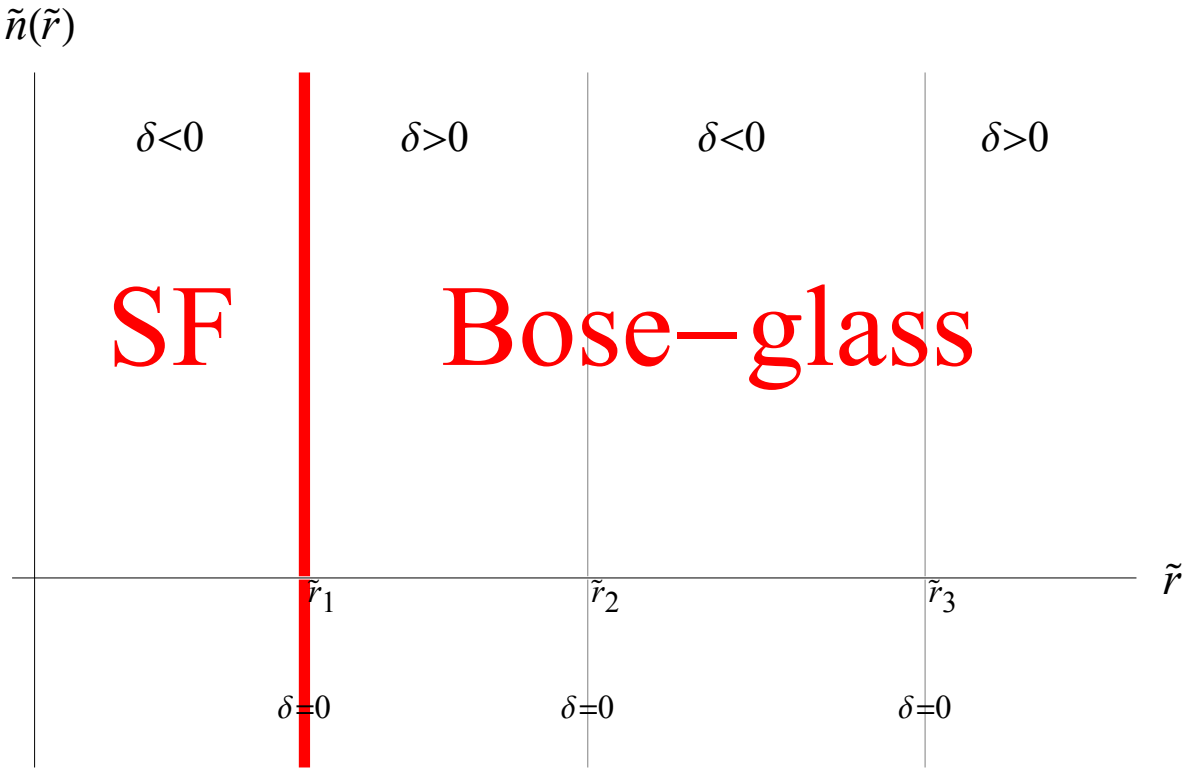
- Application of Cardan method in superfluid phase for $\tilde{n}(\tilde{r})$

$$\delta = -\frac{4\tilde{\mu}^3}{27} + \frac{4\tilde{\mu}^2\tilde{d}^2}{3} + 8\tilde{\mu}\tilde{d}^4 + 4\tilde{d}^6 + \left(\frac{4\tilde{\mu}^2}{9} - \frac{8\tilde{\mu}\tilde{d}^2}{3} - 8\tilde{d}^4\right)\tilde{r}^2 + \left(-\frac{4\tilde{\mu}}{9} + \frac{4\tilde{d}^2}{3}\right)\tilde{r}^4 + \frac{4}{27}\tilde{r}^6$$



