

Kritische Temperatur ungeordneter Bosonen

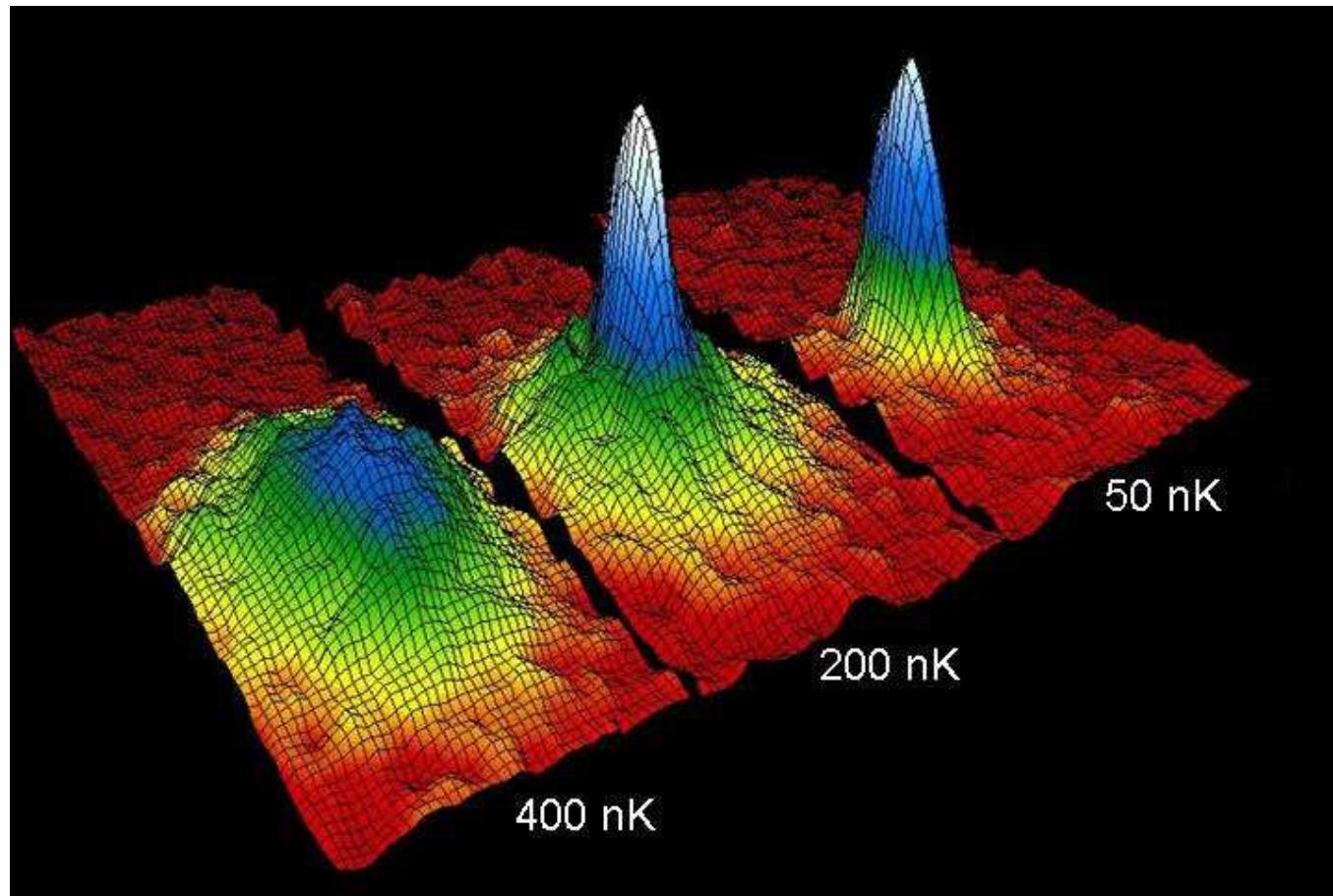
Ben Klünder



Übersicht

1. Ideales Bose-Gas in harmonischer Falle:
 - a. Großkanonische Beschreibung ohne Ordnungsparameter
 - b. Großkanonische Beschreibung mit Ordnungsparameter
2. Ungeordnetes Bose-Gas in harmonischer Falle
3. Ausblick

1.1. Bose-Einstein-Kondensation



JILA (1995), $N = 2 \cdot 10^4$, $\bar{\omega} \approx 2\pi \cdot 85$ Hz

1.2. Funktionalintegral-Formalismus

$$Z = \oint \mathcal{D}\psi^* \oint \mathcal{D}\psi e^{-\mathcal{A}[\psi^*, \psi]/\hbar}$$

$$\mathcal{A}[\psi^*, \psi] = \int_0^{\hbar\beta} d\tau \int d^Dx \psi^*(\mathbf{x}, \tau) \left[\hbar \frac{\partial}{\partial \tau} + \hat{H}(\mathbf{x}) - \mu \right] \psi(\mathbf{x}, \tau)$$

$$\hat{H}(\mathbf{x}) = -\frac{\hbar^2}{2M} \Delta + V(\mathbf{x}), \quad \hat{H}(\mathbf{x}) \psi_{\mathbf{n}}(\mathbf{x}) = E_{\mathbf{n}} \psi_{\mathbf{n}}(\mathbf{x})$$

