

Density Distribution for Ideal Trapped Bose Gases

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1. Density Matrix in Canonical Ensemble

2. Density Matrix in Grand-Canonical Ensemble

3. Critical Temperature

1. Density Matrix in Canonical Ensemble

N-Particle Propagator:

$$(\mathbf{x}_1, \dots, \mathbf{x}_N; \tau_b \mid \mathbf{x}'_1, \dots, \mathbf{x}'_N; \tau_a) = \prod_{n=1}^N \left(\int_{\mathbf{x}_n(\tau_a)=\mathbf{x}'_n}^{\mathbf{x}_n(\tau_b)=\mathbf{x}_n} \mathcal{D}^3 x_n(\tau) \right) \exp \left\{ -\frac{1}{\hbar} \mathcal{A}[\mathbf{x}_1, \dots, \mathbf{x}_N] \right\}$$

Action:

$$\mathcal{A}[\mathbf{x}_1, \dots, \mathbf{x}_N] = \sum_{n=1}^N \int_{\tau_a}^{\tau_b} d\tau' \left[\frac{M}{2} \dot{\mathbf{x}}_n^2(\tau') + \frac{M}{2} \omega^2 \mathbf{x}_n^2(\tau') \right]$$

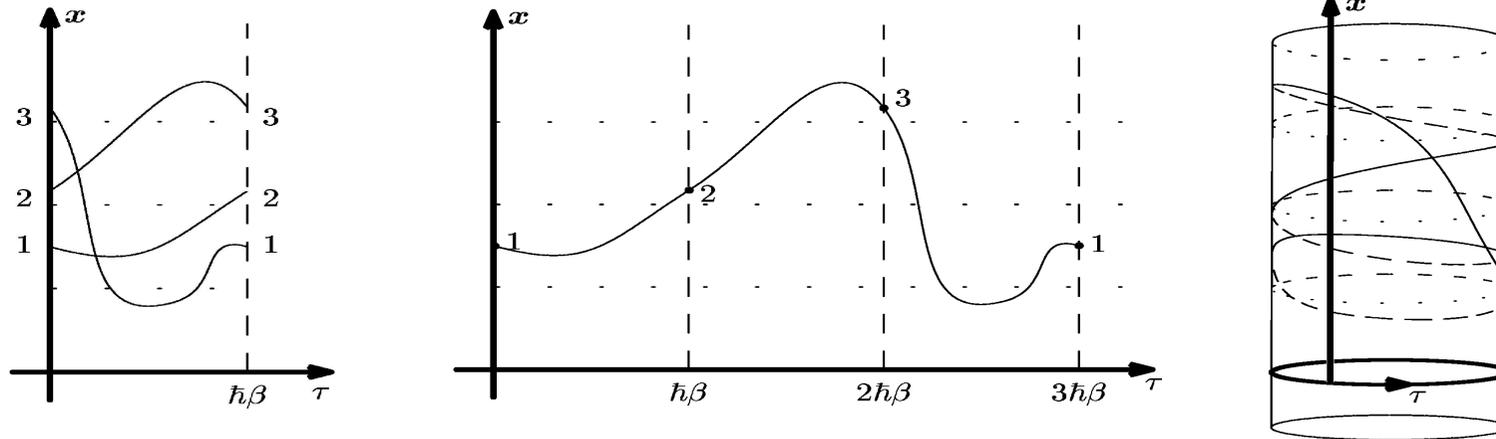
Partition Function:

$$Z_N^B(\beta) = \int d^3 x_1 \cdots d^3 x_N (\mathbf{x}_1, \dots, \mathbf{x}_N; \hbar\beta \mid \mathbf{x}_1, \dots, \mathbf{x}_N; 0)^B$$

Indistinguishability:

$$(\mathbf{x}_1, \dots, \mathbf{x}_N; \hbar\beta \mid \mathbf{x}_1, \dots, \mathbf{x}_N; 0)^B = \frac{1}{N!} \sum_P (\mathbf{x}_{P(1)}; \hbar\beta \mid \mathbf{x}_1; 0) \cdots (\mathbf{x}_{P(N)}; \hbar\beta \mid \mathbf{x}_N; 0)$$