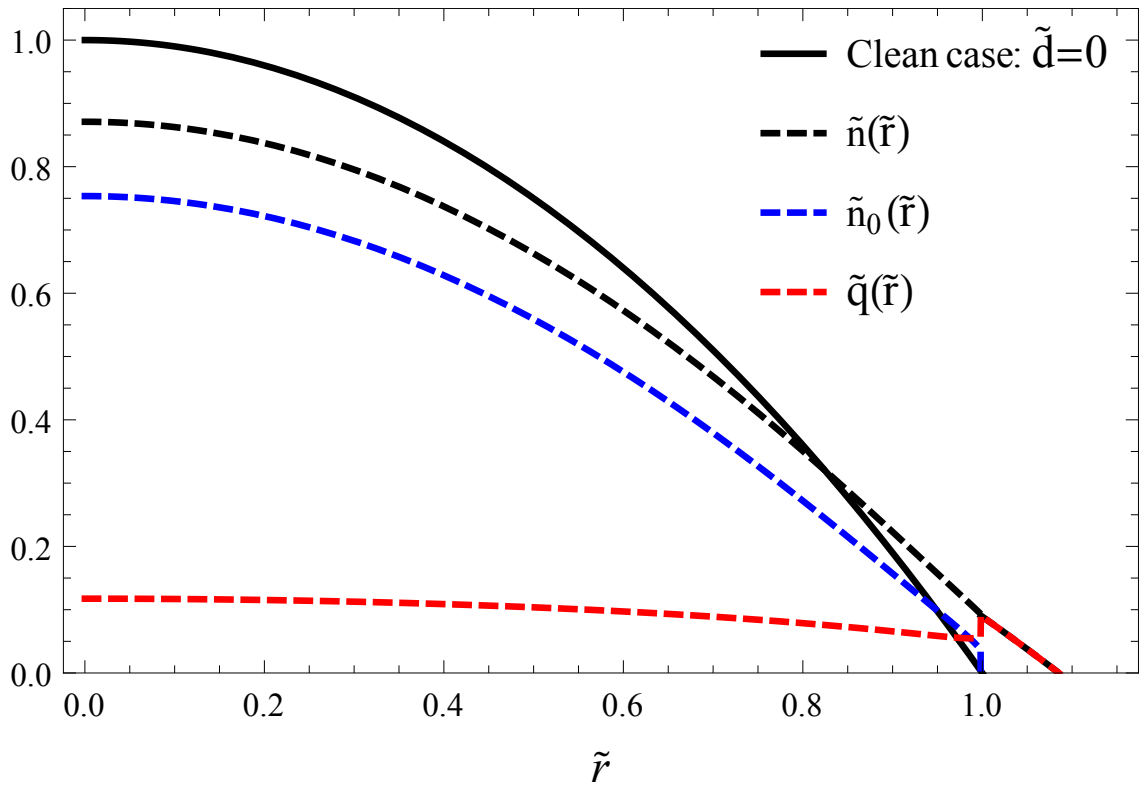
- Application of Cardan method in superfluid phase for $\tilde{n}(\tilde{r})$

$$\delta = -\frac{4\tilde{\mu}^3}{27} + \frac{4\tilde{\mu}^2\tilde{d}^2}{3} + 8\tilde{\mu}\tilde{d}^4 + 4\tilde{d}^6 + \left(\frac{4\tilde{\mu}^2}{9} - \frac{8\tilde{\mu}\tilde{d}^2}{3} - 8\tilde{d}^4\right)\tilde{r}^2 + \left(-\frac{4\tilde{\mu}}{9} + \frac{4\tilde{d}^2}{3}\right)\tilde{r}^4 + \frac{4}{27}\tilde{r}^6$$