Zerlegung: $\psi(\mathbf{x}, \tau) = \sum_{m=-\infty}^{\infty} \sum_{\mathbf{n}} c_{\mathbf{n}, m} \psi_{\mathbf{n}}(\mathbf{x}) e^{-i\omega_m \tau}, \quad \omega_m = \frac{2\pi}{\hbar\beta} m$

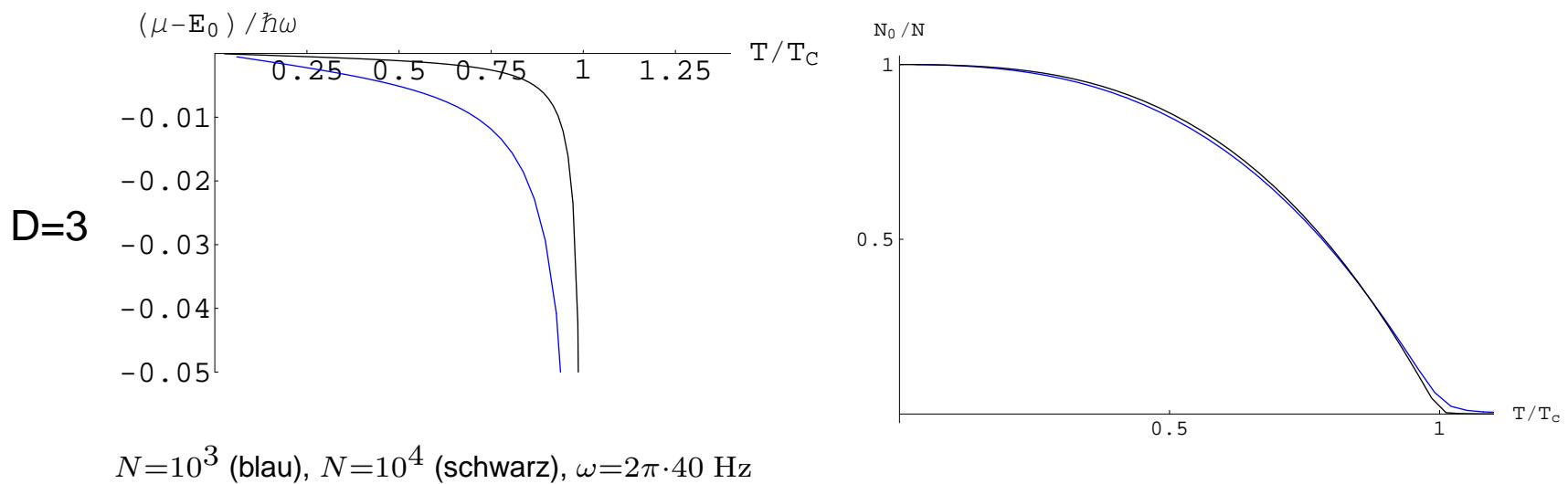
$$Z = \prod_{\mathbf{n}} \prod_{m=-\infty}^{\infty} \int dc_{\mathbf{n}, m}^* \int dc_{\mathbf{n}, m} e^{-\beta \sum_{\mathbf{n}} \sum_{m=-\infty}^{\infty} (-i\hbar\omega_m + E_{\mathbf{n}} - \mu) c_{\mathbf{n}, m}^* c_{\mathbf{n}, m}}$$

$$\Omega = -\frac{1}{\beta} \ln Z = \frac{1}{\beta} \sum_{\mathbf{n}} \ln \left[1 - e^{-\beta(E_{\mathbf{n}} - \mu)} \right]$$

1.3. Harmonische Falle $V(\mathbf{x}) = \frac{M}{2}\omega^2\mathbf{x}^2$

$$\Omega = - \sum_{k=1}^{\infty} \frac{e^{\beta(\mu-E_0)k}}{\beta k (1 - e^{-\hbar\omega\beta k})^D}, \quad E_0 = \frac{D}{2}\hbar\omega$$

$$N = -\frac{\partial \Omega}{\partial \mu} = \sum_{k=1}^{\infty} \frac{e^{\beta(\mu-E_0)k}}{(1 - e^{-\hbar\omega\beta k})^D} \quad N_0 = \frac{1}{e^{\beta(E_0-\mu)} - 1}, \quad \mu < E_0$$



1.4. Berechnung des effektiven Potentials

Ansatz: $\psi(\mathbf{x}, \tau) = \Psi(\mathbf{x}, \tau) + \delta\psi(\mathbf{x}, \tau)$, $\int d^Dx \Psi^*(\mathbf{x}, \tau)\delta\psi(\mathbf{x}, \tau) = 0$

$$Z = e^{-\mathcal{A}[\Psi^*, \Psi]/\hbar} \oint \mathcal{D}\delta\psi^* \oint \mathcal{D}\delta\psi e^{-\mathcal{A}[\delta\psi^*, \delta\psi]/\hbar}$$