Possible Permutation for 3 Particles:



Closed Cycle:

$$\begin{aligned}
 h_m(\beta) &= \int d^3 x_1 \cdots d^3 x_m (\mathbf{x}_1; \hbar\beta | \mathbf{x}_m; 0) (\mathbf{x}_m; \hbar\beta | \mathbf{x}_{m-1}; 0) (\mathbf{x}_2; \hbar\beta | \mathbf{x}_1; 0) \\
 &= Z_1(m\beta) = \sum_{\mathbf{k}} e^{-m\beta E_{\mathbf{k}}}
 \end{aligned}$$

Two-Point-Function:

$$\begin{aligned}
 h_m(\mathbf{x}_1, \mathbf{x}'_1, \beta) &= \int d^3 x_2 \cdots d^3 x_m (\mathbf{x}_1; \hbar\beta | \mathbf{x}_m; 0) (\mathbf{x}_m; \hbar\beta | \mathbf{x}_{m-1}; 0) \cdots (\mathbf{x}_2; \hbar\beta | \mathbf{x}'_1; 0) \\
 &= (\mathbf{x}_1; m\hbar\beta | \mathbf{x}'_1; 0)
 \end{aligned}$$

N-Body Density Matrix:

$$\rho_N^B(\mathbf{x}_1, \mathbf{x}'_1, \beta) = \frac{1}{Z_N^B(\beta)} \int d^3x_2 \cdots d^3x_N \frac{1}{N!} \sum_P (\mathbf{x}_{P(1)}; \hbar\beta | \mathbf{x}_1; 0) \cdots (\mathbf{x}_{P(N)}; \hbar\beta | \mathbf{x}_N; 0)$$

Relation for Density Matrix:

$$\rho_N^B(\mathbf{x}_1, \mathbf{x}'_1, \beta) = \frac{1}{N Z_N(\beta)} \sum_{m=1}^N h_m(\mathbf{x}_1, \mathbf{x}'_1, \beta) Z_{N-m}^B(\beta)$$

One-Particle Propagator:

$$(\mathbf{x}; m\hbar\beta | \mathbf{x}; 0) = \left[\frac{M\omega}{2\pi\hbar \sinh(m\hbar\beta\omega)} \right]^{3/2} \exp \left\{ -\frac{M\omega}{2\hbar \sinh(m\hbar\beta\omega)} \left[(\mathbf{x}^2 + \mathbf{x}'^2) \cosh(m\hbar\beta\omega) - 2\mathbf{x}\mathbf{x}' \right] \right\}$$

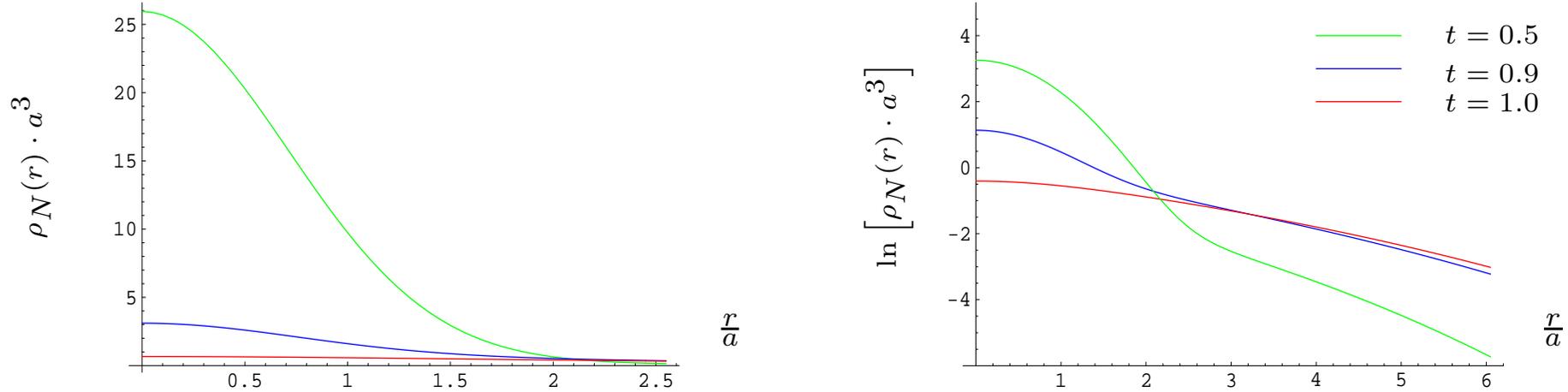
Normalization Condition:

$$\int d^3x_1 \rho_N(\mathbf{x}_1, \mathbf{x}_1, \beta) = 1$$

Recursion Relation for Partition Function:

$$Z_N^B(\beta) = \frac{1}{N} \sum_{m=1}^N Z_1(m\beta) Z_{N-m}^B(\beta), \quad Z_0(\beta) = 1$$

Density Distribution in Isotropic Harmonic Trap for $N = 1000$:



$$t = \frac{T}{T_C^0}, \quad T_C^0 = \frac{\hbar\omega}{k_B} \left[\frac{N}{\zeta(3)} \right]^{1/3}, \quad \rho_N(r) = \rho_N(\mathbf{x}, \mathbf{x}), \quad r = |\mathbf{x}|, \quad a = \sqrt{\frac{\hbar}{M\omega}}$$

2. Density Matrix in Grand-Canonical Ensemble

$$\rho_{\text{GK}}(\mathbf{x}) = \rho_0(\mathbf{x}) + \rho_T(\mathbf{x}) = N_0 |\Psi_0(\mathbf{x})|^2 + \sum_{\mathbf{k}} e^{-\beta E_{\mathbf{k}}} |\Psi_{\mathbf{k}}(\mathbf{x})|^2 e^{\beta \mu}$$

Thermal Density:

$$\rho_T(\mathbf{x}) = \frac{1}{\lambda^3} \left\{ \tilde{\zeta}_{3/2}(e^{\beta[\tilde{\mu} - V(\mathbf{x})]}) + \frac{3\hbar\beta\omega}{2} \tilde{\zeta}_{1/2}(e^{\beta[\tilde{\mu} - V(\mathbf{x})]}) + \dots \right\}$$

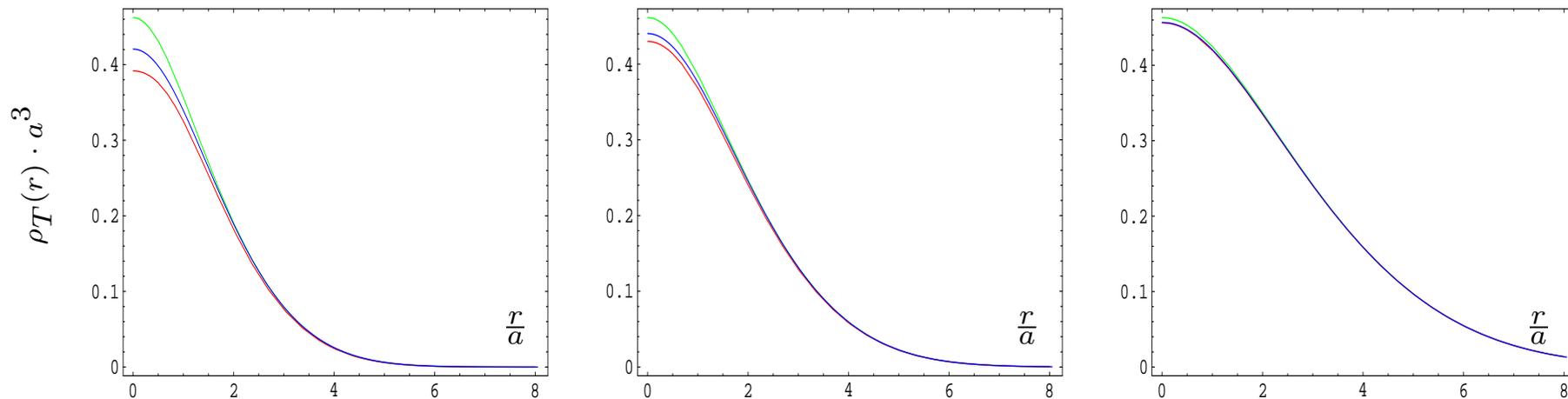
$$\tilde{\mu} = \mu - \frac{3\hbar\omega}{2}, \quad \lambda = \sqrt{\frac{2\pi\hbar^2}{Mk_B T}}$$