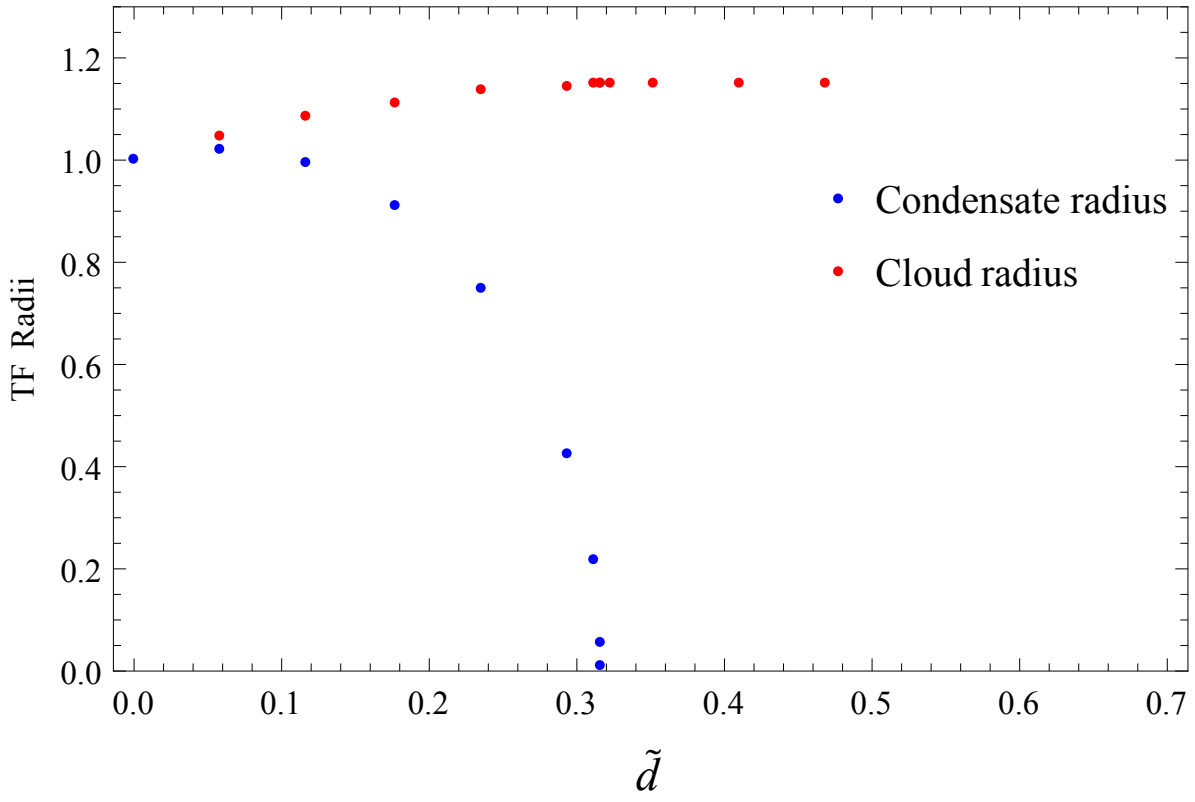
Densities

- ^{87}Rb, $N = 10^6$, $\tilde{d} = 0.117$, $\tilde{\mu} = 1.177$, $\Omega = 200\pi\mathrm{Hz}$ and $a = 5.29\mathrm{nm}$