Zerlegung: $\delta\psi(\mathbf{x}, \tau) = \sum_{m=-\infty}^{\infty} \sum_{\mathbf{n} \neq 0} c_{\mathbf{n}, m} \psi_{\mathbf{n}}(\mathbf{x}) e^{-i\omega_m \tau}$

$$\oint \mathcal{D}\delta\psi^* \oint \mathcal{D}\delta\psi = \prod_{\mathbf{n} \neq 0} \prod_{m=-\infty}^{\infty} \int dc_{\mathbf{n}, m}^* \int dc_{\mathbf{n}, m}$$

$$\begin{aligned} \Gamma[\Psi^*, \Psi] &= - \sum_{k=1}^{\infty} \frac{e^{\beta(\mu - E_0)k}}{\beta k} \left[\frac{1}{(1 - e^{-\hbar\omega\beta})^D} - 1 \right] \\ &\quad + \frac{1}{\hbar\beta} \int_0^{\hbar\beta} d\tau \int d^Dx \Psi^*(\mathbf{x}, \tau) \left[\hbar \frac{\partial}{\partial \tau} + \hat{H}(\mathbf{x}) - \mu \right] \Psi(\mathbf{x}, \tau) \end{aligned}$$

1.5. Extremalisierung

$$\frac{\delta \Gamma[\Psi_e^*, \Psi_e]}{\delta \Psi_e^*(\mathbf{x}, \tau)} = 0$$

$$\left[\hbar \frac{\partial}{\partial \tau} + \hat{H}(\mathbf{x}) - \mu \right] \Psi_e(\mathbf{x}, \tau) = 0$$

Lösung: $\Psi_e(\mathbf{x}, \tau) = \sqrt{N_0} \psi_0(\mathbf{x})$

$$(E_0 - \mu) N_0 = 0 \Rightarrow \begin{cases} N_0 = 0, & \text{Gas-Phase} \\ \mu = E_0, & \text{BEC-Phase} \end{cases}$$

$$\Omega = - \sum_{k=1}^{\infty} \frac{e^{\beta(\mu-E_0)k}}{\beta k} \left[\frac{1}{(1-e^{-\hbar\omega\beta k})^D} - 1 \right] + (E_0 - \mu) N_0$$

$$N = \sum_{k=1}^{\infty} e^{\beta(\mu-E_0)k} \left[\frac{1}{(1-e^{-\hbar\omega\beta k})^D} - 1 \right] + N_0$$

1.6. Semiklassische Näherung

$$I(A, b, D) = \sum_{k=1}^{\infty} \frac{e^{-Abk}}{(1 - e^{-bk})^D}, \quad 0 < b = \hbar\omega\beta \ll 1, \quad A > 0$$

$$\sum_{k=1}^{\infty} \xrightarrow{?} \int_0^{\infty} dk$$

$$I(A, b, D) = \sum_{k=1}^{\infty} e^{-Abk} \left[\frac{1}{(1-e^{-bk})^D} - \sum_{l=0}^{D-1} \frac{C_l(D)}{(bk)^{D-l}} \right] + \sum_{k=1}^{\infty} \sum_{l=0}^{D-1} \frac{C_l(D)e^{-Abk}}{(bk)^{D-l}}$$

Euler-MacLaurin-Formel:

$$\begin{aligned} I(A, b, D) &= \int_0^{\infty} dk e^{-Abk} \left[\frac{1}{(1-e^{-bk})^D} - \sum_{l=0}^{D-1} \frac{C_l(D)}{(bk)^{D-l}} \right] \\ &\quad + \sum_{k=1}^{\infty} \sum_{l=0}^{D-1} \frac{C_l(D)e^{-Abk}}{(bk)^{D-l}} + \mathcal{O}(b^0) \end{aligned}$$

Dimensionale Regularisierung: $D=d-\epsilon$, $d=1, 2, 3, 4 \dots$, $\epsilon > 0$