Density Distribution for Isotropic Harmonic Trap at $\frac{T}{T_C} = 1.1$:

$N = 30$

$N = 100$

$N = 1000$

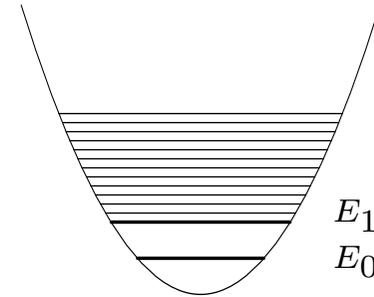


Grand-Canonical, with Finite-Size Corrections, Canonical

Treatment of Polylogarithmic Functions:

$$\tilde{\zeta}_n(\tilde{z}) = \frac{1}{\Gamma(n)} \int_{\beta(E_1 - E_0)}^{\infty} dx \frac{x^{n-1}}{\tilde{z}^{-1} e^x - 1}$$

$$\tilde{z} = e^{\beta\tilde{\mu}}$$



Thermal Partical Number:

$$N_T = \int d^3x \rho_T^B(\mathbf{x}, \mathbf{x}, \beta)$$

$$= \frac{1}{(\beta\hbar\omega)^3} \left\{ \tilde{\zeta}_3(\tilde{z}) + \frac{3\hbar\beta\omega}{2} \tilde{\zeta}_2(\tilde{z}) + \frac{(\hbar\beta\omega)^2}{8} \tilde{\zeta}_1(\tilde{z}) + \dots \right\}$$

Treatment of $\tilde{\zeta}_1(z)$:

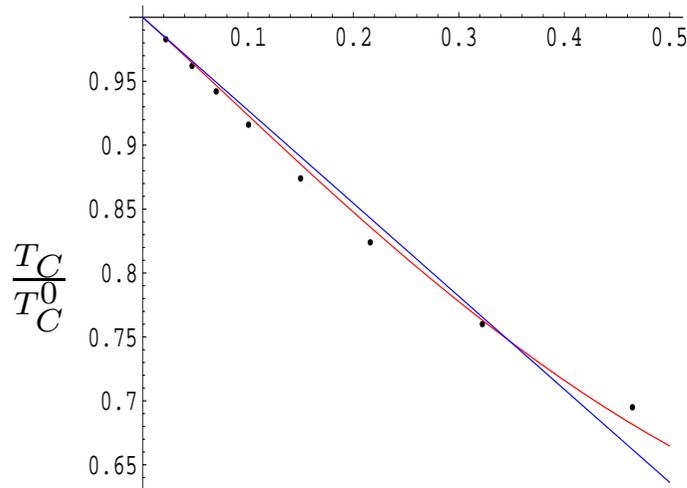
$$\tilde{\zeta}_1(\tilde{z}) = \int_{\beta(E_1 - E_0)}^{\infty} dx \frac{1}{\tilde{z}^{-1} e^x - 1} = -\log(1 - e^{-\beta(E_1 - E_0)\tilde{z}})$$

$$\tilde{\zeta}_1(1) \approx -\log(\hbar\beta\omega)$$

3. Finite-Size Corrections of Critical Temperature

Expansion for Critical Temperature:

$$T_C = \frac{\hbar\omega}{k_B} \left(\frac{N}{\zeta(3)} \right)^{1/3} \left\{ 1 - \frac{\zeta(2)}{2N^{1/3}\zeta(3)^{2/3}} + \frac{3\zeta(2)^2 + 4/3 \log[\zeta(3)/N]}{12N^{2/3}\zeta(3)^{1/3}} + \dots \right\}$$



$N^{-1/3}$

Grand-Canonical Results:

- First Order Finite-Size Correction
- Second Order Finite-Size Correction
- Canonical Result

N	T_C/T_C^0
100 000	0.98
1000	0.93
100	0.85
10	0.73