Thomas-Fermi Radii

- ^{87}Rb, $N = 10^6$, $\Omega = 200\pi\mathrm{Hz}$ and $a = 5.29\mathrm{nm}$



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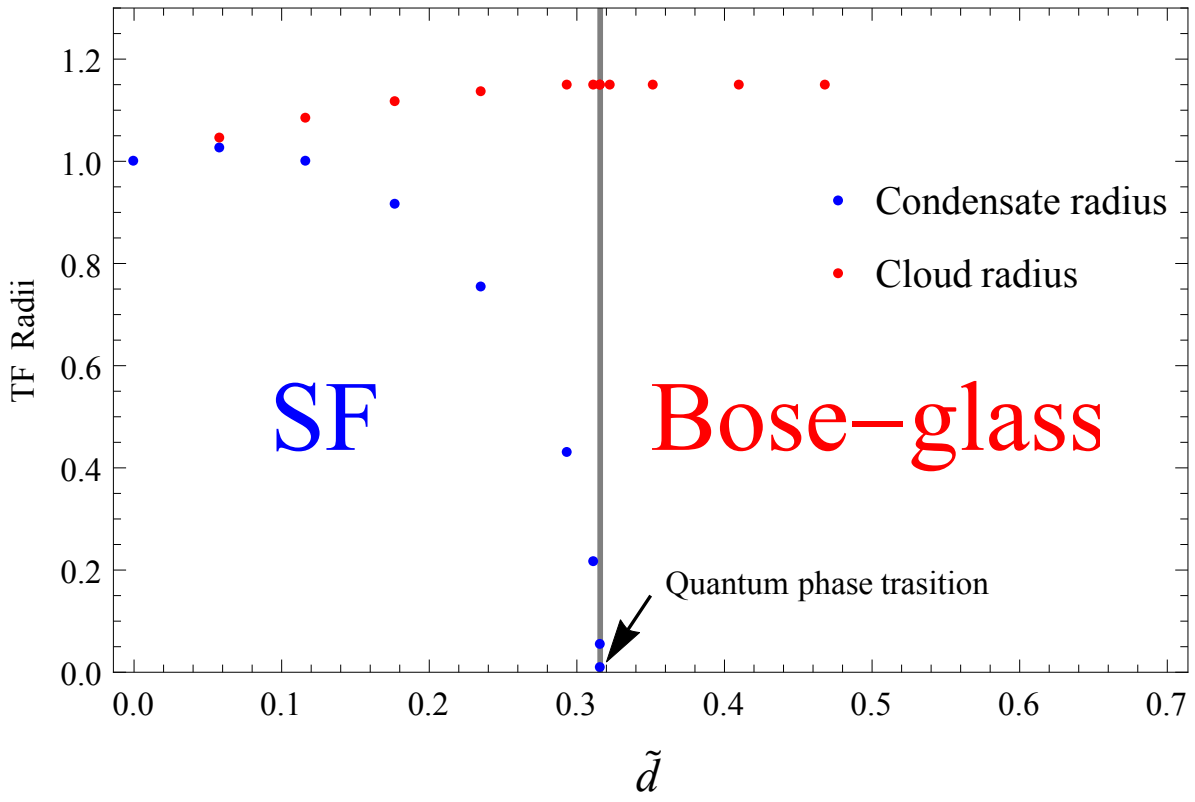


Model

- Gross-Pitaevskii equation for the ground state

$$\left[-\frac{\hbar^2\Delta}{2m} - \mu + U(x) + V(x) + \frac{g}{2}\psi^*(x)\psi(x)\right]\psi(x) = 0$$

- Assumptions
  - One dimension
  - Gaussian correlation function $R(x) = \frac{\varepsilon^2}{\sqrt{2\pi}\lambda}\exp\left\{-\frac{x^2}{2\lambda^2}\right\}$
- Condensate depletion
  - Particle density $n(x) = \overline{\psi(x)^2}$
  - Condensate density $n_0(x) = \overline{\psi(x)}^2$
  - Depletion $q(x) = n(x) - n_0(x)$

Method

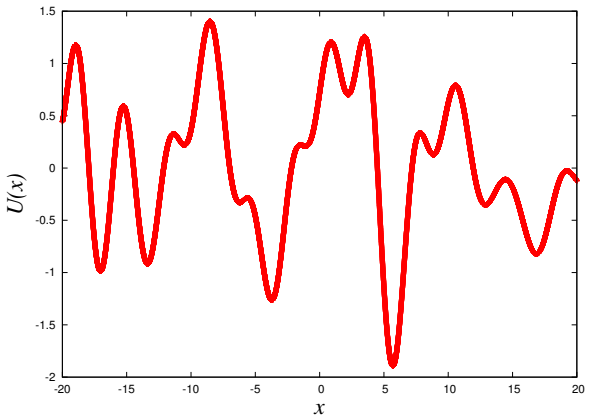
- Generating random potential

$$U(x) = \frac{1}{\sqrt{\mathcal{N}}} \sum_{n=0}^{\mathcal{N}-1} [A_n \cos(k_n x) + B_n \sin(k_n x)]$$