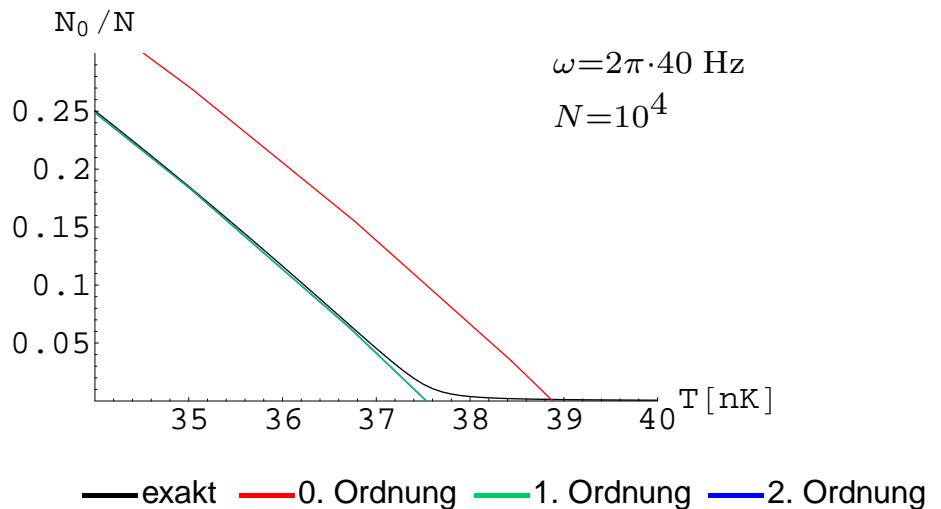
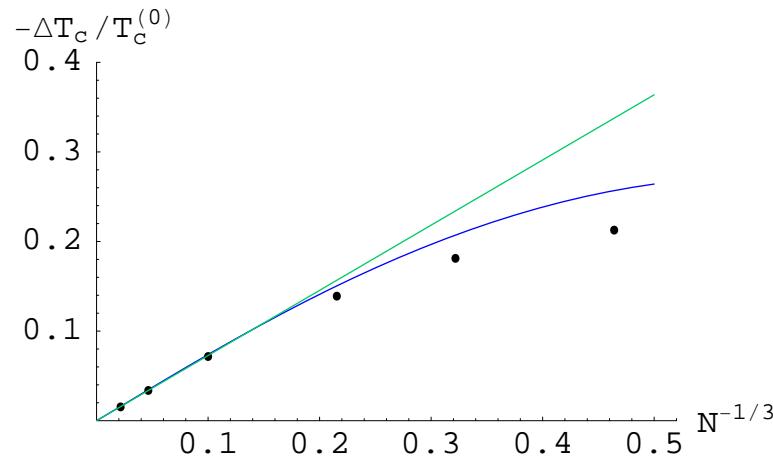
1.7. Kritische Temperatur

Thermodynamischer Limes:
 \equiv Semiklassischer Limes $\hbar\omega\beta_c^{(0)} \ll 1$

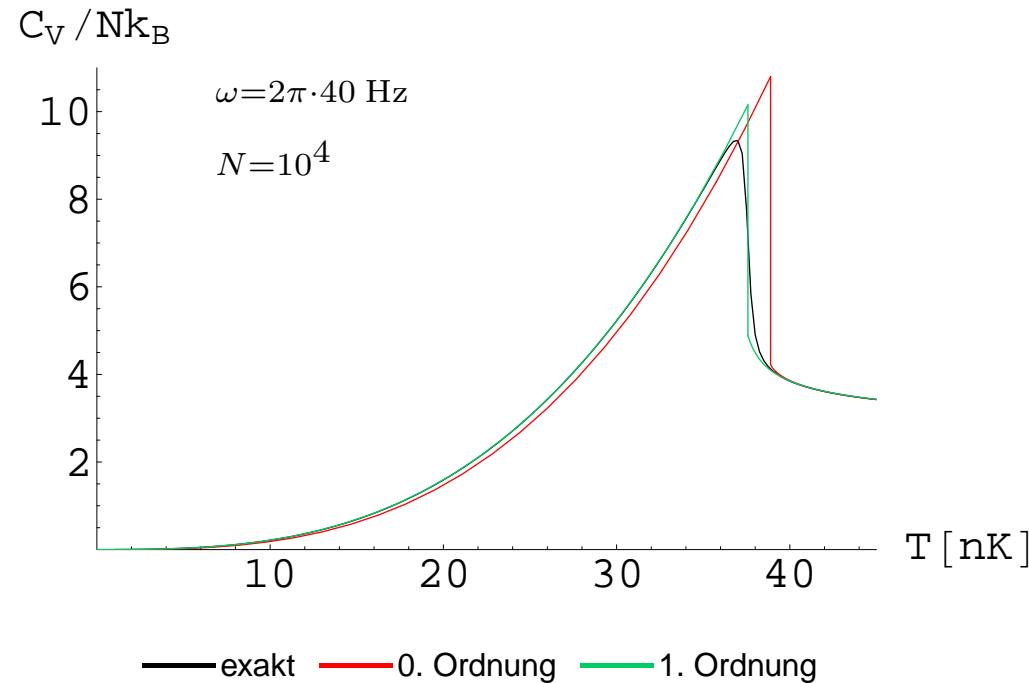
$$T_c^{(0)} = \frac{\hbar\omega}{k_B} \left(\frac{N}{\zeta(3)} \right)^{\frac{1}{3}}$$

Finite-Size-Korrekturen:

$$\begin{aligned} \frac{\Delta T_c}{T_c^{(0)}} &= - \left(\frac{\zeta(3)}{N} \right)^{\frac{1}{3}} \frac{\zeta(2)}{2\zeta(3)} \\ &\quad - \left(\frac{\zeta(3)}{N} \right)^{\frac{2}{3}} \frac{1}{3\zeta(3)} \left[-\frac{1}{3} \ln \left(\frac{\zeta(3)}{N} \right) + \gamma - \frac{19}{24} - \frac{3\zeta(2)^2}{4\zeta(3)} \right] + \dots \end{aligned}$$



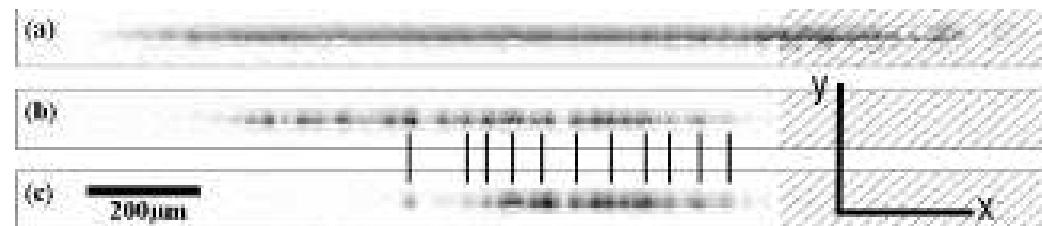
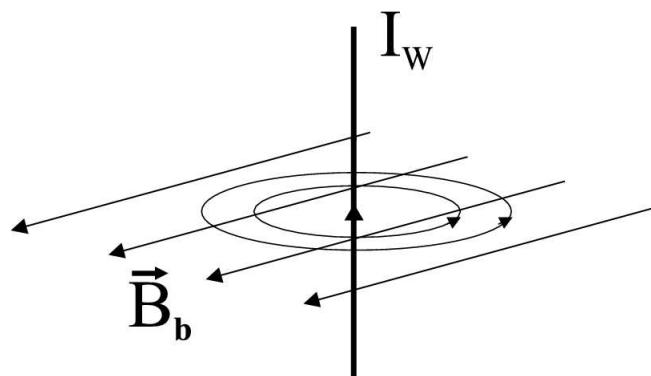
1.8. Wärmekapazität



$$\begin{aligned} & \lim_{T \uparrow T_c} C_V(T) - \lim_{T \downarrow T_c} C_V(T) \\ &= 3Nk_B \left\{ \frac{3\zeta(3)}{\zeta(2)} - \left(\frac{\zeta(3)}{N} \right)^{\frac{1}{3}} \frac{3\zeta(3)}{\zeta(2)^2} \left[-\frac{1}{2} \ln \left(\frac{\zeta(3)}{N} \right) + \frac{5}{4} + \zeta(2) + \frac{3}{2}\gamma - \frac{\zeta(2)^2}{2\zeta(3)} \right] \right\} + \dots \end{aligned}$$