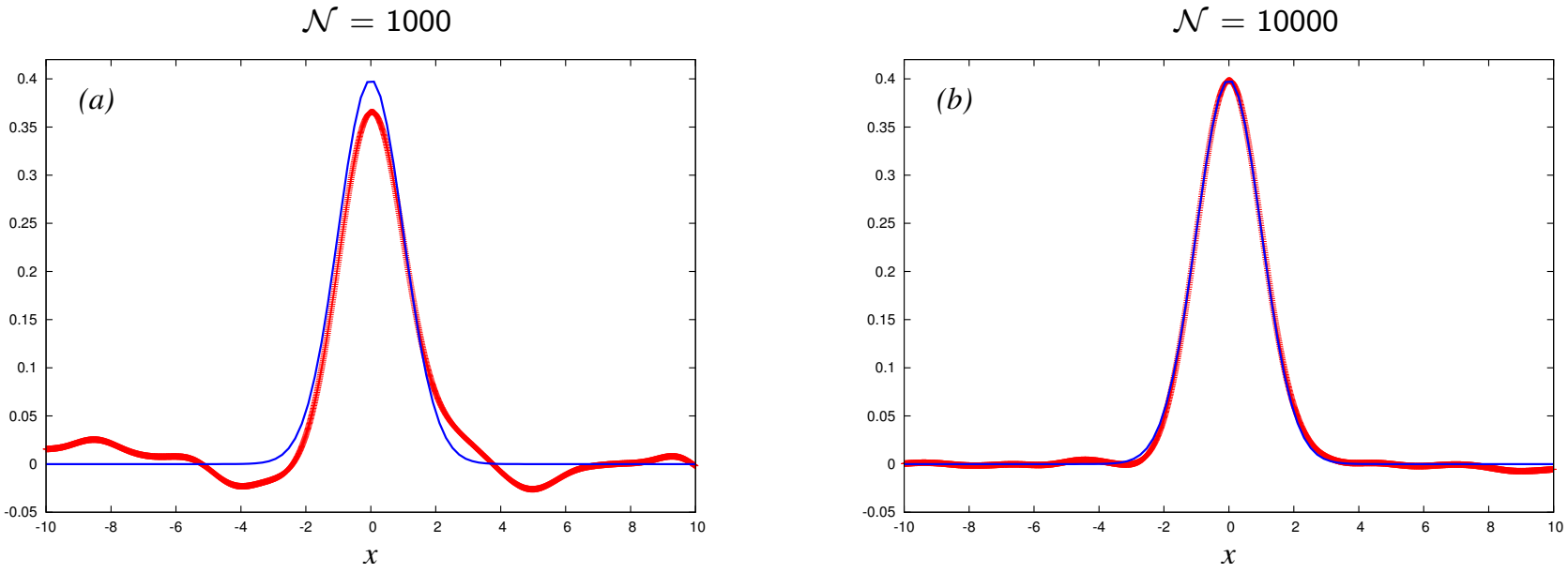
$$\overline{A_n B_n} = 0, \qquad \overline{A_n A_m} = \overline{B_n B_m} = R(0)\delta nm \quad \textit{and} \quad p(k_n) = \frac{\lambda}{\sqrt{2\pi}} \exp\left\{-\frac{\lambda^2 k_n^2}{2}\right\}$$

J. Majda and P. Kramer, Phys. Rep. **314,** 237 (1999)

- Sample potential: $\mathcal{N} = 100$, $\varepsilon = 1$ and $\lambda = 1$



- Correlation $\overline{U(x)U(0)}$ and $R(x)$: $\varepsilon = 1$ and $\lambda = 1$