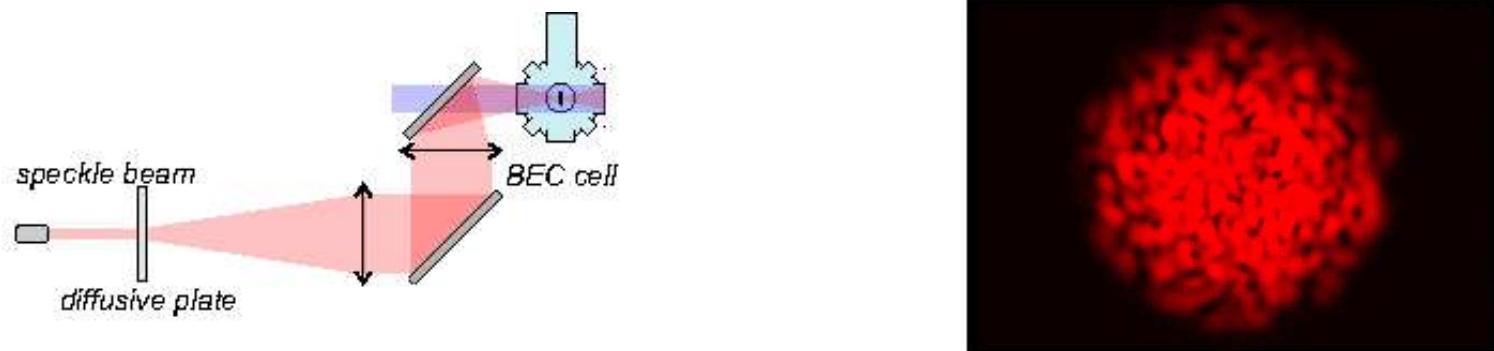
2.1. Natürliche Unordnung

Atomchips

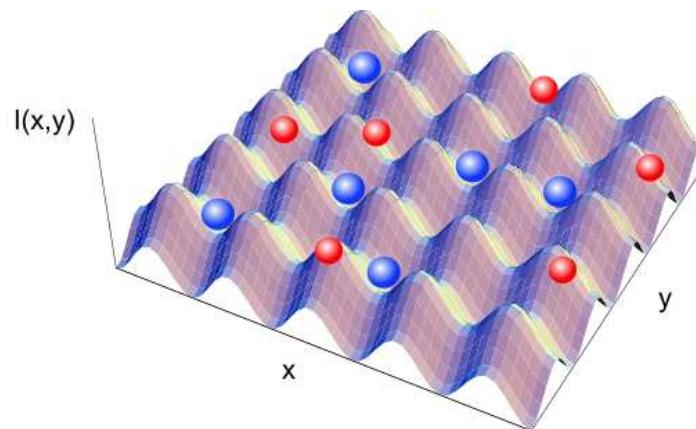


2.2. Künstliche Unordnung

Laser-Speckles



Verschiedene Atomarten

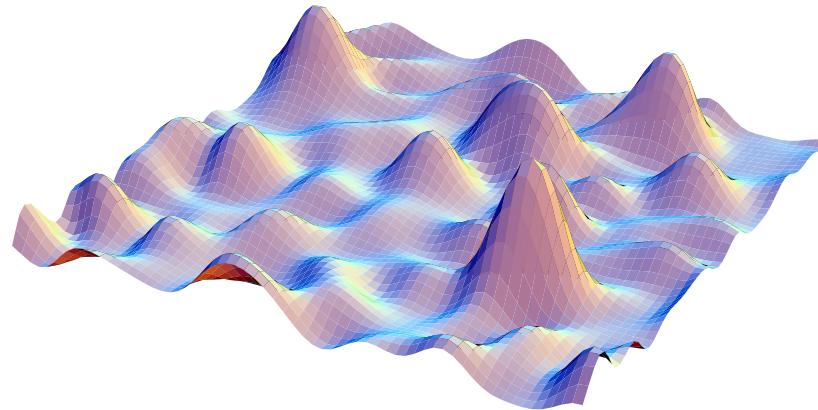


2.3. Zufallspotential

$$\overline{\bullet} = \prod_{\mathbf{x}} \int_{-\infty}^{\infty} dU(\mathbf{x}) \bullet P[U], \quad \overline{1} = 1$$

$$\overline{U(\mathbf{x})} = 0$$

$$\overline{U(\mathbf{x}_1)U(\mathbf{x}_2)} = \frac{R}{(2\pi\epsilon^2)^{D/2}} e^{-(\mathbf{x}_1 - \mathbf{x}_2)^2 / 2\epsilon^2}$$



2.4. Effektives Potential für schwache Unordnung

$$Z = \oint \mathcal{D}\psi^* \oint \mathcal{D}\psi e^{-\mathcal{A}[\psi^*, \psi]/\hbar}$$

$$\begin{aligned} \mathcal{A}[\psi^*, \psi] &= \hbar \int_0^{\hbar\beta} d\tau_{1,2} \int d^3x_{1,2} \psi^*(\mathbf{x}_1, \tau_1) G^{-1}(\mathbf{x}_1, \tau_1; \mathbf{x}_2, \tau_2) \psi(\mathbf{x}_2, \tau_2) \\ &\quad + \int_0^{\hbar\beta} d\tau_1 \int d^3x_1 U(\mathbf{x}_1) \psi^*(\mathbf{x}_1, \tau_1) \psi(\mathbf{x}_1, \tau_1) \end{aligned}$$

$$G^{-1}(\mathbf{x}_1, \tau_1; \mathbf{x}_2, \tau_2) = \frac{1}{\hbar} \delta(\mathbf{x}_1 - \mathbf{x}_2) \delta(\tau_1 - \tau_2) \left[\hbar \frac{\partial}{\partial \tau_2} - \frac{\hbar^2}{2M} \Delta_2 + V(\mathbf{x}_2) - \mu \right]$$

$$\psi_1 = \Psi_1 + \delta\psi_1 \quad \Rightarrow \quad Z = e^{-\mathcal{A}[\Psi^*, \Psi]/\hbar} \oint \mathcal{D}\delta\psi^* \oint \mathcal{D}\delta\psi e^{-\int_{12} \delta\psi_1^* G_{12}^{-1} \delta\psi_2 - \int_1 U_1 \delta\psi_1^* \delta\psi_1 / \hbar}$$

2.5. Störungstheorie

$$Z_0 = \oint \mathcal{D}\delta\psi^* \oint \mathcal{D}\delta\psi e^{-\int_{12} \delta\psi_1^* G_{12}^{-1} \delta\psi_2} = e^{-\text{Tr}\ln G^{-1}}$$

$$G_{12} = \frac{1}{Z_0} \oint \mathcal{D}\delta\psi^* \oint \mathcal{D}\delta\psi \delta\psi_1 \delta\psi_2^* e^{-\int_{12} \delta\psi_1^* G_{12}^{-1} \delta\psi_2}$$

$$\begin{aligned} Z &= \oint \mathcal{D}\delta\psi^* \oint \mathcal{D}\delta\psi \left(1 - \frac{1}{\hbar} \int_1 U_1 \delta\psi_1^* \delta\psi_1 + \frac{1}{2\hbar^2} \int_{12} U_1 U_2 \delta\psi_1^* \delta\psi_1 \delta\psi_2^* \delta\psi_2 + \dots \right) \\ &\quad \times e^{-\int_{12} \delta\psi_1^* G_{12}^{-1} \delta\psi_2} e^{-\mathcal{A}[\Psi^*, \Psi]/\hbar} \end{aligned}$$

$$\begin{aligned} \Gamma[\Psi^*, \Psi] &= -\frac{1}{\beta} \ln Z = \frac{1}{\beta} \left(\text{Tr}\ln G^{-1} + \int_{12} \Psi_1^* G_{12}^{-1} \Psi_2 \right. \\ &\quad \left. + \frac{1}{\hbar} \int_1 U_1 \Psi_1^* \Psi_1 + \frac{1}{\hbar} \int_1 U_1 G_{11} - \frac{1}{2\hbar^2} \int_{12} U_1 U_2 G_{12} G_{21} + \dots \right) \end{aligned}$$

2.6. Kritisches chemisches Potential

$$\frac{\delta \Gamma[\Psi_e^*, \Psi_e]}{\delta \Psi_e^*(\mathbf{x}, \tau)} = \left[\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2M} \Delta + V(\mathbf{x}) + U(\mathbf{x}) - \mu_c \right] \Psi_e(\mathbf{x}, \tau) = 0$$

$$\mu_c = E_0 + \mu_c^{(1)} + \mu_c^{(2)} + \dots, \quad \Psi_e(\mathbf{x}, \tau) = \sqrt{N_0} \psi_0(\mathbf{x}) + \Psi_e^{(1)}(\mathbf{x}) + \Psi_e^{(2)}(\mathbf{x}) + \dots$$

$$[\hat{H}(\mathbf{x}) - E_0] \Psi_e^{(1)}(\mathbf{x}) = [\mu_c^{(1)} - U(\mathbf{x})] \sqrt{N_0} \psi_0(\mathbf{x}) \Rightarrow \Psi_e^{(1)}(\mathbf{x}) = \dots$$

$$[\hat{H}(\mathbf{x}) - E_0] \Psi_e^{(2)}(\mathbf{x}) = [\mu_c^{(1)} - U(\mathbf{x})] \Psi_e^{(1)}(\mathbf{x}) + \mu_c^{(2)} \sqrt{N_0} \psi_0(\mathbf{x})$$

$$\mu_c^{(1)} = \int d^3x \, U(\mathbf{x}) \psi_0^*(\mathbf{x}) \psi_0(\mathbf{x})$$

$$\mu_c^{(2)} = \frac{1}{\sqrt{N_0}} \int d^3x \, [U(\mathbf{x}) - \mu_c^{(1)}] \psi_0^*(\mathbf{x}) \Psi_e^{(1)}(\mathbf{x})$$

2.7. Kritische Temperatur

Teilchenzahlgleichung: $N = N(N_0, T, \mu)$

Kritische Temperatur: $N = N(0, T_c, \mu_c)$

$$\Rightarrow T_c[U] = T_c^{(0)} + T_c^{(1)}[U] + T_c^{(2)}[U] + \dots$$

Verschiebung der kritischen Temperatur:

$$\frac{\Delta T_c[U]}{T_c^{(0)}} = \frac{T_c[U] - T_c^{(0)}}{T_c^{(0)}} = \frac{T_c^{(1)}[U] + T_c^{(2)}[U] + \dots}{T_c^{(0)}}$$

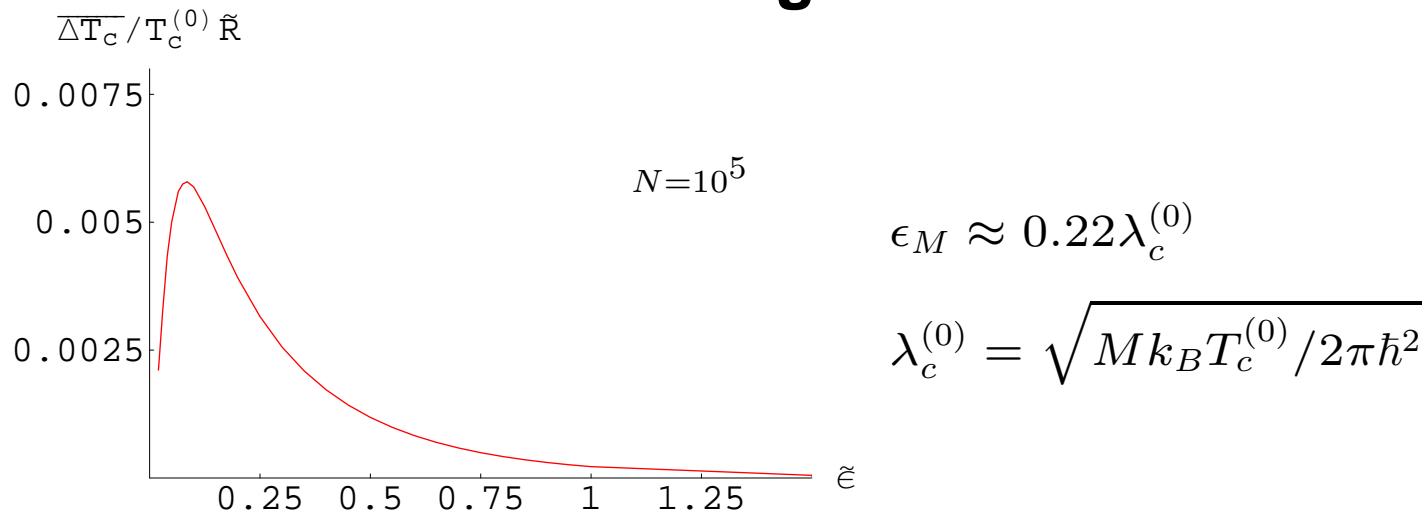
Unordnungsmittel:

Mittlere Verschiebung: $\overline{\frac{\Delta T_c[U]}{T_c^{(0)}}} = \overline{\frac{T_c^{(2)}[U]}{T_c^{(0)}}} + \dots$