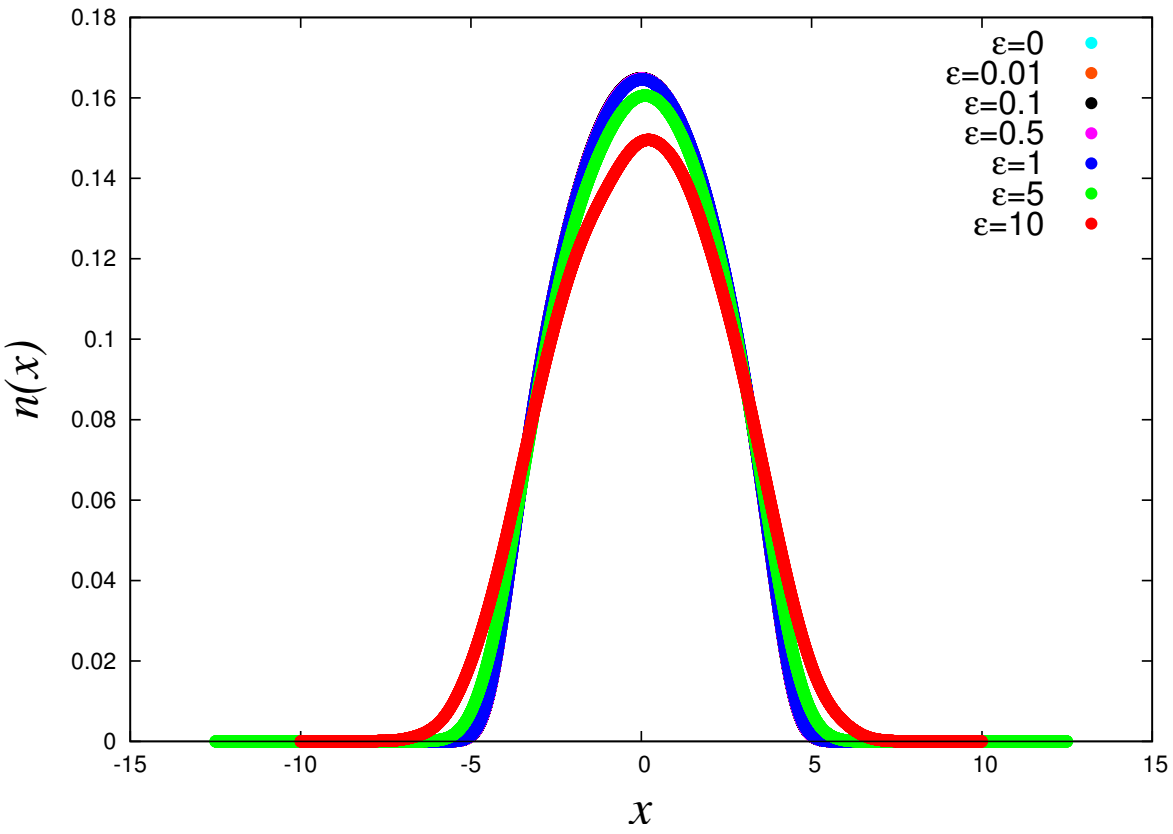


- C program for solving time-(in)dependent Gross-Pitaevskii equation in one space dimension

A. Balaž et al., Comput. Phys. Commun. **183**, 2021 (2012)

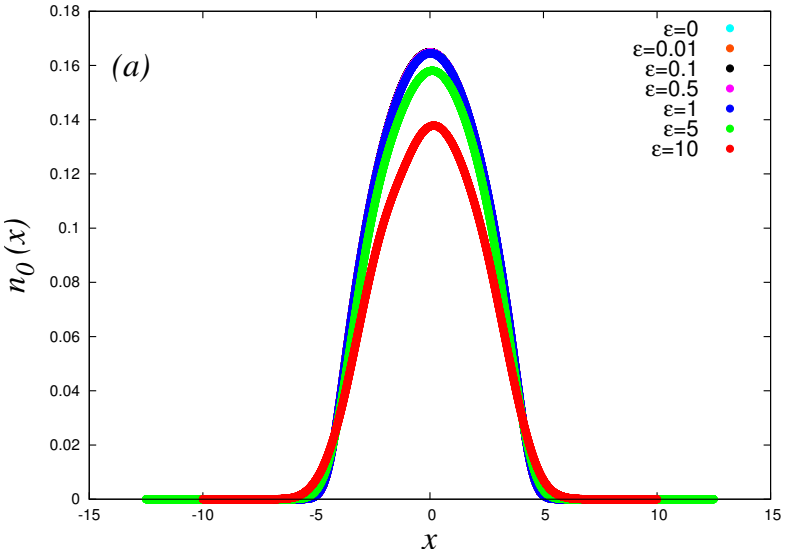
Results

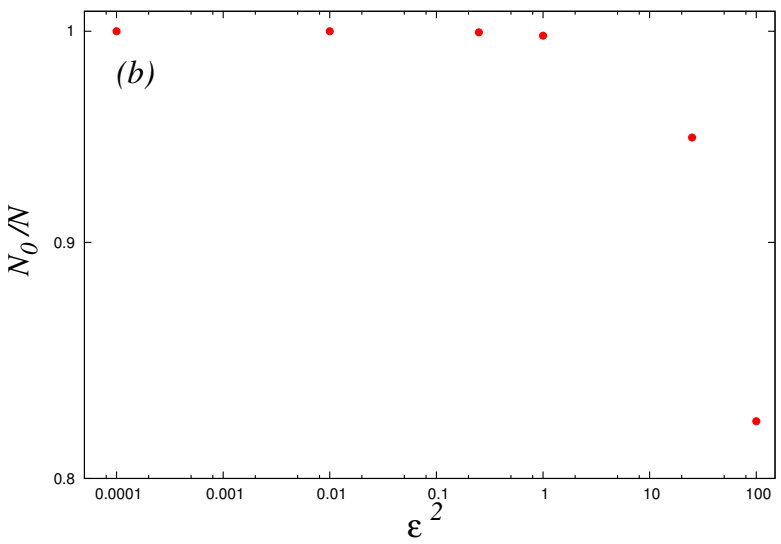
- Particle density: $\mathcal{N} = 10000$ and $\lambda = 1$



- Condensate density: $\mathcal{N} = 10000$ and $\lambda = 1$

Increasing disorder strength $\Rightarrow$ the global condensate density decreases