Varianz: $\overline{\frac{\Delta T_c[U]^2}{T_c^{(0)2}}} = \overline{\frac{T_c^{(1)}[U]^2}{T_c^{(0)2}}} + \dots$

Auswertung in semiklassischer Näherung (Kapitel 1)

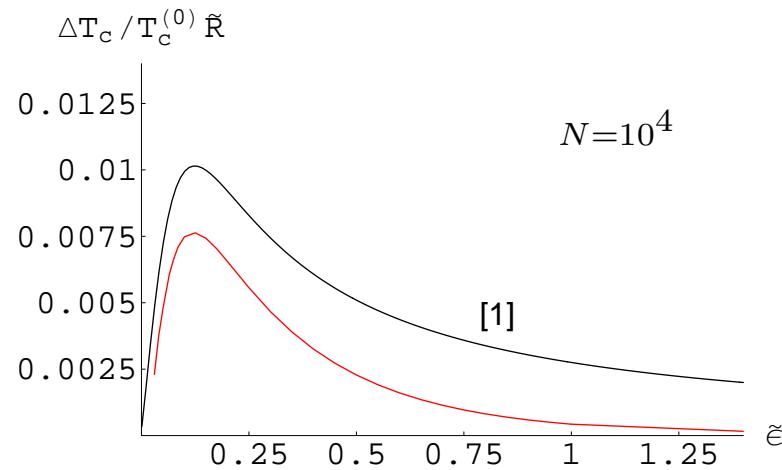
2.8. Ergebnisse I



$$\begin{aligned} \frac{\overline{\Delta T_c}}{T_c^{(0)}} &= \frac{2\tilde{R}}{3N^{1/3}\zeta(3)^{2/3}(2\pi)^{3/2}} \left\{ - \sum_{n=1}^{\infty} \frac{1}{n[n + 4\tilde{\epsilon}^2 N^{1/3}/\zeta(3)]\tilde{\epsilon}} \right. \\ &\quad + \sum_{l=1}^{\infty} \frac{(2l+1)\Gamma(l/2)}{\Gamma(l+3/2)} \int_0^{\infty} dr'_2 \int_{r'_2}^{\infty} dr'_1 \frac{2^{3/2}\zeta(2)}{\tilde{\epsilon}^2} e^{-(r'_1{}^2 + r'_2{}^2)(1/2 + 1/2\tilde{\epsilon}^2)} \\ &\quad \times I_{l+1/2}(r'_1 r'_2 / \tilde{\epsilon}^2) M_{3/4, l/2+1/4}(r'_2{}^2) W_{3/4, l/2+1/4}(r'_1{}^2) \Big\} + \dots \end{aligned}$$

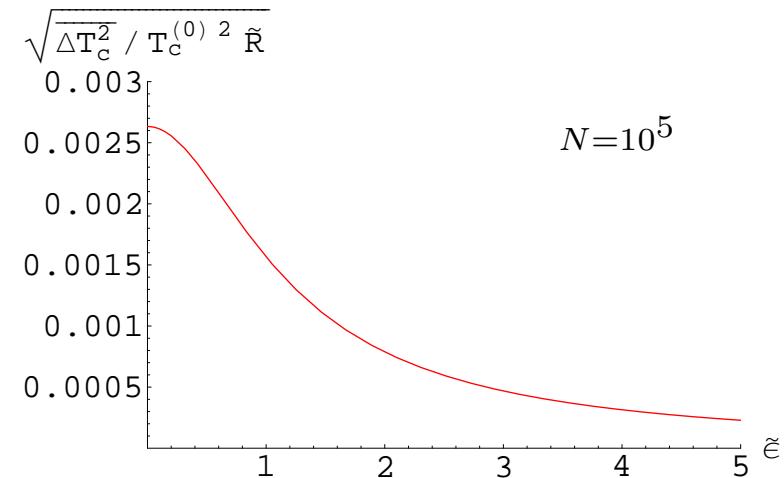
2.9. Ergebnisse II

Vergleich mit Literatur



[1] Timmer, Pelster, Graham, Europhys. Lett. **76**, 760 (2006)

Standardabweichung



$$\sqrt{\frac{\overline{\Delta T_c^2}}{T_c^{(0)2}}} = \frac{\zeta(2)\sqrt{\tilde{R}}}{3\zeta(3)^{2/3}N^{1/3}(2\pi)^{3/4}(1+\tilde{\epsilon}^2)^{3/4}} + \dots$$

3. Ausblick

- Ideales Bose-Gas in harmonischer Falle:
Finite-Size-Korrekturen thermodynamischer Größen
⇒ Erweiterung auf $D=1, 2$
⇒ Verallgemeinerung auf Teilchendichte
- Ungeordnetes Bose-Gas in harmonischer Falle:
Mittlere Verschiebung und Standardabweichung der kritischen Temperatur

⇒ Superfluide Dichte

Lopatin, Vinokur, Phys. Rev. Lett. **88**, 235503
(2002)

⇒ Kollektive Schwingungen

Falco, Pelster, Graham, Phys. Rev. A **76**,
013624 (2007)