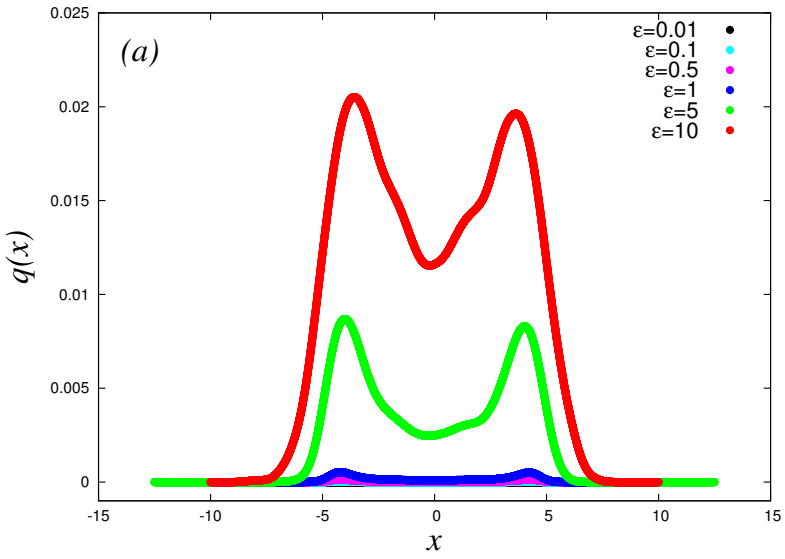


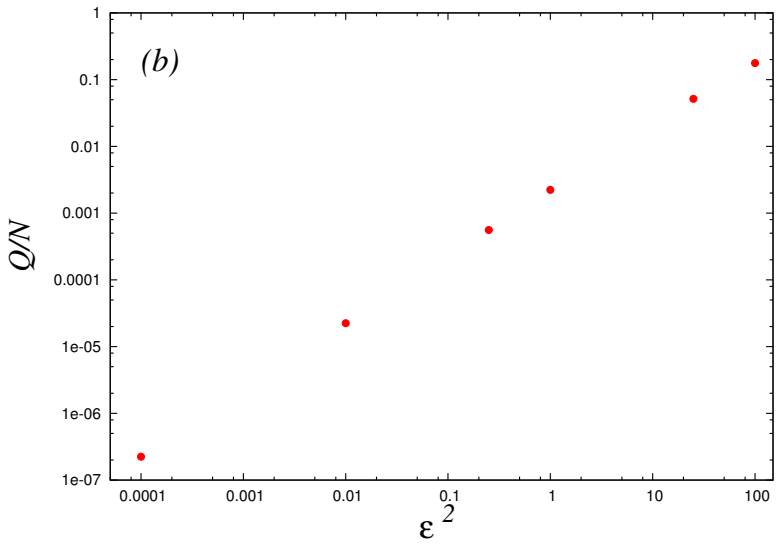


where $\begin{cases} N_0 = \int n_0(x)\,dx \\ N = \int n(x)\,dx \end{cases}$

- Condensate depletion: $\mathcal{N} = 10000$ and $\lambda = 1$

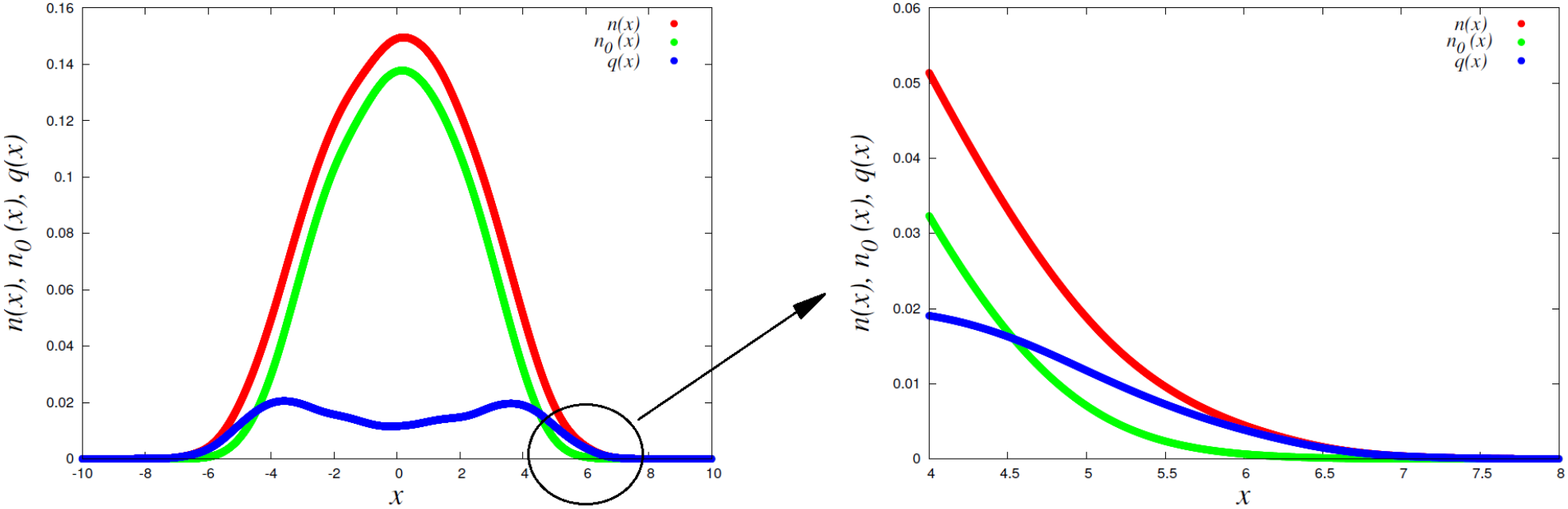
Increasing disorder strength$\Rightarrow$ depletion increases





where $Q = \int q(x)\, dx$

- Bose-glass phase: $\mathcal{N} = 10000$, $\lambda = 1$ and $\varepsilon = 10$
  Existence of a Bose-glass phase in the intermediate region



Outlook

- Comparison with Huang-Meng theory
- Perturbation method
- Beyond Thomas-Fermi approximation
- Anisotropic trap potential
- Extend the numerical study to 3 dimensions
- General interaction potential
- Finite temperature
- Time dependence of densities and Thomas-Fermi radii
- Replica Symmetry Breaking?

Thank You For Your Attention